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Chow–Heegner Points on CM Elliptic Curves and Values of p-adic L-functions

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We outline a new construction of rational points on CM elliptic curves, using cycles on higher-dimensional varieties, contingent on certain cases of the Tate conjecture. This construction admits of complex and p-adic analogs that are defined independently of the Tate conjecture. In the p-adic case, using p-adic Rankin L-functions and a p-adic Gross–Zagier type formula proved in our articles [2, 3], we show unconditionally that the points so constructed are in fact rational. In the complex case, we are unable to prove rationality (or even algebraicity) but we can verify it numerically in several cases.

1 Introduction

The theory of Heegner points supplies one of the most fruitful approaches to the Birch and Swinnerton-Dyer conjecture, leading to the best results for elliptic curves of analytic rank 1. In spite of attempts to broaden the scope of the Heegner point construction ([1, 7, 8, 20]), all provable, systematic constructions of algebraic points on elliptic curves still rely on parameterizations of elliptic curves by modular or Shimura curves. The primary goal of this article is to explore new constructions of rational points on elliptic

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curves and abelian varieties in which, loosely speaking, Heegner divisors are replaced by higher-dimensional algebraic cycles on certain modular varieties. In general, the algebraicity of the resulting points depends on the validity of ostensibly difficult cases of the Hodge or Tate conjectures. One of the main theorems of this article (Theorem 1.4 of Section 1) illustrates how these algebraicity statements can sometimes be obtained unconditionally by exploiting the connection between the relevant "generalized Heegner cycles" and values of certain p-adic Rankin L-series.

We begin with a brief sketch of the classical picture that we aim to generalize. It is known thanks to [6, 19, 21] that all elliptic curves over the rationals are modular. For an elliptic curve A of conductor N, this means that

$$L(A, s) = L(f, s), \tag{1.1}$$

where $f(z) = \sum a_n e^{2\pi i n z}$ is a cusp form of weight 2 on the Hecke congruence group $\Gamma_0(N)$. The modularity of A is established by showing that the p-adic Galois representation

$$V_p(A) := \left(\lim_{\leftarrow} A[p^n]\right) \otimes \mathbb{Q}_p = H^1_{\mathrm{et}}(\bar{A}, \mathbb{Q}_p)(1)$$

is a constituent of the first p-adic étale cohomology of the modular curve $X_0(N)$. On the other hand, the Eichler–Shimura construction attaches to f an elliptic curve quotient A_f of the Jacobian $J_0(N)$ of $X_0(N)$ satisfying $L(A_f, s) = L(f, s)$. In particular, the semisimple Galois representations $V_p(A_f)$ and $V_p(A)$ are isomorphic. It follows from Faltings' proof of the Tate conjecture for abelian varieties over number fields that A is isogenous to A_f, and therefore, there is a nonconstant morphism

$$\Phi : J_0(N) \longrightarrow A \tag{1.2}$$

of algebraic varieties over $\mathbb{Q}$, inducing, for each $F \supset \mathbb{Q}$, a map $\Phi_F : J_0(N)(F) \longrightarrow A(F)$ on F-rational points.

A key application of Φ arises from the fact that $X_0(N)$ is equipped with a distinguished supply of algebraic points corresponding to the moduli of elliptic curves with complex multiplication by an order in a quadratic imaginary field K. The images under $\Phi_{\bar{\mathbb{Q}}}$ of the degree 0 divisors supported on these points produce elements of $A(\bar{\mathbb{Q}})$ defined over abelian extensions of K, which include the so-called *Heegner points*. The Gross–Zagier formula [12] relates the canonical heights of these points to the central critical derivatives of $L(A/K, s)$ and of its twists by (unramified) abelian characters of K. This connection between algebraic points and Hasse–Weil L-series has led to the strongest known results on the Birch and Swinnerton-Dyer conjecture, most notably the theorem that

$$\mathrm{rank}(A(\mathbb{Q})) = \mathrm{ord}_{s=1} L(A, s) \text{ and } \#\text{Ш}(A/\mathbb{Q}) < \infty, \quad \text{when } \mathrm{ord}_{s=1}(L(A, s)) \leq 1,$$

which follows by combining the Gross-Zagier formula and a method of Kolyvagin [11] are combined. The theory of Heegner points is also the key ingredient in the proof of the main results in [3].

Given a variety X (defined over $\mathbb{Q}$, say), let $\mathrm{CH}^j(X)(F)$ denote the Chow group of codimension j algebraic cycles on X defined over a field F modulo rational equivalence and let $\mathrm{CH}^j(X)_0(F)$ denote the subgroup of null-homologous cycles. Write $\mathrm{CH}^j(X)$ and $\mathrm{CH}^j(X)_0$ for the corresponding functors on $\mathbb{Q}$-algebras. Via the natural equivalence $\mathrm{CH}^1(X_0(N))_0 = J_0(N)$, the map Φ of (1.2) can be recast as a natural transformation

$$\Phi : \mathrm{CH}^1(X_0(N))_0 \longrightarrow A. \tag{1.3}$$

It is tempting to generalize (1.3) by replacing $X_0(N)$ by a variety X over $\mathbb{Q}$ of dimension $d > 1$ and $\mathrm{CH}^1(X_0(N))_0$ by $\mathrm{CH}^j(X)_0$ for some $0 \leq j \leq d$. Any element Π of the Chow group $\mathrm{CH}^{d+1-j}(X \times A)(\mathbb{Q})$ induces a natural transformation

$$\Phi : \mathrm{CH}^j(X)_0 \longrightarrow A \tag{1.4}$$

sending $\Delta \in \mathrm{CH}^j(X)_0(F)$ to

$$\Phi_F(\Delta) := \pi_{A,*}(\pi_X^*(\tilde{\Delta}) \cdot \tilde{\Pi}), \tag{1.5}$$

where π_X and π_A denote the natural projections from $X \times A$ to X and A, respectively. We are interested mainly in the case where X is a Shimura variety or is closely related to a Shimura variety. (For instance, when X is the universal object or a self-fold fiber product of the universal object over a Shimura variety of PEL type.) The variety X is then referred to as a *modular variety* and the natural transformation Φ is called the *modular parameterization of A* attached to the pair (X, Π).

Modular parameterizations acquire special interest when $\mathrm{CH}^j(X)_0(\bar{\mathbb{Q}})$ is equipped with a systematic supply of special elements, such as those arising from Shimura subvarieties of X. The images in $A(\bar{\mathbb{Q}})$ of such special elements under $\Phi_{\bar{\mathbb{Q}}}$ can be viewed as "higher dimensional" analogs of Heegner points: they will be referred to as *Chow–Heegner points*. Given an elliptic curve A, it would be of interest to construct modular parameterizations to A in the greatest possible generality, study their basic properties, and explore the relations (if any) between the resulting systems of Chow–Heegner points and leading terms of L-series attached to A.

We develop this loosely formulated program in the simple but nontrivial setting where A is an elliptic curve with complex multiplication by an imaginary quadratic field K of odd discriminant $-D$, and X is a suitable family of $2r$-dimensional abelian varieties fibered over a modular curve.

For Section 1, suppose for simplicity that K has class number one and that A is the canonical elliptic curve over $\mathbb{Q}$ of conductor D^2 attached to the Hecke character defined by

$$\psi_A((a)) = \varepsilon_K(a \bmod \sqrt{-D})a,$$

where ε_K is the quadratic character of conductor D associated with the field K. (These assumptions will be significantly relaxed in the body of the paper.) Given a nonzero differential $\omega_A \in \Omega^1(A/\mathbb{Q})$, let $[\omega_A]$ denote the corresponding class in the de Rham cohomology of A.

Fix an integer $r \geq 0$, and consider the Hecke character $\psi = \psi_A^{r+1}$. The binary theta series

$$\theta_\psi := \sum_{\mathfrak{a} \subset \mathcal{O}_K} \psi_A^{r+1}(\mathfrak{a}) q^{\mathfrak{a}\bar{\mathfrak{a}}}$$

attached to ψ is a modular form of weight $r+2$ on a certain modular curve C (which is a quotient of $X_1(D)$ or $X_0(D^2)$ depending on whether r is odd or even), and has rational Fourier coefficients. Such a modular form gives rise to a regular differential $(r+1)$-form ω_{θ_ψ} on the rth *Kuga–Sato variety* over C, denoted by W_r. Let $[\omega_{\theta_\psi}]$ denote the class of ω_{θ_ψ} in the de Rham cohomology $H^{r+1}_{\mathrm{dR}}(W_r/\mathbb{Q})$. The classes of ω_{θ_ψ} and of the antiholomorphic $(r+1)$-form $\bar{\omega}_{\theta_\psi}$ generate the θ_ψ-isotypic component of $H^{r+1}_{\mathrm{dR}}(W_r/\mathbb{C})$ under the action of the Hecke correspondences.

For all $1 \leq j \leq r+1$, let $p_j : A^{r+1} \longrightarrow A$ denote the projection onto the jth factor, and let

$$[\omega_A^{r+1}] := p_1^*[\omega_A] \wedge \cdots \wedge p_{r+1}^*[\omega_A] \in H^{r+1}_{\mathrm{dR}}(A^{r+1}).$$

Our construction of Chow–Heegner points is based on the following conjecture, which is formulated (for more general K, without the class number one hypothesis) in Section 2.

Conjecture 1.1. There is an algebraic cycle $\Pi^? \in \mathrm{CH}^{r+1}(W_r \times A^{r+1})(K) \otimes \mathbb{Q}$ satisfying

$$\Pi^{?*}_{\mathrm{dR}}([\omega_A^{r+1}]) = c_{\psi,K} \cdot [\omega_{\theta_\psi}],$$

for some element $c_{\psi,K}$ in $K^\times$, where

$$\Pi^{?*}_{\mathrm{dR}} : H^{r+1}_{\mathrm{dR}}(A^{r+1}/K) \longrightarrow H^{r+1}_{\mathrm{dR}}(W_r/K)$$

is the map on de Rham cohomology induced by $\Pi^?$. □

Remark 1.2. In fact, in the special case considered above, using that A is defined over $\mathbb{Q}$ and not just K, one can arrange the cycle $\Pi^?$ to be defined over $\mathbb{Q}$ (if it exists at all!). Then $c_{\psi,K}$ lies in $\mathbb{Q}^\times$, and by appropriately scaling $\Pi^?$, we may further arrange that $c_{\psi,K} = 1$. This would simplify some of the forthcoming discussion, see for example the commutative diagram (1.13). However, we have chosen to retain the constant $c_{\psi,K}$ in the rest of the introduction in order to give the reader a better picture of the more general situation considered in the main text where the class number of K is not 1 and the curve A can be defined only over some extension of K. □

Remark 1.3. The rationale for Conjecture 1.1 is explained in Section 2.4, where it is shown to follow from the Tate conjecture on algebraic cycles. To the authors' knowledge, the existence of $\Pi^?$ is known only in the following cases:

(1) $r = 0$, where it follows from Faltings' proof of the Tate conjecture for a product of curves over number fields;
(2) $(r, D) = (1, -4)$ (see [16, Remark 2.4.1]) and $(1, -7)$, where it can be proved using the theory of Shioda–Inose structures and the fact that W_r is a singular K3 surface [10];
(3) $r = 2$ and $D = -3$, (see [18, Section 1]).

For general values of r and D, Conjecture 1.1 appears to lie rather deep and might be touted as a good "proving ground" for the validity of the Hodge and Tate conjectures. One of the main results of this paper—Theorem 1.4—uses p-adic methods to establish *unconditionally* a consequence of Conjecture 1.1, leading to the construction of rational points on A. The complex calculations of the last section likewise lend numerical support for a (ostensibly deeper) complex analog of Theorem 1.4. Sections 3 and 4 may therefore be viewed as providing indirect support (of a theoretical and experimental nature, respectively) for the validity of Conjecture 1.1. □

We next make the simple (but key) remark that the putative cycle $\Pi^?$ is also an element of the Chow group $\mathrm{CH}^{r+1}(X_r \times A) \otimes \mathbb{Q}$, where X_r is the $(2r+1)$-dimensional variety

$$X_r := W_r \times A^r.$$

Viewed in this way, the cycle $\Pi^?$ gives rise to a modular parameterization

$$\Phi^? : \mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}} := \mathrm{CH}^{r+1}(X_r)_0 \otimes \mathbb{Q} \longrightarrow A \otimes \mathbb{Q} \tag{1.6}$$

as in (1.4), that is defined over K, that is, there is a natural map

$$\Phi^?_F : \mathrm{CH}^{r+1}(X_r)_0(F) \longrightarrow A(F) \otimes \mathbb{Q}$$

for any field F containing K. Furthermore, it satisfies the equation

$$\Phi^{?*}_{\mathrm{dR}}(\omega_A) = c_{\psi,K} \cdot \omega_{\theta_\psi} \wedge \eta_A^r. \tag{1.7}$$

Here η_A is the unique element of $H^1_{\mathrm{dR}}(A/K)$ satisfying

$$[\lambda]^*\eta_A = \bar{\lambda}\eta_A, \quad \text{for all } \lambda \in \mathcal{O}_K, \quad \langle \omega_A, \eta_A \rangle = 1, \tag{1.8}$$

where $[\lambda]$ denotes the element of $\mathrm{End}_K(A)$ corresponding to λ. (See Proposition 2.11 for details.)

Article [2] introduced and studied a collection of null-homologous, r-dimensional algebraic cycles on X_r, that is, elements of the source $\mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}$ of the map (1.6), referred to as *generalized Heegner cycles*. These cycles, whose precise definition is recalled in Section 2.5, extend the notion of Heegner cycles on Kuga–Sato varieties considered in [13, 17, 22]. They are indexed by isogenies $\varphi : A \longrightarrow A'$ and are defined over abelian extensions of K. It can be shown that they generate a subspace of $\mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(K^{\mathrm{ab}})$ of infinite dimension. The map $\Phi^?_{K^{\mathrm{ab}}}$ (if it exists) transforms these generalized Heegner cycles into points of $A(K^{\mathrm{ab}}) \otimes \mathbb{Q}$. It is natural to expect that the resulting collection $\{\Phi^?_{K^{\mathrm{ab}}}(\Delta_\varphi)\}_{\varphi:A\longrightarrow A'}$ of Chow–Heegner points generates an infinite-dimensional subspace of $A(K^{\mathrm{ab}})\otimes\mathbb{Q}$ and that it gives rise to an "Euler system" in the sense of Kolyvagin.

In the classical situation where $r=0$, the variety X_r is just a modular curve and (as already mentioned) the existence of $\Phi^?$ follows from Faltings' proof of the Tate conjecture for products of curves. When $r\geq 1$, Section 3 uses p-adic methods to show that an alternative cohomological construction of $\Phi^?_{K^{\mathrm{ab}}}(\Delta_\varphi)$ gives rise in many cases to algebraic points on A with the expected field of rationality and offers, therefore, some theoretical evidence for the existence of $\Phi^?$. We now describe this construction briefly.

Let p be a rational prime split in K and fix a prime $\mathfrak{p}$ of K above p. As explained in Remark 2.12 of Section 2.4, even without the Tate conjecture, one can still define a natural $G_K:=\mathrm{Gal}(\bar{K}/K)$-equivariant projection

$$\Phi^*_{\mathrm{et},\mathfrak{p}}:H^{2r+1}_{\mathrm{et}}(\bar{X}_r,\mathbb{Q}_p)(r+1)\longrightarrow H^1_{\mathrm{et}}(\bar{A},\mathbb{Q}_p)(1)=V_p(A), \tag{1.9}$$

where $V_p(A)$ is the p-adic Galois representation arising from the p-adic Tate module of A. A priori, this last map is well defined only up to an element in $\mathbb{Q}_p^\times$. We normalize it by embedding K in $\mathbb{Q}_p$ via $\mathfrak{p}$ and requiring that the map

$$\Phi^*_{\mathrm{dR},\mathfrak{p}}:H^{2r+1}_{\mathrm{dR}}(X_r/\mathbb{Q}_p)\longrightarrow H^1_{\mathrm{dR}}(A/\mathbb{Q}_p)$$

obtained by applying to $\Phi^*_{\mathrm{et},\mathfrak{p}}$ the comparison functor between p-adic étale cohomology and de Rham cohomology over p-adic fields satisfies

$$\Phi^*_{\mathrm{dR},\mathfrak{p}}(\omega_{\theta_\psi}\wedge\eta_A^r)=\omega_A, \tag{1.10}$$

where ω_{θ_ψ} and ω_A are as in Conjecture 1.1, and η_A is defined in (1.8). We can then define the following p-adic avatars of $\Phi^?$ without invoking the Tate conjecture:

(a) *The map* Φ^{et}_F:

Let F be a field containing K. The Chow group $\mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(F)$ of null-homologous cycles is equipped with the p-adic étale Abel–Jacobi map over F:

$$\mathrm{AJ}^{\mathrm{et}}_F:\mathrm{CH}^{r+1}(X_r)(F)_{0,\mathbb{Q}}\longrightarrow H^1(F,H^{2r+1}_{\mathrm{et}}(\bar{X}_r,\mathbb{Q}_p)(r+1)), \tag{1.11}$$

where $H^1(F, M)$ denotes the continuous Galois cohomology of a $G_F := \mathrm{Gal}(\bar{F}/F)$-module M. The maps (1.11) and (1.9) can be combined to give a map

$$\Phi_F^{\mathrm{et}} : \mathrm{CH}^{r+1}(X_r)(F)_{0,\mathbb{Q}} \longrightarrow H^1(F, V_p(A)), \tag{1.12}$$

which is the counterpart in p-adic étale cohomology of the conjectural map $\Phi_F^?$. More precisely, the map Φ_F^{et} is related to $\Phi_F^?$ (when the latter can be shown to exist) by the commutative diagram

$$\begin{tikzcd} & & A(F)\otimes\mathbb{Q} \arrow[d, "\delta"] \\ \mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(F) \arrow[urr, dashed, "\Phi_F^?"] \arrow[r, "\Phi_F^{\mathrm{et}}"] & H^1(F, V_p(A)) \arrow[r, dashed, "\cdot c_{\psi,K}"] & H^1(F, V_p(A)) \end{tikzcd} \tag{1.13}$$

where

$$\delta : A(F) \otimes \mathbb{Q} \longrightarrow H^1(F, V_p(A)) \tag{1.14}$$

is the projective limit of the connecting homomorphisms arising in the p^n-descent exact sequences of Kummer theory, and $c_{\psi,K}$ is the element in $K^\times$ from Conjecture 1.1 viewed as living in $\mathbb{Q}_p^\times$ via the embedding of K in $\mathbb{Q}_p$ corresponding to $\mathfrak{p}$.
(b) *The map* $\Phi_F^{(v)}$: When F is a number field, (1.13) suggests that the image of Φ_F^{et} is contained in the Selmer group of A over F, and this can indeed be shown to be the case. In fact, one can show that for every finite place v of F, the image of $\Phi_{F_v}^{\mathrm{et}}$ is contained in the images of the local connecting homomorphisms

$$\delta_v : A(F_v) \otimes \mathbb{Q} \longrightarrow H^1(F_v, V_p(A)).$$

In particular, fixing a place v of F and replacing F by its v-adic completion F_v, we can *define* a map $\Phi_F^{(v)}$ by the commutativity of the following local counterpart of the diagram (1.13):

$$\begin{tikzcd} & A(F_v)\otimes\mathbb{Q} \arrow[d, "\delta_v"] \\ \mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(F_v) \arrow[ur, bend left, "\Phi_F^{(v)}"] \arrow[r, "\Phi_{F_v}^{\mathrm{et}}"] & H^1(F_v, V_p(A)) \end{tikzcd} \tag{1.15}$$

As will be explained in greater detail in Section 3, when v is a place lying over $\mathfrak{p}$, the map $\Phi_F^{(v)}$ can also be defined by p-adic integration, via the comparison theorems between the p-adic étale cohomology and the de Rham cohomology of varieties over p-adic fields.

The main theorem of this paper, which is proved in Section 3, relates the Selmer classes of the form $\Phi_F^{\mathrm{et}}(\Delta)$ when F is a number field and Δ is a generalized Heegner cycle to global points in $A(F)$. We will state only a special case of the main result, postponing the more general statements to Section 3.2. Assume for Theorem 1.4 that the field K has odd discriminant, that the sign in the functional equation for $L(\psi_A, s)$ is -1, so that the Hasse–Weil L-series $L(A/\mathbb{Q}, s) = L(\psi_A, s)$ vanishes to odd order at $s = 1$, and that the integer r is odd. In that case, the theta series θ_ψ belongs to the space $S_{r+2}(\Gamma_0(D), \varepsilon_K)$ of cusp forms on $\Gamma_0(D)$ of weight $r+2$ and character $\varepsilon_K := (\frac{\cdot}{D})$. In particular, the variety W_r is essentially the rth Kuga–Sato variety over the modular curve $X_0(D)$. Furthermore, the L-series $L(\psi_A^{2r+1}, s)$ has sign $+1$ in its functional equation, and $L(\psi_A^{2r+1}, s)$ therefore vanishes to *even order* at the central point $s = r+1$.

Theorem 1.4. Let Δ_r be the generalized Heegner cycle in $\mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(K)$ attached to the identity isogeny $1 : A \longrightarrow A$. The cohomology class $\Phi_K^{\mathrm{et}}(\Delta_r)$ belongs to $\delta(A(K) \otimes \mathbb{Q})$. More precisely, there is a point $P_D \in A(K) \otimes \mathbb{Q}$ (depending on D but not on r) such that

$$\Phi_K^{\mathrm{et}}(\Delta_r) = \sqrt{-D} \cdot m_{D,r} \cdot \delta(P_D),$$

where $m_{D,r} \in \mathbb{Z}$ satisfies

$$m_{D,r}^2 = \frac{2r!(2\pi\sqrt{D})^r}{\Omega(A)^{2r+1}} L(\psi_A^{2r+1}, r+1),$$

and $\Omega(A)$ is a complex period attached to A. The point P_D is of infinite order if and only if

$$L'(\psi_A, 1) \neq 0. \qquad \square$$

This result is proved, in a more general form, in Theorems 3.3 and 3.5 of Section 3.2. For an even more general (but less precise) statement in which the simplifying assumptions imposed in Theorem 1.4 are considerably relaxed, see Theorem 3.4.

Remark 1.5. When $L(A, s)$ has a simple zero at $s = 1$, it is known a priori that the Selmer group $\mathrm{Sel}_p(A/K)$ is of rank 1 over $K \otimes \mathbb{Q}_p$ and agrees with $\delta(A(K) \otimes \mathbb{Q}_p)$. It follows directly that

$$\Phi_K^{\mathrm{et}}(\Delta_r) \text{ belongs to } \delta(A(K) \otimes \mathbb{Q}_p).$$

The first part of Theorem 1.4 is *significantly stronger* in that it involves the rational vector space $A(K) \otimes \mathbb{Q}$ rather than its p-adification. This stronger statement is not a formal consequence of the one-dimensionality of the Selmer group. Indeed, its proof relies on invoking [3, Theorem 2] after relating the local point $\Phi_{K_\mathfrak{p}}^{(\mathfrak{p})}(\Delta) \in A(K_\mathfrak{p}) \otimes \mathbb{Q} = A(\mathbb{Q}_p) \otimes \mathbb{Q}$ to the special value $\mathscr{L}_p(\psi_A^*)$ of the Katz two-variable p-adic L-function that arises in that theorem. □

Finally, we discuss the picture over the complex numbers. Section 4.1 describes a complex homomorphism

$$\Phi_\mathbb{C} : \mathrm{CH}^{r+1}(X_r)_0(\mathbb{C}) \longrightarrow A(\mathbb{C}),$$

which is defined analytically by integration of differential forms on $X_r(\mathbb{C})$, without invoking Conjecture 1.1, but agrees with $\Phi_\mathbb{C}^?$ (up to multiplication by some nonzero element in $\mathcal{O}_K$) when the latter exists. This map is defined using the complex Abel–Jacobi map on cycles introduced and studied by Griffiths and Weil and is the complex analog of the homomorphism $\Phi_{K_\mathfrak{p}}^{(\mathfrak{p})}$. The existence of the global map $\Phi_K^?$ predicted by the Hodge or Tate conjecture would imply the following algebraicity statement:

Conjecture 1.6. Let H be a subfield of K^{ab} and let $\Delta_\varphi \in \mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(H)$ be a generalized Heegner cycle defined over H. Then (after fixing an embedding of H into $\mathbb{C}$),

$$\Phi_\mathbb{C}(\Delta_\varphi) \text{ belongs to } A(H) \otimes \mathbb{Q},$$

and

$$\Phi_\mathbb{C}(\Delta_\varphi^\sigma) = \Phi_\mathbb{C}(\Delta_\varphi)^\sigma \quad \text{for all } \sigma \in \mathrm{Gal}(H/K).$$ □

While ostensibly weaker than Conjecture 1.1, Conjecture 1.6 has the virtue of being more readily amenable to experimental verification. Section 4 explains how the images of generalized Heegner cycles under $\Phi_\mathbb{C}$ can be computed numerically to high accuracy, and illustrates, for a few such Δ_φ, how the points $\Phi_\mathbb{C}(\Delta_\varphi)$ can be recognized as algebraic points defined over the predicted class fields. In particular, extensive numerical verifications of Conjecture 1.6 are carried out, for fairly large values of r.

On the theoretical side, this conjecture appears to lie deeper than its p-adic counterpart, and we were unable to provide any theoretical evidence for it beyond the fact that it follows from the Hodge or Tate conjectures. It might be argued that calculations of the sort that are performed in Section 4 provide independent numerical confirmation of these conjectures for certain specific Hodge and Tate cycles on the $(2r+2)$-dimensional varieties $W_r \times A^{r+1}$, for which the corresponding algebraic cycles seem hard to produce unconditionally.

Conventions regarding number fields and embeddings: Throughout this article, all number fields that arise are viewed as embedded in a fixed algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. A complex embedding $\bar{\mathbb{Q}} \longrightarrow \mathbb{C}$ and p-adic embeddings $\bar{\mathbb{Q}} \longrightarrow \mathbb{C}_p$ for each rational prime p are also fixed from the outset, so that any finite extension of $\mathbb{Q}$ is simultaneously realized as a subfield of $\mathbb{C}$ and of $\mathbb{C}_p$.

2 Motives and Chow–Heegner Points

The goal of the first three sections of this chapter is to recall the construction of the motives attached to Hecke characters and to modular forms. The remaining three sections are devoted to the definition of Chow–Heegner points on CM elliptic curves, as the image of generalized Heegner cycles by modular parameterizations attached to CM forms.

2.1 Motives for rational and homological equivalence

We begin by laying down our conventions regarding motives, following [9]. We will work with either Chow motives or Grothendieck motives. For X, a nonsingular variety over a number field F, let $C^m(X)$ denote the group of algebraic cycles of codimension m on X defined over F. Let $\sim$ denote rational equivalence in $C^m(X)$, and set

$$\mathcal{C}^m(X) := \mathrm{CH}^m(X) = C^m(X)/\sim .$$

Given two nonsingular varieties X and Y over F and a number field E, we define the groups of correspondences

$$\mathrm{Corr}^m(X, Y) := \mathrm{CH}^{\dim X + m}(X \times Y) \quad \mathrm{Corr}^m(X, Y)_E := \mathrm{Corr}^m(X, Y) \otimes_{\mathbb{Z}} E.$$

Definition 2.1. A *motive* over F with coefficients in E is a triple (X, e, m) where X/F is a nonsingular projective variety, $e \in \mathrm{Corr}^0(X, X)_E$ is an idempotent, and m is an integer. □

Definition 2.2. The category $\mathcal{M}_{F,E}$ of *Chow motives* is the category whose objects are motives over F with coefficients in E, with morphisms defined by

$$\mathrm{Hom}_{\mathcal{M}_{F,E}}((X, e, m), (Y, f, n)) = f \circ \mathrm{Corr}^{n-m}(X, Y)_{\mathbb{Q}} \circ e. \qquad \square$$

The category $\mathcal{M}_{F,E}^{\mathrm{hom}}$ of *Grothendieck motives* is defined in exactly the same way, but with homological equivalence replacing rational equivalence. We will denote the corresponding groups of cycle classes by $\mathcal{C}^r(X)_0$, $\mathrm{Corr}_0^m(X, Y)$, $\mathrm{Corr}_0^m(X, Y)_E$, etc.

Since rational equivalence is finer than homological equivalence, there is a natural functor

$$\mathcal{M}_{F,E} \to \mathcal{M}_{F,E}^{\mathrm{hom}},$$

so that every Chow motive gives rise to a Grothendieck motive. Furthermore, the category of Grothendieck motives is equipped with natural *realization functors* arising from any cohomology theory satisfying the Weil axioms. We now recall the description of the image of a motive $M = (X, e, m)$ over F with coefficients in E under the most important realizations:

The Betti realization: Recall that our conventions about number fields supply us with an embedding $F \longrightarrow \mathbb{C}$. The Betti realization is defined in terms of this embedding by

$$M_B := e \cdot (H^*(X(\mathbb{C}), \mathbb{Q})(m) \otimes E).$$

It is a finite-dimensional E-vector space with a natural E-Hodge structure arising from the comparison isomorphism between the singular cohomology and the de Rham cohomology over $\mathbb{C}$.

The ℓ-adic realization: Let $\bar{X}$ denote the base change of X to $\bar{\mathbb{Q}}$. The ℓ-adic cohomology of $\bar{X}$ gives rise to the ℓ-adic étale realization of M:

$$M_\ell := e \cdot (H^*_{\mathrm{et}}(\bar{X}, \mathbb{Q}_\ell(m)) \otimes E).$$

It is a free $E \otimes \mathbb{Q}_\ell$-module of finite rank equipped with a continuous linear G_F-action.

The de Rham realization: The de Rham realization of M is defined by

$$M_{\mathrm{dR}} := e \cdot (H^*_{\mathrm{dR}}(X/F)(m) \otimes_{\mathbb{Q}} E),$$

where $H^*_{\mathrm{dR}}(X/F)$ denotes the algebraic de Rham cohomology of X. The module M_{dR} is a free $E \otimes F$-module of finite rank equipped with a decreasing, separated and exhaustive Hodge filtration.

Moreover, there are natural comparison isomorphisms

$$M_B \otimes_{\mathbb{Q}} \mathbb{C} \simeq M_{\mathrm{dR}} \otimes_F \mathbb{C}, \tag{2.1}$$

$$M_B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \simeq M_\ell, \tag{2.2}$$

which are $E \otimes \mathbb{C}$-linear and $E \otimes \mathbb{Q}_\ell$-linear, respectively. Thus,

$$\mathrm{rank}_E M_B = \mathrm{rank}_{E\otimes F} M_{\mathrm{dR}} = \mathrm{rank}_{E\otimes \mathbb{Q}_\ell}(M_\ell),$$

and this common integer is called the E-rank of the motive M.

Remark 2.3. If F is a p-adic field, one also has a comparison isomorphism

$$M_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR},p} \simeq M_{\mathrm{dR}} \otimes_F B_{\mathrm{dR},p}, \tag{2.3}$$

where $B_{\mathrm{dR},p}$ is Fontaine's ring of p-adic periods, which is endowed with a decreasing, exhaustive filtration and a continuous G_F-action. This comparison isomorphism is compatible with natural filtrations and G_F-actions on both sides. □

Remark 2.4. Our definition of motives with coefficients coincides with *Language B* of Deligne [9]. There is an equivalent way of defining motives with coefficients (the *Language A*) where the objects are motives M in $\mathcal{M}_{F,\mathbb{Q}}$ equipped with the structure of an E-module: $E \to \mathrm{End}(M)$, and morphisms are those that commute with the E-action. We refer the reader to Section 2.1 of loc. cit. for the translation between these points of view. □

2.2 The motive of a Hecke character

In this section, we recall how to attach a motive to an algebraic Hecke character ψ of an quadratic imaginary field K of infinity type $(r, 0)$. (The reader is referred to [3, Section 2.1] for out notations and conventions regarding algebraic Hecke characters.) This generalizes the exposition of [3, Section 2.2], where we recall how an abelian variety with complex multiplication is attached to a Hecke character of K of infinity type $(1, 0)$. For more general algebraic Hecke characters that are not of type $(1, 0)$, one no longer has an associated abelian variety. Nevertheless, such a character still gives rise to a motive over K with coefficients in the field generated by its values.

Suppose that $\psi : \mathbb{A}_K^\times \to \mathbb{C}^\times$ is such a Hecke character and let E_ψ be the field generated over K by the values of ψ on the finite idèles. Pick a finite Galois extension F of K such that $\psi_F := \psi \circ \mathrm{N}_{F/K}$ satisfies the equation

$$\psi_F = \psi_A^r,$$

where ψ_A is the Hecke character of F with values in K associated with an elliptic curve A/F with complex multiplication by $\mathcal{O}_K$.

We construct motives $M(\psi_F) \in \mathcal{M}_{F,\mathbb{Q}}$, $M(\psi_F)_K \in \mathcal{M}_{F,K}$ associated to ψ_F by considering an appropriate piece of the middle cohomology of the variety A^r over F. Similar to Section 1, write $[\alpha]$ for the element of $\mathrm{End}_F(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ corresponding to an element $\alpha \in K$. Define an idempotent $e_r = e_r^{(1)} \circ e_r^{(2)} \in \mathrm{Corr}^0(A^r, A^r)_{\mathbb{Q}}$ by setting

$$e_r^{(1)} := \left(\frac{\sqrt{-D} + [\sqrt{-D}]}{2\sqrt{-D}}\right)^{\otimes r} + \left(\frac{\sqrt{-D} - [\sqrt{-D}]}{2\sqrt{-D}}\right)^{\otimes r}, \quad e_r^{(2)} := \left(\frac{1 - [-1]}{2}\right)^{\otimes r}.$$

Let $M(\psi_F)$ be the motive in $\mathcal{M}_{F,\mathbb{Q}}$ defined by

$$M(\psi_F) := (A^r, e_r, 0),$$

and let $M(\psi_F)_K$ denote the motive in $\mathcal{M}_{F,K}$ obtained (in Language A) by making K act on $M(\psi_F)$ via its diagonal action on A^r. The ℓ-adic étale realization $M(\psi_F)_{K,\ell}$ is free of rank 1 over $K \otimes \mathbb{Q}_\ell$, and G_F acts on it via ψ_F, viewed as a $(K \otimes \mathbb{Q}_\ell)^\times$-valued Galois character:

$$M(\psi_F)_\ell = e_r H^r_{\mathrm{et}}(\bar{A}^r, \mathbb{Q}_\ell) = (K \otimes \mathbb{Q}_\ell)(\psi_F).$$

The de Rham realization $M(\psi_F)_{K,\mathrm{dR}}$ is a free one-dimensional $F \otimes_{\mathbb{Q}} K$-vector space, generated as an F-vector space by the classes of

$$\omega_A^r := e_r(\omega_A \wedge \cdots \wedge \omega_A) \quad \text{and} \quad \eta_A^r := e_r(\eta_A \wedge \cdots \wedge \eta_A),$$

where η_A is the unique class in $H^1_{\mathrm{dR}}(A/F)$ satisfying

$$[\alpha]^* \eta_A = \bar{\alpha}\eta_A \text{ for all } \alpha \in K, \quad \text{and} \quad \langle \omega_A, \eta_A \rangle = 1.$$

The Hodge filtration on $M(\psi_F)_{\mathrm{dR}}$ is given by

$$\mathrm{Fil}^0 M(\psi_F)_{\mathrm{dR}} = M(\psi_F)_{\mathrm{dR}} = F \cdot \omega_A^r + F \cdot \eta_A^r,$$
$$\mathrm{Fil}^1 M(\psi_F)_{\mathrm{dR}} = \cdots = \mathrm{Fil}^r M(\psi_F)_{\mathrm{dR}} = F \cdot \omega_A^r,$$
$$\mathrm{Fil}^{r+1} M(\psi_F)_{\mathrm{dR}} = 0.$$

It can be shown that after extending coefficients to E_ψ, the motive $M(\psi_F)_K$ descends to a motive $M(\psi) \in \mathcal{M}_{K,E_\psi}$, whose ℓ-adic realization is a free rank 1 module over $E_\psi \otimes \mathbb{Q}_\ell$ on which G_K acts via the character ψ. In this article, however, we shall only make use of the motives $M(\psi_F)$ and $M(\psi_F)_K$.

2.3 Deligne–Scholl motives

Let $S_{r+2}(\Gamma_0(N), \varepsilon)$ be the space of cusp forms on $\Gamma_0(N)$ of weight $r+2$ and nebentype character ε. In this section, we will let ψ be a Hecke character of K of infinity type $(r+1, 0)$. This Hecke character gives rise to a theta series

$$\theta_\psi = \sum_{n=1}^{\infty} a_n(\theta_\psi) q^n \in S_{r+2}(\Gamma_0(N), \varepsilon)$$

as in [3, Proposition 3.13], with $N := D \cdot \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f})$ and $\varepsilon := \varepsilon_\psi \cdot \varepsilon_K$, where ε_ψ is the central character of ψ (see [3, Definition 2.2]) and ε_K is the quadratic Dirichlet character associated to the extension $K/\mathbb{Q}$. Observe that the subfield E_{θ_ψ} of $\bar{\mathbb{Q}}$ generated by the Fourier coefficients $a_n(\theta_\psi)$ is always contained in E_ψ and if ψ is a self-dual character (see [3, Definition 3.4]) then E_{θ_ψ} is a totally real field, and $E_\psi = E_{\theta_\psi} K$.

Deligne has attached to θ_ψ a compatible system $\{V_\ell(\theta_\psi)\}$ of two-dimensional ℓ-adic representations of $G_{\mathbb{Q}}$ with coefficients in $E_{\theta_\psi} \otimes \mathbb{Q}_\ell$, such that for any prime $p \nmid N\ell$, the characteristic polynomial of the Frobenius element at p is given by

$$X^2 - a_p(\theta_\psi)X + \varepsilon(p)p^{r+1}.$$

This representation is realized in the middle ℓ-adic cohomology of a variety that is fibered over a modular curve. More precisely, let $\Gamma := \Gamma_\varepsilon(N) \subset \Gamma_0(N)$ be the congruence subgroup of $\mathbf{SL}_2(\mathbb{Z})$ attached to f, defined by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ such that } \varepsilon(a) = 1 \right\}. \tag{2.4}$$

Writing $\mathcal{H}$ for the Poincaré upper half place of complex numbers with a strictly positive imaginary part and $\mathcal{H}^*$ for $\mathcal{H} \cup \mathbf{P}_1(\mathbb{Q})$, let C denote the modular curve whose complex points are identified with $\Gamma \backslash \mathcal{H}^*$. Let W_r be the rth Kuga–Sato variety over C. It is a canonical compactification and desingularization of the r-fold self-product of the universal elliptic curve over C. (See, for example, [2], Chapter 2 and the Appendix for more details on this definition.)

Remark 2.5. Article [2] is written using $\Gamma_1(N)$ level structures. The careful reader may therefore wish to replace $\Gamma = \Gamma_\varepsilon(N)$ by $\Gamma_1(N)$ throughout the rest of the paper and make the obvious modifications. For example, in the definition of $P_\psi^?(\chi)$ in (2.26), one would need to take a trace before summing over $\mathrm{Pic}(\mathcal{O}_c)$. This is explained in more detail in [5, Section 4.2]. □

Theorem 2.6 (Scholl)**.** There is a projector $e_{\theta_\psi} \in \mathrm{Corr}_0^0(W_r, W_r) \otimes E_{\theta_\psi}$ whose associated Grothendieck motive $M(\theta_\psi) := (W_r, e_{\theta_\psi}, 0)$ satisfies (for all ℓ)

$$M(\theta_\psi)_\ell \simeq V_\ell(\theta_\psi)$$

as $E_{\theta_\psi}[G_{\mathbb{Q}}]$-modules. □

We remark that $M(\theta_\psi)$ is a motive over $\mathbb{Q}$ with coefficients in E_{θ_ψ} and that its ℓ-adic realization $M(\theta_\psi)_\ell$ is identified with $e_{\theta_\psi}(H_{\mathrm{et}}^{r+1}(\bar{W}_r, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}} E_{\theta_\psi})$. The de Rham realization

$$M(\theta_\psi)_{\mathrm{dR}} = e_{\theta_\psi} H^{r+1}_{\mathrm{dR}}(W_r/E_{\theta_\psi})$$

is a two-dimensional E_{θ_ψ}-vector space equipped with a canonical decreasing, exhaustive and separated Hodge filtration. This vector space and its associated filtration can be described concretely in terms of the cusp form θ_ψ as follows.

Let C^0 denote the complement in C of the subscheme formed by the cusps. Setting $W_r^0 := W_r \times_C C^0$, there is a natural analytic uniformization

$$W_r^0(\mathbb{C}) = (\mathbb{Z}^{2r} \rtimes \Gamma)\backslash(\mathbb{C}^r \times \mathcal{H}),$$

where the action of $\mathbb{Z}^{2r}$ on $\mathbb{C}^r \times \mathcal{H}$ is given by

$$(m_1, n_1, \ldots, m_r, n_r)(w_1, \ldots, w_r, \tau) := (w_1 + m_1 + n_1\tau, \ldots, w_r + m_r + n_r\tau, \tau), \tag{2.5}$$

and Γ acts by the rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(w_1, \ldots, w_r, \tau) = \left(\frac{w_1}{c\tau + d}, \ldots, \frac{w_r}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right). \tag{2.6}$$

The holomorphic $(r+1)$-form

$$\omega_{\theta_\psi} := (2\pi \mathrm{i})^{r+1}\theta_\psi(\tau)\, dw_1 \cdots dw_r d\tau \tag{2.7}$$

on $W_r^0(\mathbb{C})$ extends to a regular differential on W_r. This differential is defined over the field E_{θ_ψ}, by the q-expansion principle, and hence lies in $H^{r+1}_{\mathrm{dR}}(W_r/E_{\theta_\psi})$. Its class generates the $(r+1)$-st step in the Hodge filtration of $M(\theta_\psi)_{\mathrm{dR}}$, which is given by

$$\mathrm{Fil}^0 M(\theta_\psi)_{\mathrm{dR}} = M(\theta_\psi)_{\mathrm{dR}},$$

$$\mathrm{Fil}^1 M(\theta_\psi)_{\mathrm{dR}} = \cdots = \mathrm{Fil}^{r+1} M(\theta_\psi)_{\mathrm{dR}} = E_{\theta_\psi} \cdot \omega_{\theta_\psi},$$

$$\mathrm{Fil}^{r+2} M(\theta_\psi)_{\mathrm{dR}} = 0.$$

The following proposition compares the Deligne–Scholl motive associated to θ_ψ with the CM motives constructed in the previous section in the main case of interest to us. We will suppose that ψ is a self-dual Hecke character of K of infinity type $(r+1, 0)$, and as in the previous section that F is a finite Galois extension of K such that $\psi_F = \psi_A^{r+1}$ for some elliptic curve A over F with CM by $\mathcal{O}_K$.

Proposition 2.7. For every finite prime ℓ, the ℓ-adic representations associated to the motives $M(\theta_\psi)|_F$ and $M(\psi_F) \otimes_{\mathbb{Q}} E_{\theta_\psi}$ are isomorphic as $(E_{\theta_\psi} \otimes \mathbb{Q}_\ell)[G_F]$-modules. □

Proof. It suffices to check this after further tensoring with E_ψ (over E_{θ_ψ}). Note that the ℓ-adic realization of $M(\theta_\psi)|_F \otimes_{E_{\theta_\psi}} E_\psi$ is a rank-2 $(E_\psi \otimes \mathbb{Q}_\ell)[G_F]$-module on which G_F acts as $\psi_{F,\ell} \oplus \psi^*_{F,\ell}$, where ψ^* is the Hecke character of K obtained from ψ by composing with complex conjugation on $\mathbb{A}_K^\times$. On the other hand, since $E_\chi = KE_{\theta_\chi} \simeq K \otimes_{\mathbb{Q}} E_{\theta_\chi}$, the ℓ-adic realization of $M(\psi_F) \otimes_{\mathbb{Q}} E_\psi$ is a rank-2 $(E_\psi \otimes \mathbb{Q}_\ell)[G_F]$-module on which G_F acts as $\psi_\ell \oplus \bar{\psi}_\ell$. However, the characters ψ^* and $\bar{\psi}$ are equal since ψ is self-dual, so the result follows. ■

2.4 Modular parameterizations attached to CM forms

In this section, we will explain how the Tate conjectures imply the existence of algebraic cycle classes generalizing those in Conjecture 1.1 of Section 1. Recall the Chow groups $\mathrm{CH}^d(V)(F)$ defined in Section 1.

Conjecture 2.8 (Tate). Let V be a smooth projective variety over a number field F. Then the ℓ-adic étale cycle class map

$$\mathrm{cl}_\ell : \mathrm{CH}^j(V)(F) \otimes \mathbb{Q}_\ell \longrightarrow H^{2j}_{\mathrm{et}}(\bar{V}, \mathbb{Q}_\ell)(j)^{G_F} \tag{2.8}$$

is surjective. □

A class in the target of (2.8) is called an *ℓ-adic Tate cycle*. The Tate conjecture will be used in our constructions through the following simple consequence.

Lemma 2.9. Let V_1 and V_2 be smooth projective varieties of dimension d over a number field F, and let $e_j \in \mathrm{Corr}^0(V_j, V_j) \otimes E$ (for $j = 1, 2$) be idempotents satisfying

$$e_j H^*_{\mathrm{et}}(\bar{V}_j, \mathbb{Q}_\ell) \otimes E = e_j H^d_{\mathrm{et}}(\bar{V}_j, \mathbb{Q}_\ell) \otimes E, \quad j = 1, 2.$$

Let $M_j := (V_j, e_j, 0)$ be the associated motives over F with coefficients in E, and suppose that the ℓ-adic realizations of M_1 and M_2 are isomorphic as $(E \otimes \mathbb{Q}_\ell)[G_F]$-modules.

If Conjecture 2.8 is true for $V_1 \times V_2$, then there exists a correspondence $\Pi \in \mathrm{CH}^d(V_1 \times V_2)(F) \otimes E$ for which

(1) the induced morphism

$$\Pi_\ell^*: (M_1)_\ell \longrightarrow (M_2)_\ell \tag{2.9}$$

of ℓ-adic realizations is an isomorphism of $E \otimes \mathbb{Q}_\ell[G_F]$-modules;

(2) the induced morphism

$$\Pi_{\mathrm{dR}}^*: (M_1)_{\mathrm{dR}} \longrightarrow (M_2)_{\mathrm{dR}} \tag{2.10}$$

is an isomorphism of $E \otimes F$-vector spaces.

□

Proof. Let

$$h: e_1 H^d_{\mathrm{et}}(\bar{V}_1, E \otimes \mathbb{Q}_\ell) \simeq e_2 H^d_{\mathrm{et}}(\bar{V}_2, E \otimes \mathbb{Q}_\ell)$$

be any isomorphism of $(E \otimes \mathbb{Q}_\ell)[G_F]$-modules. It corresponds to a Tate cycle

$$\begin{aligned} Z_h &\in (H^d_{\mathrm{et}}(\bar{V}_1, E \otimes \mathbb{Q}_\ell)^\vee \otimes H^d_{\mathrm{et}}(\bar{V}_2, E \otimes \mathbb{Q}_\ell))^{G_F} \\ &= (H^d_{\mathrm{et}}(\bar{V}_1, E \otimes \mathbb{Q}_\ell(d)) \otimes H^d_{\mathrm{et}}(\bar{V}_2, E \otimes \mathbb{Q}_\ell))^{G_F} \\ &\subset (H^{2d}_{\mathrm{et}}(\overline{V_1 \times V_2}, E \otimes \mathbb{Q}_\ell(d)))^{G_F}, \end{aligned}$$

where the superscript $^\vee$ in the first line denotes the $E \otimes \mathbb{Q}_\ell$-linear dual, the second line follows from the Poincaré duality, and the third from the Künneth formula. By Conjecture 2.8, there are elements $\alpha_1, \ldots, \alpha_t \in E \otimes \mathbb{Q}_\ell$ and cycles $\Pi_1, \ldots, \Pi_t \in \mathrm{CH}^d(V_1 \times V_2)(F)$ satisfying

$$Z_h = \sum_{j=1}^t \alpha_j \mathrm{cl}_\ell(\Pi_j).$$

After multiplying Z_h by a suitable power of ℓ, we may assume, without loss of generality, that the coefficients α_j belong to $\mathcal{O}_E \otimes \mathbb{Z}_\ell$. If $(\beta_1, \ldots, \beta_t) \in \mathcal{O}_E^t$ is any vector that is sufficiently close to $(\alpha_1, \ldots, \alpha_t)$ in the ℓ-adic topology, then the corresponding algebraic cycle

$$\Pi := \sum_{j=1}^t \beta_j \cdot \Pi_j \quad \in \mathrm{CH}^d(V_1 \times V_2)(F) \otimes E$$

satisfies condition 1 in the statement of Lemma 2.9. Condition 2 is verified by embedding F into one of its ℓ-adic completions F_λ and applying Fontaine's comparison functor to (2.9) in which source and targets are de Rham representations of G_{F_λ}. This shows that Π^*_{dR} induces an isomorphism on the de Rham cohomology over $F_\lambda \otimes E$, and part 2 follows. ■

The following proposition (in which, to ease notations, we identify differential forms with their image in de Rham cohomology) justifies Conjecture 1.1 of Section 1. Notations are as in Sections 2.2 and 2.3, with ψ a self-dual Hecke character of infinity type $(r+1,0)$.

Proposition 2.10. If the Tate conjecture is true for $W_r \times A^{r+1}$, then there is an algebraic cycle $\Pi^? \in \mathrm{CH}^{r+1}(W_r \times A^{r+1})(F) \otimes E_{\theta_\psi}$ such that

$$\Pi^{?*}_{\mathrm{dR}}(\omega_A^{r+1}) = c_{\psi,F} \cdot \omega_{\theta_\psi}, \tag{2.11}$$

for some $c_{\psi,F} \in (F \otimes_{\mathbb{Q}} E_{\theta_\psi})^\times$. □

Proof. Let M_1 and M_2 be the motives $M(\psi_F) \otimes_{\mathbb{Q}} E_{\theta_\psi}$ and $M(\theta_\psi)|_F$ in $\mathcal{M}_{F,E_{\theta_\psi}}$. By Proposition 2.7, the ℓ-adic realizations of M_1 and M_2 are isomorphic. Part (1) of Lemma 2.9 implies, assuming the validity of Conjecture 2.8, the existence of a correspondence $\Pi^?$ in $\mathrm{CH}^{r+1}(W_r \times A^{r+1})(F) \otimes E_{\theta_\psi}$ which induces an isomorphism on the ℓ-adic and de Rham realizations of M_1 and M_2. The isomorphism on de Rham realizations respects the Hodge filtrations and therefore sends the class ω_A^{r+1} to a unit $F \otimes_{\mathbb{Q}} E_{\theta_\psi}$-rational multiple of ω_{θ_ψ}, hence the proposition follows. ■

Note that the ambient F-variety $Z := W_r \times A^{r+1} = W_r \times A^r \times A$ in which the correspondence $\Pi^?$ is contained is equipped with three obvious projection maps

$$\begin{tikzcd} & Z \arrow[dl, "\pi_0"'] \arrow[d, "\pi_1"] \arrow[dr, "\pi_2"] & \\ W_r & A^r & A \end{tikzcd}$$

Let X_r be the F-variety

$$X_r = W_r \times A^r.$$

After setting

$$\pi_{01} = \pi_0 \times \pi_1 : Z \longrightarrow X_r, \quad \pi_{12} = \pi_1 \times \pi_2 : Z \longrightarrow A^r \times A,$$

we recall the simple (but key!) observation already made in Section 1 that $\Pi^?$ can be viewed as a correspondence in two different ways, via the diagrams:

$$\begin{tikzcd} & Z \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_{12}"] & \\ W_r & & A^r \times A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & Z \arrow[dl, "\pi_{01}"'] \arrow[dr, "\pi_2"] & \\ X_r & & A \end{tikzcd}$$

In order to maintain a notational distinction between these two ways of viewing $\Pi^?$, the correspondence from X_r to A attached to the cycle $\Pi^?$ is denoted by $\Phi^?$ instead of $\Pi^?$. It induces a natural transformation of functors on F-algebras:

$$\Phi^? : \mathrm{CH}^{r+1}(X_r)_0 \otimes E_{\theta_\psi} \longrightarrow \mathrm{CH}^1(A)_0 \otimes E_{\theta_\psi} = A \otimes E_{\theta_\psi}, \tag{2.12}$$

where $A \otimes E_{\theta_\psi}$ is the functor from the category of F-algebras to the category of E_{θ_ψ}-vector spaces which to L associates $A(L) \otimes E_\psi$. The natural transformation $\Phi^?$ is referred to as the *modular parameterization* attached to the correspondence $\Phi^?$. For any F-algebra L, we will also write

$$\Phi^?_L : \mathrm{CH}^{r+1}(X_r)_0(L) \otimes E_{\theta_\psi} \longrightarrow A(L) \otimes E_{\theta_\psi} \tag{2.13}$$

for the associated homomorphism on L-rational points (modulo torsion).

Like the class $\Pi^?$, the correspondence $\Phi^?$ also induces a functorial $F \otimes_{\mathbb{Q}} E_{\theta_\psi}$-linear map on de Rham cohomology, given by

$$\Phi^{?*}_{\mathrm{dR}} : H^1_{\mathrm{dR}}(A/F) \otimes E_{\theta_\psi} \longrightarrow H^{2r+1}_{\mathrm{dR}}(X_r/F) \otimes E_{\theta_\psi}. \tag{2.14}$$

Recall that $\eta_A \in H^1_{\mathrm{dR}}(A/F)$ is defined as in (1.8) of Section 1.

Proposition 2.11. The image of the class $\omega_A \in \Omega^1(A/F) \subset H^1_{\mathrm{dR}}(A/F)$ under $\Phi^?_{\mathrm{dR}}$ is given by

$$\Phi^{?*}_{\mathrm{dR}}(\omega_A) = c_{\psi,F} \cdot \omega_{\theta_\psi} \wedge \eta_A^r,$$

where $c_{\psi,F}$ is as in Proposition 2.10. □

Proof. Suppose that

$$\Pi^{?} = \sum_j m_j Z_j$$

is an E_{θ_ψ}-linear combination of codimension $(r+1)$ subvarieties of Z. The cycle class map is given by

$$\mathrm{cl}_{\Pi^?} : H^{2r+2}_{\mathrm{dR}}(Z/F) \otimes E_{\theta_\psi} \longrightarrow F \otimes E_{\theta_\psi},$$

where

$$\mathrm{cl}_{\Pi^?}(\omega) = \sum_j \mathrm{cl}_{Z_j}(\omega) \otimes m_j.$$

By Proposition 2.10 and the construction of $\Pi^?_{\mathrm{dR}}$, we have

$$\Pi^{?*}_{\mathrm{dR}}(\omega_A^{r+1}) = c_{\psi,F} \cdot \omega_{\theta_\psi}, \tag{2.15}$$

and

$$\Pi^{?*}_{\mathrm{dR}}(\eta_A^j \omega_A^{r+1-j}) = 0, \qquad \text{for } 1 \le j \le r. \tag{2.16}$$

By definition of $\Pi^?_{\mathrm{dR}}$, Equation (2.15) can be rewritten as

$$\mathrm{cl}_{\Pi^?}(\pi_0^*(\alpha) \wedge \pi_{12}^*(\omega_A^{r+1})) = \langle \alpha, c_{\psi,F} \cdot \omega_{\theta_\psi} \rangle_{W_r}, \quad \text{for all } \alpha \in H^{r+1}_{\mathrm{dR}}(W_r/F) \otimes E_{\theta_\psi}, \tag{2.17}$$

while (2.16) shows that

$$\mathrm{cl}_{\Pi^?}(\pi_0^*(\alpha) \wedge \pi_{12}^*(\eta_A^j \omega_A^{r+1-j})) = 0, \quad \text{when } 1 \le j \le r. \tag{2.18}$$

Equation (2.17) can also be rewritten as

$$\mathrm{cl}_{\Phi^?}(\pi_{01}^*(\alpha \wedge \omega_A^r) \wedge \pi_2^*(\omega_A)) = \langle \alpha \wedge \omega_A^r, c_{\psi,F} \cdot \omega_{\theta_\psi} \wedge \eta_A^r \rangle_{X_r}, \tag{2.19}$$

while Equation (2.18) implies that, for all $\alpha \in H^{r+1}_{\mathrm{dR}}(W_r/F) \otimes E_{\theta_\psi}$ and all $1 \le j \le r$,

$$\mathrm{cl}_{\Phi^?}(\pi_{01}^*(\alpha \wedge \eta_A^j \omega_A^{r-j}) \wedge \pi_2^*(\omega_A)) = 0 = \langle \alpha \wedge \eta_A^j \omega_A^{r-j}, \omega_{\theta_\psi} \wedge \eta_A^r \rangle_{X_r}. \tag{2.20}$$

In light of the definition of the map $\Phi^{?*}_{\mathrm{dR}}$, Equations (2.19) and (2.20) imply that

$$\Phi^{?*}_{\mathrm{dR}}(\omega_A) = c_{\psi,F} \cdot \omega_{\theta_\psi} \wedge \eta^r_A.$$

The proposition follows. ∎

Remark 2.12. We note that given a rational prime ℓ and a prime λ of F above ℓ such that $F_\lambda = \mathbb{Q}_\ell$, the maps induced by the putative correspondences $\Pi^?$ and $\Phi^?$ in ℓ-adic and de Rham cohomology (the latter over F_λ) can be defined regardless of the existence of these correspondences, at least up to a global constant independent of λ. Indeed, let Π^*_ℓ be any isomorphism

$$\Pi^*_\ell : (M_1)_\ell \simeq (M_2)_\ell$$

of $(E_{\theta_\psi} \otimes \mathbb{Q}_\ell)[G_F]$-modules. By the comparison theorem, this gives rise to an $E_{\theta_\psi} \otimes_{\mathbb{Q}} F_\lambda$-linear isomorphism of de Rham realizations:

$$\Pi^*_{\mathrm{dR},\lambda} : M_{1,\mathrm{dR}} \otimes_F F_\lambda \longrightarrow M_{2,\mathrm{dR}} \otimes_F F_\lambda,$$

mapping ω_A^{r+1} to a (unit) $E_\psi \otimes_{\mathbb{Q}} F_\lambda$-rational multiple of ω_{θ_ψ}. Since $F_\lambda = \mathbb{Q}_\ell$, we can rescale Π^*_ℓ uniquely such that $\Pi^*_{\mathrm{dR},\lambda}$ satisfies:

$$\Pi^*_{\mathrm{dR},\lambda}(\omega_A^{r+1}) = \omega_{\theta_\psi}.$$

Now as in the proof of Lemma 2.9, the Tate cycle corresponding to the normalized isomorphism Π^*_ℓ can be viewed as a nonzero element of

$$(H^1_{\mathrm{et}}(\bar{A}, E_{\theta_\psi} \otimes \mathbb{Q}_\ell)^\vee \otimes H^{2r+1}(\bar{X}_r, E_{\theta_\psi} \otimes \mathbb{Q}_\ell)(r))^{G_F},$$

and hence gives rise to a map

$$\Phi^*_{\mathrm{et},\lambda} : H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbb{Q}_\ell)(r+1) \otimes_{\mathbb{Q}} E_{\theta_\psi} \longrightarrow H^1_{\mathrm{et}}(\bar{A}, \mathbb{Q}_\ell)(1) \otimes_{\mathbb{Q}} E_{\theta_\psi} = V_\ell(A) \otimes_{\mathbb{Q}} E_{\theta_\psi}. \tag{2.21}$$

By the comparison isomorphism, one gets a map

$$\Phi^*_{\mathrm{dR},\lambda} : H^1_{\mathrm{dR}}(A/F_\lambda) \otimes E_{\theta_\psi} \longrightarrow H^{2r+1}_{\mathrm{dR}}(X_r/F_\lambda) \otimes E_{\theta_\psi}. \tag{2.22}$$

The same proof as in Proposition 2.11 shows that

$$\Phi^*_{\mathrm{dR},\lambda}(\omega_A) = \omega_{\theta_\psi} \wedge \eta_A^r.$$

Note that if $\Phi^?$ exists, then the map $\Phi^*_{\mathrm{dR},\lambda}$ differs from $\Phi^?_{\mathrm{dR}}$ exactly by the global constant $c_{\psi,F}$. □

Remark 2.13. Consider the following special case (see also [3, Section 3.7]) in which the following assumptions are made:

(1) The quadratic imaginary field K has class number one, odd discriminant, and unit group of order two. This implies that $K = \mathbb{Q}(\sqrt{-D})$, where $D := -\mathrm{Disc}(K)$ belongs to the finite set

$$S := \{7, 11, 19, 43, 67, 163\}.$$

(2) Let ψ_0 be the so-called *canonical Hecke character* of K of infinity type $(1,0)$ given by the formula

$$\psi_0((a)) = \varepsilon_K(a \bmod \mathfrak{d}_K)a, \tag{2.23}$$

where $\mathfrak{d}_K = (\sqrt{-D})$. The character ψ_0 determines (uniquely, up to an isogeny) an elliptic curve $A/\mathbb{Q}$ satisfying

$$\mathrm{End}_K(A) = \mathcal{O}_K, \quad L(A/\mathbb{Q}, s) = L(\psi_0, s).$$

After fixing A, we will also write ψ_A instead of ψ_0. It can be checked that the conductor of ψ_A is equal to $\mathfrak{d}_K$, and that

$$\psi_A^* = \bar{\psi}_A, \quad \psi_A\psi_A^* = \mathbf{N}_K, \quad \varepsilon_{\psi_A} = \varepsilon_K,$$

so that ψ_A is self-dual.

Suppose that $\psi = \psi_A^{r+1}$. In this case, the aforementioned setup simplifies drastically since $E_{\theta_\psi} = \mathbb{Q}$, and we may choose $F = K$. The modular parameterization $\Phi^?$ arises from a class in $\mathrm{CH}^{r+1}(X_r \times A)(K) \otimes \mathbb{Q}$ and induces a natural transformation of functors on

K-algebras:

$$\Phi^{?} : \mathrm{CH}^{r+1}(X_r)_0 \otimes \mathbb{Q} \longrightarrow A \otimes \mathbb{Q}. \tag{2.24}$$

□

2.5 Generalized Heegner cycles and Chow–Heegner points

Recall the notation $\Gamma := \Gamma_\varepsilon(N) \subset \Gamma_0(N)$ in (2.4) The associated modular curve $C = X_\varepsilon(N)$ has a model over $\mathbb{Q}$ obtained by realizing C as the solution to a moduli problem, which we now describe. Given an abelian group G of exponent N, denote by G^* the set of elements of G of order N. This set of "primitive elements" is equipped with a natural free action by $(\mathbb{Z}/N\mathbb{Z})^\times$, which is transitive when G is cyclic.

Definition 2.14. A *Γ-level structure* on an elliptic curve E is a pair (C_N, t), where

(1) C_N is a cyclic subgroup scheme of E of order N,
(2) t is an orbit in C_N^* for the action of $\ker \varepsilon$.

If E is an elliptic curve defined over a field L, then the Γ-level structure (C_N, t) on E is defined over the field L if C_N is a group scheme over L and t is fixed by the natural action of $\mathrm{Gal}(\bar{L}/L)$. □

The curve C coarsely classifies the set of isomorphism classes of triples (E, C_N, t), where E is an elliptic curve and (C_N, t) is a Γ-level structure on E. When Γ is torsion-free (which occurs, for example, when ε is odd and N is divisible by a prime of the form $4n+3$ and a prime of the form $3n+2$), the curve C is even a fine moduli space; for any field L, one then has

$$C(L) = \{\text{Triples } (E, C_N, t) \text{ defined over } L\}/L\text{-isomorphism}.$$

Since the datum of t determines the associated cyclic group C_N, we sometimes drop the latter from the notation, and write (E, t) instead of (E, C_N, t) when convenient.

We assume now that $\mathcal{O}_K$ contains a cyclic ideal $\mathfrak{N}$ of norm N. Since $N = D \cdot \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{f}_\psi)$, this condition is equivalent to requiring that $\mathfrak{f}_\psi$ is a (possibly empty) product

$$\mathfrak{f}_\psi = \prod_i \mathfrak{q}_i^{n_i},$$

where $\mathfrak{q}_i$ is a prime ideal in $\mathcal{O}_K$ lying over a rational prime q_i split in K and the q_i are pairwise coprime. The group scheme $A[\mathfrak{N}]$ of $\mathfrak{N}$-torsion points in A is a cyclic subgroup scheme of A of order N. A Γ-level structure on A of the form $(A[\mathfrak{N}], t)$ is said to be *of Heegner type* (associated to the ideal $\mathfrak{N}$).

Fixing a choice t of Γ-level structure on A attached to $\mathfrak{N}$, the datum of (A, t) determines a point P_A on $C(\tilde{F})$ for some abelian extension $\tilde{F}$ of K, and a canonical embedding ι_A of A^r into the fiber in W_r above P_A. We will assume henceforth that the extension F of K has been chosen large enough so that $F \supseteq \tilde{F}$. More generally then, if $\varphi : A \longrightarrow A'$ is an isogeny defined over F whose kernel intersects $A[\mathfrak{N}]$ trivially (i.e., an isogeny of elliptic curves with Γ-level structure), then the pair $(A', \varphi(t))$ determines a point $P_{A'} \in C(F)$ and an embedding $\iota_\varphi : (A')^r \longrightarrow W_r$ which is defined over F. We associate to such an isogeny φ a codimension $r+1$ cycle Υ_φ on the variety X_r by letting $\mathrm{Graph}(\varphi) \subset A \times A'$ denote the graph of φ and setting

$$\Upsilon_\varphi := \mathrm{Graph}(\varphi)^r \subset (A \times A')^r \stackrel{\simeq}{\longrightarrow} (A')^r \times A^r \subset W_r \times A^r,$$

where the last inclusion is induced from the pair $(\iota_{A'}, \mathrm{id}_A^r)$. We then set

$$\Delta_\varphi := \epsilon_X \Upsilon_\varphi \in \mathrm{CH}^{r+1}(X_r)_0(F), \tag{2.25}$$

where ϵ_X is the idempotent given in [2, Equation (2.2.1)], viewed as an element of the ring $\mathrm{Corr}^0(X_r, X_r)$ of algebraic correspondences from X_r to itself.

Definition 2.15. The *Chow–Heegner point* attached to the data (ψ, φ) is the point

$$P_\psi^?(\varphi) := \Phi_F^?(\Delta_\varphi) \in A(F) \otimes E_{\theta_\psi} = A(F) \otimes_{\mathcal{O}_K} E_\psi. \qquad \square$$

Note that this definition is only a conjectural one, since the existence of the homomorphism $\Phi_F^?$ depends on the existence of the algebraic cycle $\Pi^?$.

We now discuss some specific examples of φ that will be relevant to us. Let c be a positive integer and suppose that F contains the ring class field of K of conductor c. An isogeny $\varphi_0 : A \longrightarrow A_0$ (defined over F) is said to be a *primitive isogeny of conductor* c if it is of degree c and if the endomorphism ring $\mathrm{End}(A_0)$ is isomorphic to the order $\mathcal{O}_c$ in K of conductor c. The kernel of a primitive isogeny necessarily intersects $A[\mathfrak{N}]$ trivially, that is, such a φ_0 is an isogeny of elliptic curves with Γ-level structure. The corresponding Chow–Heegner point $P_\psi^?(\varphi_0)$ is said to be of *conductor* c.

Once φ_0 is fixed, one can also consider an infinite collection of Chow–Heegner points indexed by certain projective $\mathcal{O}_c$-submodules of $\mathcal{O}_c$. More precisely, let $\mathfrak{a}$ be such a projective module for which

$$A_0[\mathfrak{a}] \cap \varphi_0(A[\mathfrak{N}]) = 0,$$

and let

$$\varphi_{\mathfrak{a}} : A_0 \longrightarrow A_{\mathfrak{a}} := A_0 / A_0[\mathfrak{a}]$$

denote the canonical isogeny of elliptic curves with Γ-level structure given by the theory of complex multiplication. Since the isogeny $\varphi_{\mathfrak{a}}$ is defined over F, the Chow–Heegner point

$$P_{\psi}^{?}(\mathfrak{a}) := P_{\psi}^{?}(\varphi_{\mathfrak{a}}\varphi_0) = \Phi_F^{?}(\Delta_{\mathfrak{a}}), \quad \text{where } \Delta_{\mathfrak{a}} = \Delta_{\varphi_{\mathfrak{a}}\varphi_0},$$

belongs to $A(F) \otimes E_{\theta_\psi}$ as well.

Lemma 2.16. For all elements $\lambda \in \mathcal{O}_c$ that are prime to $\mathfrak{N}$, we have

$$P_{\psi}^{?}(\lambda\mathfrak{a}) = \varepsilon(\lambda \bmod \mathfrak{N})\lambda^r P_{\psi}^{?}(\mathfrak{a}) \quad \text{in } A(F) \otimes_{\mathcal{O}_K} E_{\psi}.$$

More generally, for any $\mathfrak{b}$,

$$\varphi_{\mathfrak{a}}(P_{\psi}^{?}(\mathfrak{a}\mathfrak{b})) = \psi(\mathfrak{a}) P_{\psi}^{?}(\mathfrak{b})^{\sigma_{\mathfrak{a}}},$$

where $\sigma_{\mathfrak{a}}$ is the Frobenius element in $\mathrm{Gal}(F/K)$ attached to $\mathfrak{a}$. □

Proof. Let $P_{\mathfrak{a}}$ be the point of $C(F)$ attached to the elliptic curve $A_{\mathfrak{a}}$ with Γ-level structure, and recall that $\pi^{-1}(P_{\mathfrak{a}})$ is the fiber above $P_{\mathfrak{a}}$ for the natural projection $\pi : X_r \longrightarrow C$. The algebraic cycle

$$\Delta_{\lambda\mathfrak{a}} - \varepsilon(\lambda)\lambda^r \Delta_{\mathfrak{a}}$$

is entirely supported in the fiber $\pi^{-1}(P_{\mathfrak{a}})$, and its image in the homology of this fiber under the cycle class map is 0. The result follows from this using the fact that the image of a cycle Δ supported on a fiber $\pi^{-1}(P)$ depends only on the point P and on the image of Δ in the homology of the fiber. The proof of the general case is similar. ■

Now pick a rational integer c prime to N and recall that we have defined in [3, Section 3.2] a set of Hecke characters of K denoted by $\Sigma_{cc}(c, \mathfrak{N}, \varepsilon)$. (In loc. cit., we required c to be prime to pN, where p is a fixed prime split in K; however, this is not a key part of the definition, and in this paper we shall pick such a p later.) The set $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ can be expressed as a disjoint union

$$\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon) = \Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon) \cup \Sigma^{(2)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon),$$

where $\Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ and $\Sigma^{(2)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ denote the subsets consisting of characters of infinity type $(k + j, -j)$ with $1 - k \leq j \leq -1$ and $j \geq 0$, respectively. If p is a rational prime split in K and prime to cN, we shall denote by $\hat{\Sigma}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ the completion of $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ relative to the p-adic compact open topology on $\Sigma_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$, which is defined in [2, Section 5.2]. We note that the set $\Sigma^{(2)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ of classical central critical characters "of type 2" is dense in $\hat{\Sigma}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$.

Let χ be a Hecke character of K of infinity type $(r, 0)$ such that $\chi \mathbf{N}_K$ belongs to $\Sigma^{(1)}_{\mathrm{cc}}(c, \mathfrak{N}, \varepsilon)$ (so that χ is self-dual as well) and let $E_{\psi,\chi}$ denote the field generated over K by the values of ψ and χ. By Lemma 2.16, the expression

$$\chi(\mathfrak{a})^{-1} P^?_\psi(\mathfrak{a}) \in A(F) \otimes_{\mathcal{O}_K} E_{\psi,\chi}$$

depends only on the image of $\mathfrak{a}$ in the class group $G_c := \mathrm{Pic}(\mathcal{O}_c)$. Hence, we can define the Chow–Heegner point attached to the theta series θ_ψ and the character χ by summing over this class group:

$$P^?_\psi(\chi) := \sum_{\mathfrak{a} \in \mathrm{Pic}(\mathcal{O}_c)} \chi^{-1}(\mathfrak{a}) P^?_\psi(\mathfrak{a}) \in A(F) \otimes_{\mathcal{O}_K} E_{\psi,\chi}. \tag{2.26}$$

The Chow–Heegner point $P^?_\psi(\chi)$ thus defined belongs (conjecturally) to $A(F) \otimes_{\mathcal{O}_K} E_{\psi,\chi}$.

2.6 A special case

We now specialize the Chow–Heegner point construction to a simple but illustrative case, in which the hypotheses of Remark 2.13 are imposed. Thus, $\psi = \psi_A^{r+1}$, and the modular parameterization $\Phi^?$ gives a homomorphism from $\mathrm{CH}^{r+1}(X_r)(K)$ to $A(K) \otimes \mathbb{Q}$. We further assume that

(1) The integer r is odd. This implies that ψ is an unramified Hecke character of infinity type $(r + 1, 0)$ with values in K, and that its associated theta series θ_ψ belongs to $S_{r+2}(\Gamma_0(D), \varepsilon_K)$.

(2) The character χ as above is a Hecke character of infinity type $(r, 0)$, and

$$\chi\mathbf{N}_K \text{ belongs to } \Sigma_{\mathrm{cc}}^{(1)}(c, \mathfrak{d}_K, \varepsilon_K),$$

with c prime to D. Bertolini et al. [3, proof of Lemma 3.34] shows that any such χ can be written as

$$\chi = \psi_A^r \chi_0^{-1},$$

where χ_0 is a ring class character of K of conductor dividing c.

Under these conditions, we have

$$\Gamma = \Gamma_{\varepsilon_K}(D) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(D) \text{ such that } \varepsilon_K(a) = 1 \right\}.$$

Furthermore, the action of G_K on the cyclic group $A[\mathfrak{d}_K](\bar{K})$ is via the Dth cyclotomic character, and therefore, a Γ-level structure of Heegner type on the curve A is necessarily defined over K. The corresponding Γ-level structures on A_0 and on $A_{\mathfrak{a}}$ are, therefore, defined over the ring class field H_c. It follows that the generalized Heegner cycles Δ_φ belong to $\mathrm{CH}^{r+1}(X_r)_{0,\mathbb{Q}}(H_c)$, for any isogeny φ of conductor c, and therefore—assuming the existence of $\Phi^{?-}$–that

$$P_\psi^?(\mathfrak{a}) \text{ belongs to } A(H_c) \otimes_{\mathcal{O}_K} K, \quad P_\psi^?(\chi) \text{ belongs to } (A(H_c) \otimes_{\mathcal{O}_K} E_\chi)^{\chi_0},$$

where the χ_0-component $(A(H_c) \otimes_{\mathcal{O}_K} E_\chi)^{\chi_0}$ of the Mordell–Weil group over the ring class field H_c is defined by

$$(A(H_c) \otimes_{\mathcal{O}_K} E_\chi)^{\chi_0} := \{P \in A(H_c) \otimes_{\mathcal{O}_K} E_\chi \text{ such that } \sigma P = \chi_0(\sigma) P, \quad \forall \sigma \in \mathrm{Gal}(H_c/K)\}. \tag{2.27}$$

3 Chow–Heegner Points Over $\mathbb{C}_p$

3.1 The p-adic Abel–Jacobi map

The construction of the point $P_\psi^?(\chi)$ is only conjectural, since it depends on the existence of the cycle $\Pi^?$ and the corresponding map $\Phi^?$. In order to obtain unconditional results, we will replace the conjectural map $\Phi^?$ by its analog in p-adic étale cohomology.

Let F_0 denote the finite Galois extension of K which was denoted by F in Section 2.2. Recall that $\psi \circ \mathrm{N}_{F_0/K} = \psi_A^{r+1}$, where ψ_A is the Hecke character associated to an elliptic curve A/F_0 with CM by $\mathcal{O}_K$. Fix a rational prime p that does not divide the level N of θ_ψ, and such that there exists a prime v_0 of F_0 above p with $F_{0,v_0} = \mathbb{Q}_p$. Recall that the choice of the place v_0 above p in F_0 allows us to define a normalized map

$$\Phi^*_{\mathrm{et},p} : H^{2r+1}_{\mathrm{et}}(\bar{X}_r, \mathbb{Q}_p(r+1)) \otimes_{\mathbb{Q}} E_{\theta_\psi} \to H^1_{\mathrm{et}}(\bar{A}, \mathbb{Q}_p(1)) \otimes_{\mathbb{Q}} E_{\theta_\psi} = V_p(A) \otimes_{\mathbb{Q}} E_{\theta_\psi} = V_p(A) \otimes_K E_\psi$$

of $E_{\theta_\psi} \otimes \mathbb{Q}_p[G_{F_0}]$-modules as in Equation (2.21).

Let F be any finite extension of K containing F_0 such that the generalized Heegner cycle Δ_φ is defined over F. The global cohomology class

$$\kappa_\psi(\varphi) := \Phi^*_{\mathrm{et},p}(\mathrm{AJ}(\Delta_\varphi)) \in H^1(F, V_p(A)) \otimes E_{\theta_\psi}$$

belongs to the pro-p Selmer group of A over F, tensored with E_{θ_ψ} (see [14, Theorem 3.1.1]), and is defined independently of any conjectures. Furthermore, if the correspondence $\Phi^?$ exists, then Proposition 2.11 implies that

$$\kappa_\psi(\varphi) = c_{\psi,F_0} \cdot \delta(P^?_\psi(\varphi)), \tag{3.1}$$

where

$$\delta : A(F) \otimes E_{\theta_\psi} \longrightarrow H^1(F, V_p(A)) \otimes E_{\theta_\psi}$$

is the connecting homomorphism of Kummer theory, and c_{ψ,F_0} is an element in $(F_0 \otimes E_{\theta_\psi})^\times \hookrightarrow (\mathbb{Q}_p \otimes E_{\theta_\psi})^\times$.

Let v be a place of F above v_0. Since $\kappa_\psi(\varphi)$ belongs to the Selmer group of A over F, there is a local point in $A(F_v) \otimes E_{\theta_\psi}$, denoted $P^{(v)}_\psi(\varphi)$, such that

$$\kappa_\psi(\varphi)_{|G_{F_v}} = \delta_v(P^{(v)}_\psi(\varphi)).$$

More generally, as in (115) of Section 1, there exists a map

$$\Phi^{(v)}_F : \mathrm{CH}^{r+1}(X_r)_0(F_v) \longrightarrow A(F_v) \otimes E_{\theta_\psi}$$

such that

$$\Phi^{(v)}_F(\Delta_\varphi) = P^{(v)}_\psi(\varphi).$$

The map $\Phi^{(v)}_F$ is the p-adic counterpart of the conjectural map $\Phi^?_F$.

In light of Proposition 2.10 and of the construction of Chow–Heegner points given in Definition 2.15, the following conjecture is a concrete consequence of the Tate (or Hodge) conjecture for the variety $X_r \times A$.

Conjecture 3.1. The local points $P_\psi^{(v)}(\varphi) \in A(F_v) \otimes E_{\theta_\psi}$ lie in $\Lambda \cdot (A(F) \otimes E_{\theta_\psi})$, where $\Lambda := (F_0 \otimes E_{\theta_\psi})^\times \hookrightarrow (\mathbb{Q}_p \otimes E_{\theta_\psi})^\times$. □

The goal of this chapter is to prove Conjecture 3.1 in many cases. The proof exploits the connection between the local points $P_\psi^{(v)}(\varphi)$ and the special values of *two different types* of p-adic L-functions: the Katz p-adic L-function attached to K and the p-adic Rankin L-function attached to θ_ψ described in [3, Sections 3.1 and 3.2], respectively. The reader should consult these sections for the notations and basic interpolation properties defining these two types of p-adic L-functions.

We begin by relating $P_\psi^{(v)}(\varphi)$ to p-adic Abel–Jacobi maps. The *p-adic Abel–Jacobi map* attached to the elliptic curve A/F_v is a homomorphism

$$\mathrm{AJ}_A : \mathrm{CH}^1(A)_{0,\mathbb{Q}}(F_v) \longrightarrow \Omega^1(A/F_v)^\vee, \tag{3.2}$$

where the superscript of $\vee$ on the right denotes the F_v-linear dual. Under the identification of $\mathrm{CH}^1(A)_{0,\mathbb{Q}}(F_v)$ with $A(F_v) \otimes \mathbb{Q}$, it is determined by the relation

$$\mathrm{AJ}_A(P)(\omega) = \log_\omega(P), \tag{3.3}$$

where $\omega \in \Omega^1(A/F_v)$ and

$$\log_\omega : A(F_v) \otimes \mathbb{Q} \longrightarrow F_v$$

denotes the formal group logarithm on A attached to this choice of regular differential. It can be extended by E_{θ_ψ}-linearity to a map from $A(F_v) \otimes E_{\theta_\psi}$ to $F_v \otimes E_\psi$.

There is also a p-adic Abel–Jacobi map on null-homologous algebraic cycles

$$\mathrm{AJ}_{X_r} : \mathrm{CH}^{r+1}(X_r)_0(F_v) \longrightarrow \mathrm{Fil}^{r+1} H^{2r+1}_{\mathrm{dR}}(X_r/F_v)^\vee$$

attached to the variety X_r, where Fil^j refers to the jth step in the Hodge filtration on algebraic de Rham cohomology. Details on the definition of AJ_{X_r} can be found in [2, Section 3], where it is explained how AJ_{X_r} can be calculated via p-adic integration.

In light of Remark 2.12, the functoriality of the Abel–Jacobi maps is expressed in the following commutative diagram relating AJ_A and AJ_{X_r}:

$$\begin{tikzcd} \mathrm{CH}^{r+1}(X_r)_0(F_v) \arrow[r, "\mathrm{AJ}_{X_r}"] \arrow[d, "\Phi_F^{(v)}"] & \mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r/F_v)^\vee \arrow[d, "\Phi_{\mathrm{dR},v}^{*,\vee}"'] \\ A(F_v) \otimes E_{\theta_\psi} \arrow[r, "\mathrm{AJ}_A"] & \Omega^1(A/F_v)^\vee \otimes E_{\theta_\psi} \end{tikzcd} \tag{3.4}$$

Proposition 3.2. For all isogenies $\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$ of elliptic curves with Γ-level structure,

$$\log_{\omega_A}(P_\psi^{(v)}(\varphi)) = \mathrm{AJ}_{X_r}(\Delta_\varphi)(\omega_{\theta_\psi} \wedge \eta_A^r). \qquad \square$$

Proof. By Equation (3.3) and the definition of $P_\psi^{(v)}(\varphi)$,

$$\log_{\omega_A}(P_\psi^{(v)}(\varphi)) = \mathrm{AJ}_A(P_\psi^{(v)}(\varphi))(\omega_A) = \mathrm{AJ}_A(\Phi_F^{(v)}(\Delta_\varphi))(\omega_A). \tag{3.5}$$

The commutative diagram (3.4) shows that

$$\mathrm{AJ}_A(\Phi_F^{(v)}(\Delta_\varphi))(\omega_A) = \mathrm{AJ}_{X_r}(\Delta_\varphi)(\Phi_{\mathrm{dR},v}^*(\omega_A)) = \mathrm{AJ}_{X_r}(\Delta_\varphi)(\omega_{\theta_\psi} \wedge \eta_A^r). \tag{3.6}$$

Proposition 3.2 now follows from (3.5) and (3.6). ■

We will study the local points $P_\psi^{(v)}(\varphi)$ via the formula of Proposition 3.2.

3.2 Rationality of Chow–Heegner points over $\mathbb{C}_p$

We begin by placing ourselves in the setting of Section 2.6, in which

$$\psi = \psi_A^{r+1}, \quad \chi = \psi_A^r \chi_0,$$

where χ_0 is a ring class character of K of conductor c. In this case, we can take $F_0 = K$. Let p be a prime split in K and fix a prime $\mathfrak{p}$ of K above p. We set

$$P_{A,r}^{(\mathfrak{p})}(\chi_0) := P_{\psi_A^{r+1}}^{(\mathfrak{p})}(\psi_A^r \chi_0) = P_\psi^{(\mathfrak{p})}(\chi),$$

the latter being defined analogously to (2.26). The next theorem is one of the main results of this paper.

Theorem 3.3. There exists a global point $P_{A,r}(\chi_0) \in (A(H_c) \otimes_{\mathcal{O}_K} E_\chi)^{\chi_0}$ satisfying

$$\log^2_{\omega_A}(P^{(\mathrm{p})}_{A,r}(\chi_0)) = \log^2_{\omega_A}(P_{A,r}(\chi_0)) \pmod{E_\chi^\times}.$$

Furthermore, the point $P_{A,r}(\chi_0)$ is of infinite order if and only if

$$L'(\psi_A\chi_0^{-1}, 1) \neq 0, \quad L(\psi_A^{2r+1}\chi_0, r+1) \neq 0.$$

$\square$

Proof. By Proposition 3.2,

$$\log_{\omega_A}(P^{(\mathrm{p})}_{A,r}(\chi)) = \mathrm{AJ}_{X_r}(\Delta_\psi(\chi))(\omega_{\theta_\psi} \wedge \eta_A^r), \tag{3.7}$$

for an explicit cycle $\Delta_\psi(\chi) \in \mathrm{CH}^{r+1}(X_r)_0 \otimes E_\chi$. Theorem 5.13 of [2] with $f = \theta_\psi$ and $j = 0$ gives

$$\mathrm{AJ}_{X_r}(\Delta_\psi(\chi))(\omega_{\theta_\psi} \wedge \eta_A^r)^2 = \frac{L_p(\theta_\psi, \chi\mathbf{N}_K)}{\Omega_p(A)^{2r}} \pmod{E_\chi^\times}, \tag{3.8}$$

where $L_p(\theta_\psi, \chi\mathbf{N}_K)$ and $\Omega_p(A)$ are, respectively, the p-adic Rankin L-function attached to θ_ψ and the p-adic period attached to A as described in [3, Sections 3.2 and 2.4]. The fact that θ_ψ has Fourier coefficients in $\mathbb{Q}$ and that its Nebentype character ε_K is trivial when restricted to K implies that the field $E_{\psi,\chi,\varepsilon_K}$ occurring in [3, Corollary 3.18 of Section 3.4] is equal to E_χ. Therefore, this corollary implies that

$$\begin{aligned}\frac{L_p(\theta_\psi, \chi\mathbf{N}_K)}{\Omega_p(A)^{2r}} &= \mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{-1}\chi\mathbf{N}_K) \times \frac{\mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{*-1}\chi\mathbf{N}_K)}{\Omega_p(A)^{2r}} \pmod{E_\chi^\times}\\ &= \frac{\mathscr{L}_{p,c\mathfrak{d}_K}(\nu^*)}{\Omega_p(A)^{-1}} \times \frac{\mathscr{L}_{p,c\mathfrak{d}_K}(\psi_A^{2r+1}\chi_0\mathbf{N}_K^{-r})}{\Omega_p(A)^{2r+1}} \pmod{E_\chi^\times},\end{aligned} \tag{3.9}$$

where the factors $\mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{-1}\chi\mathbf{N}_K)$ and $\mathscr{L}_{p,c\mathfrak{d}_K}(\psi^{*-1}\chi\mathbf{N}_K)$ are values of the Katz two-variable p-adic L-function with conductor $c\mathfrak{d}_K$, following the notations that are adopted in [3, Section 3.1]. The character $\nu^* = \psi_A^*\chi_0$ lies in the region $\Sigma^{(1)}_{\mathrm{sd}}(c\mathfrak{d}_K)$ described in Section 3.1 of loc. cit. and is of type $(0, 1)$. Hence, Theorem 3.30 of Section 3.6 of loc.

cit. can be invoked. This theorem gives a global point $P_A(\chi_0) \in (A(H_c) \otimes_{\mathcal{O}_K} E_\chi)^{\chi_0}$ that is of infinite order if and only if $L'(\psi_A \chi_0^{-1}, 1) \neq 0$ and satisfies

$$\mathscr{L}_{p,c\mathfrak{d}_K}(\psi_A^* \chi_0) = \Omega_p(A)^{-1} \mathfrak{g}(\chi_0) \log^2_{\omega_A}(P_A(\chi_0)) \pmod{E_\chi^\times}. \tag{3.10}$$

Furthermore, the character $\psi_A^{2r+1} \chi_0 \mathbf{N}_K^{-r}$ belongs to the domain $\Sigma_{\mathrm{sd}}^{(2)}(c\mathfrak{d}_K)$ of classical interpolation for the Katz p-adic L-function. Proposition 2.15 and Lemma 2.14 in Section 2.3 of [3] show that the p-adic period attached to this central critical character is given by

$$\Omega_p((\psi_A^{2r+1} \chi_0 \mathbf{N}_K^{-r})^*) = \Omega_p(A)^{2r+1} \mathfrak{g}(\chi_0)^{-1} \pmod{E_\chi^\times}. \tag{3.11}$$

Corollary 3.3 of Section 3.1 of loc. cit. then implies that up to multiplication by a nonzero element of E_χ,

$$\mathscr{L}_{p,c\mathfrak{d}_K}(\psi_A^{2r+1} \chi_0 \mathbf{N}_K^{-r}) = \begin{cases} 0 & \text{if } L(\psi_A^{2r+1} \chi_0, r+1) = 0, \\ \Omega_p(A)^{2r+1} \mathfrak{g}(\chi_0)^{-1} & \text{otherwise.} \end{cases} \tag{3.12}$$

After setting

$$P_{A,r}(\chi_0) = \begin{cases} 0 & \text{if } L(\psi_A^{2r+1} \chi_0, r+1) = 0, \\ P_A(\chi_0) & \text{otherwise,} \end{cases} \tag{3.13}$$

equations (3.10) and (3.12) can be used to rewrite (3.9) as

$$\frac{L_p(\theta_\psi, \chi \mathbf{N}_K)}{\Omega_p(A)^{2r}} = \log^2_{\omega_A}(P_{A,r}(\chi_0)) \pmod{E_\chi^\times}. \tag{3.14}$$

Theorem 3.3 now follows when (3.7), (3.8), and (3.14) are combined. ■

We now state a more general, but less precise, version of Theorem 3.3. Let ψ and χ be two self-dual characters of K of infinity types $(r+1, 0)$ and $(r, 0)$, respectively, as in Section 2.5. Let $F_{\psi,\chi}$ be the subfield of $\bar{\mathbb{Q}}$ generated over K by F and $E_{\psi,\chi}$, and let $\nu := \psi \chi^{-1}$, so that ν is a self-dual Hecke character of K of infinity type $(1, 0)$ attached to the pair (ψ, χ).

Theorem 3.4. There exists a global point $P_\psi(\chi) \in A(F) \otimes_{\mathcal{O}_K} E_{\psi,\chi}$ such that

$$\log^2_{\omega_A}(P^{(v)}_\psi(\chi)) = \log^2_{\omega_A}(P_\psi(\chi)) \pmod{F^\times_{\psi,\chi}},$$

for all differentials $\omega_A \in \Omega^1(A/F)$. This point is nonzero if and only if

$$L'(\nu, 1) \neq 0 \quad \text{and} \quad L(\psi\chi^{*-1}, 1) \neq 0. \qquad \square$$

Proof. The proof proceeds along the same lines as (but is simpler than) the proof of Theorem 3.3. This earlier proof applies to a more special setting but derives a more precise result, in which it becomes necessary to keep a more careful track of the fields of scalars involved. To prove Theorem 3.4, it suffices to rewrite the proof of Theorem 3.3 with $E^\times_\chi$ replaced by $F^\times_{\psi,\chi}$ and $(\psi_A^{r+1}, \psi_A^r\chi_0)$ replaced by (ψ, χ). Note that Equations (3.10) and (3.11) hold modulo the larger group $F^\times_{\psi,\chi}$ without the Gauss sum factors which can therefore be ignored. ■

We now specialize the setting of Theorem 3.3 even further by assuming that $\chi_0 = 1$ is the trivial character, so that $\psi = \psi_A^{r+1}$ and $\chi = \psi_A^r$, and set

$$P^{(\mathfrak{p})}_{A,r} := P^{(\mathfrak{p})}_{\psi_A^{r+1}}(\psi_A^r).$$

In this case, the coefficient field E_χ is equal to K, and Theorem 3.3 asserts the existence of a point $P_{A,r} \in A(K) \otimes \mathbb{Q}$ such that

$$\log^2_{\omega_A}(P^{(\mathfrak{p})}_{A,r}) = \log^2_{\omega_A}(P_{A,r}) \pmod{K^\times}.$$

It is instructive to refine the argument used in the proof of Theorem 3.3 to resolve the ambiguity by the nonzero scalar in $K^\times$, in order to examine the dependence on r of the local point $P^{(\mathfrak{p})}_{A,r}$. This is the content of the next result.

Theorem 3.5. For all odd $r \geq 1$, the Chow–Heegner point $P^{(\mathfrak{p})}_{A,r}$ belongs to $A(K) \otimes \mathbb{Q}$ and is given by the formula

$$\log^2_{\omega_A}(P^{(\mathfrak{p})}_{A,r}) = \ell(r) \cdot \log^2_{\omega_A}(P_A), \tag{3.15}$$

where $\ell(r) \in \mathbb{Z}$ satisfies

$$\ell(r) = \pm \frac{r!(2\pi)^r}{(2\sqrt{D})^r \Omega(A)^{2r+1}} L(\psi_A^{2r+1}, r+1),$$

and P_A is a generator of $A(K) \otimes \mathbb{Q}$ depending only on A but not on r. □

Proof. As in the proof of Theorem 3.3, we combine (3.7) and Theorem 5.13 of [2] with $(f, j) = (\theta_{\psi_A^{r+1}}, 0)$ and $\chi \mathbf{N}_K = \psi_A^r \mathbf{N}_K$ playing the role of χ, to obtain

$$\log^2_{\omega_A}(P_\psi^{(\mathfrak{p})}(\chi)) = (1 - (p\chi(\bar{\mathfrak{p}}))^{-1} a_p(\theta_\psi) + (p\chi(\bar{\mathfrak{p}}))^{-2} p^{r+1})^{-2} \frac{L_p(\theta_\psi, \chi\mathbf{N}_K)}{\Omega_p(A)^{2r}}. \tag{3.16}$$

Since $\chi(\bar{\mathfrak{p}}) = \psi_A(\bar{\mathfrak{p}})^r$ and $a_p(\theta_\psi) = \psi_A^{r+1}(\bar{\mathfrak{p}}) + \psi_A^{r+1}(\mathfrak{p})$, the Euler factor appearing in (3.16) is given by

$$(1 - \psi_A^{-1}(\mathfrak{p}))^{-2}(1 - \psi_A^{2r+1}(\mathfrak{p}) p^{-r-1})^{-2}.$$

Therefore,

$$\log^2_{\omega_A}(P_\psi^{(\mathfrak{p})}(\chi)) = (1 - \psi_A^{-1}(\mathfrak{p}))^{-2}(1 - \psi_A^{2r+1}(\mathfrak{p}) p^{-r-1})^{-2} \frac{L_p(\theta_\psi, \chi\mathbf{N}_K)}{\Omega_p(A)^{2r}}. \tag{3.17}$$

On the other hand, by Bertolini et al. [3, Theorem 3.17] with $c = 1$ and $j = 0$

$$\frac{L_p(\theta_\psi, \chi\mathbf{N}_K)}{\Omega_p(A)^{2r}} = \frac{w(\theta_\psi, \chi)^{-1}}{2^r} \times \mathscr{L}_p(\psi_A^*) \times \frac{\mathscr{L}_p(\psi_A^{2r+1}\mathbf{N}_K^{-r})}{\Omega_p(A)^{2r}}, \tag{3.18}$$

where we write $\mathscr{L}_p$ for $\mathscr{L}_{p,\mathfrak{d}_K}$. By Bertolini et al. [2, Lemma 5.3], the norm 1 scalar $w(\theta_\psi, \chi)$ belongs to K, and is divisible only by the primes above $\sqrt{-D}$. Therefore, it is a unit in $\mathcal{O}_K$, and hence is equal to ± 1. We obtain

$$\frac{L_p(\theta_\psi, \chi\mathbf{N}_K)}{\Omega_p(A)^{2r}} = \frac{\pm 1}{2^r} \times \frac{\mathscr{L}_p(\psi_A^*)}{\Omega_p(A)^{-1}} \times \frac{\mathscr{L}_p(\psi_A^{2r+1}\mathrm{N}_K^{-r})}{\Omega_p(A)^{2r+1}}. \tag{3.19}$$

Let $P_A = P_A(1) \in A(K) \otimes \mathbb{Q}$ be as in (3.10), but chosen specifically so that

$$\frac{\mathscr{L}_p(\psi_A^*)}{\Omega_p(A)^{-1}} = (1 - \psi_A^{-1}(\mathfrak{p}))^2 \log^2_{\omega_A}(P_A). \tag{3.20}$$

By the interpolation property for the Katz L-function given, for instance, in [3, Proposition 3.5 of Section 3.1] with $j = r$ and $\nu = \psi_A^{2r+1}\mathbf{N}_K^{-r} = \psi_A^{r+1}\psi_A^{*-r}$,

$$\frac{\mathscr{L}_p(\psi_A^{2r+1}\mathbf{N}_K^{-r})}{\Omega_p(A)^{2r+1}} = (1-\psi_A(\mathfrak{p})^{2r+1}p^{-r-1})^2 \times \frac{r!(2\pi)^r L((\psi_A^*)^{2r+1}\mathbf{N}_K^{-r-1}, 0)}{\sqrt{D}^r\,\Omega(A)^{2r+1}}. \tag{3.21}$$

After substituting Equations (3.20) and (3.21) into (3.19) and using the fact that

$$L((\psi_A^*)^{2r+1}\mathbf{N}_K^{-r-1}, 0) = L(\psi_A^{2r+1}, r+1),$$

we find

$$\begin{aligned}(1-\psi_A^{-1}(\mathfrak{p}))^{-2}(1-\psi_A(\mathfrak{p})^{2r+1}p^{-r-1})^{-2} \times \frac{L_p(\theta_\psi, \chi\mathbf{N}_K)}{\Omega_p(A)^{2r}}\\ = \frac{\pm 1}{2^r}\log^2_{\omega_A}(P_A) \times \frac{r!(2\pi)^r L(\psi_A^{2r+1}, r+1)}{\sqrt{D}^r\,\Omega(A)^{2r+1}}.\end{aligned}$$

Hence, by (3.17), we obtain

$$\log^2_{\omega_A}(P_\psi^{(\mathfrak{p})}(\chi)) = \pm\frac{r!(2\pi)^r}{(2\sqrt{D})^r\,\Omega(A)^{2r+1}} \times L(\psi_A^{2r+1}, r+1) \times \log^2_{\omega_A}(P_A).$$

The result follows since $\ell(r)$ is shown to be an integer in [15]. ■

4 Chow–Heegner Points Over $\mathbb{C}$

4.1 The complex Abel–Jacobi map

For simplicity, we will confine ourselves in this section to working under the hypotheses that were made in Remark 2.13 where K is assumed, in particular, to have discriminant $-D$, with

$$D \in S := \{7, 11, 19, 43, 67, 163\}.$$

Let us suppose for the moment that an algebraic correspondence $\Pi^? \in \mathrm{CH}^{r+1}(W_r \times A^{r+1}) \otimes \mathbb{Q}$ as in Proposition 2.10 exists. By taking an integer multiple of this correspondence, we may assume that it has integer coefficients. As before then, viewing it as a correspondence

$$\Phi^? \in \mathrm{CH}^{r+1}(X_r \times A),$$

where $X_r = W_r \times A^r$, we get a *modular parameterization* also denoted by $\Phi^?$:

$$\Phi^? : \mathrm{CH}^{r+1}(X_r)_0 \longrightarrow \mathrm{CH}^1(A)_0 = A.$$

By Propositions 2.10 and 2.11, we have (with $\psi := \psi_A^{r+1}$)

$$\Pi_{\mathrm{dR}}^{?*}(\omega_A^{r+1}) = c_{\psi,K} \cdot \omega_{\theta_\psi}, \quad \Phi_{\mathrm{dR}}^{?*}(\omega_A) = c_{\psi,K} \cdot \omega_{\theta_\psi} \wedge \eta_A^r, \tag{4.1}$$

for some scalar $c_{\psi,K} \in K^\times$. This scalar can be viewed as playing the role of the Manin-constant in the context of the modular parameterization of A by $\mathrm{CH}^{r+1}(X_r)_0$.

Question 4.1. When is it possible to choose an integral cycle $\Pi^?$ so that $c_{\psi,K} = 1$? □

The difficulty in computing the modular parameterization $\Phi^?$ and the resulting Chow–Heegner points arises from the fact that it is hard in general to explicitly produce the correspondence $\Phi^?$, or even to prove its existence. In this section, we shall see that it is possible to define a complex avatar $\Phi_{\mathbb{C}}$ of $\Phi^?$ unconditionally and compute it numerically to great precision in several examples. Note that if the cycle $\Phi^?$ exists, then Equation (4.1) shows that $c_{\psi,K} \cdot \omega_{\theta_\psi} \wedge \eta_A^{r+1}$ is an integral Hodge class on $W_r \times A^{r+1}$. The construction of $\Phi_{\mathbb{C}}$ is based on the observation that one can show the following independently using a period computation, as in [16, Chapter 5, Theorem 2.4].

Proposition 4.2. There exists a scalar $c_r \in K^\times$ such that $\Xi := c_r \cdot \omega_{\theta_\psi} \wedge \eta_A^{r+1}$ is an integral Hodge class on $W_r \times A^{r+1}$. □

Let us fix such a scalar $c_r \in K^\times$. Clearly, we may assume that c_r is in fact in $\mathcal{O}_K$. Let

$$\mathrm{AJ}_A^\infty : \mathrm{CH}^1(A)_0(\mathbb{C}) \longrightarrow \frac{\mathrm{Fil}^1 H_{\mathrm{dR}}^1(A/\mathbb{C})^\vee}{\mathrm{Im}\, H_1(A(\mathbb{C}), \mathbb{Z})} \tag{4.2}$$

be the classical complex Abel–Jacobi map attached to A, where the superscript $\vee$ now denotes the complex linear dual. The map AJ_A^∞ is defined by the rule

$$\mathrm{AJ}_A^\infty(\Delta)(\omega) = \int_{\partial^{-1}\Delta} \omega, \tag{4.3}$$

the integral on the right being taken over any one-chain on $A(\mathbb{C})$ having the degree zero divisor Δ as boundary. This classical Abel–Jacobi map admits of a higher dimensional generalization for null-homologous cycles on X_r introduced by Griffiths and Weil:

$$\mathrm{AJ}_{X_r}^{\infty} : \mathrm{CH}^{r+1}(X_r)_0(\mathbb{C}) \longrightarrow \frac{\mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r/\mathbb{C})^{\vee}}{\mathrm{Im} H_{2r+1}(X_r(\mathbb{C}), \mathbb{Z})}, \tag{4.4}$$

defined just as in (4.3), but where $\mathrm{AJ}_{X_r}^{\infty}(\Delta)(\omega)$ is now defined by integrating any smooth representative of the de Rham cohomology class ω against a $(2r+1)$-chain on $X_r(\mathbb{C})$ having Δ as boundary. (Cf. the description in [4, Section 4] for example.) The map $\mathrm{AJ}_{X_r}^{\infty}$ is the complex analog of the p-adic Abel–Jacobi map AJ_{X_r} that was introduced and studied in Section 3.

If the Hodge conjecture holds, there is an algebraic cycle $\Phi^? = \Pi^? \in \mathrm{CH}^{r+1}(X_r \times A) \otimes \mathbb{Q}$ whose cohomology class equals Ξ. If further $\Phi^?$ has integral coefficients, then we have the following commutative diagram, which is the complex counterpart of (3.4) and which expresses the functoriality of the Abel–Jacobi maps under correspondences:

$$\begin{tikzcd}
\mathrm{CH}^{r+1}(X_r)_0(\mathbb{C}) \arrow[r, "\mathrm{AJ}_{X_r}^{\infty}"] \arrow[d, dotted, "\Phi_{\mathbb{C}}^?"] & \mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r/\mathbb{C})^{\vee}/\mathrm{Im} H_{2r+1}(X_r(\mathbb{C}), \mathbb{Z}) \arrow[d, "(\Phi_{\mathrm{dR},\mathbb{C}}^*)^{\vee}"] \\
\mathrm{CH}^1(A)_0(\mathbb{C}) \arrow[r, "\mathrm{AJ}_A^{\infty}"] & \Omega^1(A/\mathbb{C})^{\vee}/\mathrm{Im} H_1(A(\mathbb{C}), \mathbb{Z})
\end{tikzcd} \tag{4.5}$$

where the map $\Phi_{\mathrm{dR},\mathbb{C}}^*$ is *defined* to be the one induced by the integral Hodge class Ξ. Note that by construction

$$\Phi_{\mathrm{dR},\mathbb{C}}^*(\omega_A) = c_r \cdot \omega_{\theta_\psi} \wedge \eta_A^r.$$

Since AJ_A^{∞} is an isomorphism, in the absence of knowing the Hodge conjecture, we can simply define the complex analog $\Phi_{\mathbb{C}}$ of $\Phi_F^{(v)}$ as the unique map from $\mathrm{CH}^{r+1}(X_r)_0(\mathbb{C})$ to $A(\mathbb{C})$ for which the diagram above (with $\Phi_{\mathbb{C}}^?$ replaced by $\Phi_{\mathbb{C}}$) commutes.

We will now discuss how the map $\Phi_{\mathbb{C}}$ can be computed in practice. Recall the distinguished element ω_A of $\Omega^1(A/\mathbb{C})$ and let

$$\Lambda_A := \left\{ \int_{\gamma} \omega_A, \gamma \in H_1(A(\mathbb{C}), \mathbb{Z}) \right\} \subset \mathbb{C}$$

be the associated period lattice. Recall that $\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$ is an isogeny of elliptic curves with Γ-level structure if

$$\varphi(t_A) = t' \quad \text{and} \quad \varphi^*(\omega') = \omega_A.$$

The following proposition, which is the complex counterpart of Proposition 3.2, expresses the Abel–Jacobi image of the complex point $P_\psi(\varphi) := \Phi_{\mathbb{C}}(\Delta_\varphi)$ in terms of the Abel–Jacobi map on X_r.

Proposition 4.3. For all isogenies $\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$ of elliptic curves with Γ-level structure,

$$\mathrm{AJ}_A^\infty(P_\psi(\varphi))(\omega_A) = c_r \cdot \mathrm{AJ}_{X_r}^\infty(\Delta_\varphi)(\omega_{\theta_\psi} \wedge \eta_A^r) \pmod{\Lambda_{\omega_A}}.$$

□

Proof. The proof is the same as for Proposition 3.2. By definition of $P_\psi(\varphi)$ combined with the commutative diagram (4.5),

$$\mathrm{AJ}_A^\infty(P_\psi(\varphi))(\omega_A) = \mathrm{AJ}_A^\infty(\Phi_{\mathbb{C}}(\Delta_\varphi))(\omega_A) = \mathrm{AJ}_{X_r}^\infty(\Delta_\varphi)(\Phi_{\mathrm{dR},\mathbb{C}}^*(\omega_A)).$$

Since $\Phi_{\mathrm{dR},\mathbb{C}}^*(\omega_A) = c_r \cdot \omega_{\theta_\psi} \wedge \eta_A^r$, Proposition 4.3 follows. ■

Remark 4.4. In the aforementioned proposition and elsewhere in the article, we assume that Δ_φ has been multiplied by a nonzero integer so as to have integral coefficients. □

We now turn to giving an explicit formula for the right-hand side of the equation in Proposition 4.3. To do this, let $\Lambda_{\omega'} \subset \mathbb{C}$ be the period lattice associated to the differential ω' on A'. Note that Λ_{ω_A} is contained in $\Lambda_{\omega'}$ with index $\deg(\varphi)$.

Definition 4.5. A basis (ω_1, ω_2) of $\Lambda_{\omega'}$ is said to be *admissible* relative to (A', t') if

(1) the ratio $\tau := \omega_1/\omega_2$ has a positive imaginary part;
(2) via the identification $\frac{1}{N}\Lambda_{\omega'}/\Lambda_{\omega'} = A'(\mathbb{C})[N]$, the N-torsion point ω_2/N belongs to the orbit t'. □

Given an arbitrary cusp form $f \in S_{r+2}(\Gamma_0(N), \varepsilon)$, consider the cohomology class

$$\omega_f \wedge \eta_A^r = (2\pi \mathrm{i})^{r+1} f(z)\, \mathrm{d}z\, \mathrm{d}w^r \wedge \eta_A^r \in \mathrm{Fil}^{r+1} H_{\mathrm{dR}}^{2r+1}(X_r/\mathbb{C}).$$

Proposition 4.6. Let Δ_φ be the generalized Heegner cycle corresponding to the isogeny

$$\varphi : (A, t_A, \omega_A) \longrightarrow (A', t', \omega')$$

of elliptic curves with Γ-level structure, let (ω_1, ω_2) be an admissible basis for $\Lambda_{\omega'}$, and let $\tau = \omega_1/\omega_2$. Then

$$\mathrm{AJ}_{X_r}^{\infty}(\Delta_\varphi)(\omega_f \wedge \eta_A^r) = \omega_2^{-r} \frac{(2\pi \mathrm{i})^{r+1}}{(\tau - \bar{\tau})^r} \int_{\mathrm{i}\infty}^{\tau} (z - \bar{\tau})^r f(z)\, \mathrm{d}z. \tag{4.6}$$

□

Proof. We begin by observing that replacing ω_A by a scalar multiple $\lambda\omega_A$ multiplies both the left and right-hand sides of (4.6) by λ^{-r}. Hence, we may assume, after possibly rescaling $\Lambda_{\omega'}$, that the admissible basis (ω_1, ω_2) is of the form $(2\pi \mathrm{i}\tau, 2\pi \mathrm{i})$ with $\tau \in \mathcal{H}$. The case $j = 0$ in [4, Theorem 8.2] then implies that

$$\begin{aligned} \mathrm{AJ}_{X_r}^{\infty}(\Delta_\varphi)(\omega_f \wedge \eta_A^r) &= \frac{2\pi \mathrm{i}}{(\tau - \bar{\tau})^r} \int_{\mathrm{i}\infty}^{\tau} (z - \bar{\tau})^r f(z)\, \mathrm{d}z \\ &= \omega_2^{-r} \frac{(2\pi \mathrm{i})^{r+1}}{(\tau - \bar{\tau})^r} \int_{\mathrm{i}\infty}^{\tau} (z - \bar{\tau})^r f(z)\, \mathrm{d}z. \end{aligned}$$

The proposition follows. ■

Theorem 4.7. Let $P_\psi(\varphi)$ be the Chow–Heegner point corresponding to the generalized Heegner cycle Δ_φ. With notations as in Proposition 4.6,

$$\mathrm{AJ}_{A}^{\infty}(P_\psi(\varphi))(\omega_A) = c_r \cdot \omega_2^{-r} \frac{(2\pi \mathrm{i})^{r+1}}{(\tau - \bar{\tau})^r} \int_{\mathrm{i}\infty}^{\tau} (z - \bar{\tau})^r \theta_\psi(z)\, \mathrm{d}z \pmod{\Lambda_{\omega_A}}. \tag{4.7}$$

□

Proof. This is an immediate corollary of Propositions 4.3 and 4.6. ■

Table 1. The canonical elliptic curve A

D	a_1	a_2	a_3	a_4	a_6	$\Omega(A)$	P_A
7	1	-1	0	-107	552	$1.93331170\ldots$	–
11	0	-1	1	-7	10	$4.80242132\ldots$	$(4, 5)$
19	0	0	1	-38	90	$4.19055001\ldots$	$(0, 9)$
43	0	0	1	-860	9707	$2.89054107\ldots$	$(17, 0)$
67	0	0	1	-7370	243528	$2.10882279\ldots$	$(\frac{201}{4}, \frac{-71}{8})$
163	0	0	1	-2174420	1234136692	$0.79364722\ldots$	$(850, -69)$

In the following, we shall describe some numerical evidence for the rationality of the points $P_\psi(\varphi)$. Since the constant c_r lies in $\mathcal{O}_K \setminus \{0\}$ and since A has CM by $\mathcal{O}_K$, it will suffice in the following to show rationality assuming $c_r = 1$.

4.2 Numerical experiments

We now describe some numerical evaluations of Chow–Heegner points. As it stands, the elliptic curve A of conductor D^2 attached to the canonical Hecke character $\psi_A = \psi_0$ is determined only up to isogeny, and we pin it down by specifying that A is described by the minimal Weierstrass equation

$$A: y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

where the coefficients $a_1, \ldots, a_6$ are given in Table 1.
The penultimate column in Table 1 gives an approximate value for the positive real period $\Omega(A)$ attached to the elliptic curve A and its Néron differential ω_A. In all cases, the Néron lattice Λ_A attached to (A, ω_A) is generated by the periods

$$\omega_1 := \left(\frac{D + \sqrt{-D}}{2D}\right) \Omega(A), \quad \omega_2 := \Omega(A), \tag{4.8}$$

and (ω_1, ω_2) is an admissible basis for Λ_A in the sense of Definition 4.5. The elliptic curve A has Mordell–Weil rank 0 over $\mathbb{Q}$ when $D = 7$ and rank 1 otherwise. A specific generator P_A for $A(\mathbb{Q}) \otimes \mathbb{Q}$ is given in the last column of Table 1.

Table 2. The constants m_r for $1 \le r \le 15$

	11	19	43	67	163
1	1	1	1	1	1
3	2	6	36	114	2172
5	−8	−16	440	6920	3513800
7	14	−186	−19026	−156282	3347376774
9	304	4176	−8352	−34999056	−238857662304
11	−352	−33984	33708960	3991188960	−3941159174330400
13	76648	545064	−2074549656	46813903656	1904546981028802344
15	274736	40959504	47714214240	−90863536574160	828743785015597346448O

4.2.1 *Chow–Heegner points of level* 1

For $D \in S := \{11, 19, 43, 67, 163\}$, the elliptic curve A has rank 1 over $\mathbb{Q}$. Let $r \geq 1$ be an odd integer. As already remarked, it suffices to check rationality assuming $c_r = 1$. By Theorem 4.7, the Chow–Heegner point $P_{A,r}$ attached to the class of the diagonal $\Delta \subset (A \times A)^r$ is given by

$$\mathrm{AJ}_A^{\infty}(P_{A,r})(\omega_A) = J_r := \omega_2^{-r} \frac{(2\pi \mathrm{i})^{r+1}}{(\tau - \bar{\tau})^r} \int_{\mathrm{i}\infty}^{\tau} (z - \bar{\tau})^r \theta_{\psi}(z)\, \mathrm{d}z, \tag{4.9}$$

where (ω_1, ω_2) is the admissible basis of Λ_A given in (4.8) and $\tau = \frac{\omega_1}{\omega_2} = \frac{D+\sqrt{-D}}{2D}$. Hence, the complex point $P_{A,r}$ can be computed as the natural image of the complex number J_r under the Weierstrass uniformization.

We have calculated the complex points $P_{A,r}$ for all $D \in S$ and all $r \leq 15$, to roughly 200 digits of decimal accuracy. The calculations indicate that

$$P_{A,r} \stackrel{?}{=} \sqrt{-D} \cdot m_r \cdot P_A \pmod{A(\mathbb{C})[\iota_r]}, \tag{4.10}$$

where P_A is the generator of $A(\mathbb{Q}) \otimes \mathbb{Q}$ given in Table 1, ι_r is a small integer, and m_r is the rational integer listed in Table 2, in which the columns correspond to $D \in S$ and the rows to the odd r between 1 and 15.

The first 6 lines in this table, corresponding to $1 \leq r \leq 11$, are in perfect agreement with the values that appear in the third table of [15, Section 3.1]. This coincidence,

Table 3. The ambiguity factor ι_r for $1 \le r \le 31$

r	11	19	43	67	163	r	11	19	43	67	163
1	3	1	1	1	1	17	3^3	7	1	19	1
3	$3 \cdot 5$	5	1	1	1	19	$3 \cdot 5^2$	$5^2 \cdot 11$	11	1	1
5	$2 \cdot 3^2$	$2 \cdot 7$	2	2	2	21	$3 \cdot 23$	23	23	23	1
7	$2 \cdot 7$	5	1	1	1	23	$3^2 \cdot 5$	$5 \cdot 7$	13	1	1
9	3	11	11	1	1	25	3	1	1	1	1
11	$3^2 \cdot 5$	$5 \cdot 7$	13	1	1	27	$3 \cdot 5$	5	1	29	1
13	3	1	1	1	1	29	$3^2 \cdot 31$	$7 \cdot 11$	$11 \cdot 31$	1	1
15	$3 \cdot 5$	$5 \cdot 17$	17	17	1	31	$3 \cdot 5$	$5 \cdot 17$	17	1	1

combined with [15, Theorem 3.1], suggests the following conjecture, which is consistent with the p-adic formulae obtained in Theorem 3.5.

Conjecture 4.8. For all $D \in S$ and all odd $r \ge 1$, the Chow–Heegner point $P_{A,r}$ belongs to $A(K) \otimes \mathbb{Q}$ and is given by the formula

$$P_{A,r} = \sqrt{-D} \cdot m_r \cdot P_A, \tag{4.11}$$

where $m_r \in \mathbb{Z}$ satisfy the formula

$$m_r^2 = \frac{2r!(2\pi\sqrt{D})^r}{\Omega(A)^{2r+1}} L(\psi_A^{2r+1}, r+1),$$

and P_A is the generator of $A(\mathbb{Q}) \otimes \mathbb{Q}$ given in Table 1. □

The optimal values of ι_r that were observed experimentally are recorded in Table 3, for $1 \le r \le 31$.

Remark 4.9. The data in Table 3 suggest that the term ι_r in (4.10) is divisible only by primes that are less than or equal to $r+2$. One might, therefore, venture to guess that the primes ℓ dividing ι_r are only those for which the mod ℓ Galois representation attached to ψ_A^{r+1} has very small image, or perhaps nontrivial G_K-invariants. □

4.2.2 *Chow–Heegner points of prime level*

We may also consider (for a fixed D and a fixed odd integer r) the Chow–Heegner points on A attached to nontrivial isogenies φ. For instance, let $\ell \neq D$ be a prime. There are $\ell+1$ distinct isogenies $\varphi_j : A \longrightarrow A'_j$ of degree ℓ (with $j = 0, 1, \ldots, \ell-1, \infty$) attached to the lattices $\Lambda'_0, \ldots, \Lambda'_{\ell-1}, \Lambda'_\infty$ containing Λ_A with index ℓ. These lattices are generated by the admissible bases

$$\Lambda'_j = \mathbb{Z}\left(\frac{\omega_1 + j\omega_2}{\ell}\right) \oplus \mathbb{Z}\omega_2, \qquad \Lambda'_\infty = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\frac{\omega_2}{\ell}.$$

The elliptic curves A'_j and the isogenies φ are defined over the ring class field H_ℓ of K of conductor ℓ. Let

$$J_r(\ell, j) := \ell^r \omega_2^{-r} \frac{(2\pi \mathrm{i})^{r+1}}{(\tau - \bar{\tau})^r} \int_{\mathrm{i}\infty}^{\frac{\tau+j}{\ell}} \left(z - \frac{\bar{\tau}+j}{\ell}\right)^r \theta_{D,r}(z)\, \mathrm{d}z, \qquad 0 \le j \le \ell - 1,$$

$$J_r(\ell, \infty) := \varepsilon_K(\ell) \omega_2^{-r} \frac{(2\pi \mathrm{i})^{r+1}}{(\tau - \bar{\tau})^r} \int_{\mathrm{i}\infty}^{\ell\tau} (z - \ell\bar{\tau})^r \theta_\psi(z)\, \mathrm{d}z$$

be the associated complex invariants and let $P_{A,r}(\ell, j)$ and $P_{A,r}(\ell, \infty)$ denote the corresponding points in $\mathbb{C}/\Lambda_A = A(\mathbb{C})$.

We have attempted to verify the following conjecture numerically.

Conjecture 4.10. For all $\ell \neq D$ and all $j \in \mathbf{P}_1(\mathbb{F}_\ell)$, some (nonzero) multiple of the complex points $P_{A,r}(\ell, j)$ belong to the Mordell–Weil group $A(H_\ell)$. □

We have tested this prediction numerically for $r = 1$ and all

$$D \in S, \qquad \ell = 2, 3, 5, 7, 11,$$

as well as in a few cases where $r = 3$. Such calculations sometimes required several hundred digits of numerical precision, together with a bit of trial and error. The necessity for this arose because Conjecture 4.10 only predicts that some multiple of the points $P_{A,r}(\ell, j)$ belong to $A(H_\ell)$, as one would expect from Remark 4.4 as well as the possibility that the constant c_r is not 1. One finds in practice that these complex points do need to be multiplied by an (typically small) integer in order to belong to $A(H_\ell)$. Furthermore, the resulting global points appear (as suggested by (4.11) in the case $\ell = 1$) to be divisible by $\sqrt{-D}$, and this causes their heights to be rather large. It is, therefore, better in practice to divide the $P_{A,r}(\ell, j)$ by $\sqrt{-D}$, which introduces a further ambiguity of $A(\mathbb{C})[\sqrt{-D}]$ in the resulting global point. The conjecture that was eventually tested numerically is the following nontrivial strengthening of Conjecture 4.10:

Conjecture 4.11. Given integers $n \in \mathbb{Z}^{\geq 1}$ and $0 \leq s \leq D-1$, let

$$J_r'(\ell, j) = n \cdot \frac{J_r(\ell, j) - s\omega_1}{\sqrt{-D}}, \quad 0 \leq j \leq \ell - 1,$$

$$J_r'(\ell, \infty) = n \cdot \frac{J_r(\ell, \infty) - s\varepsilon_K(\ell)\ell^r\omega_1}{\sqrt{-D}},$$

and let $P'_{A,r}(\ell, j) \in A(\mathbb{C})$ be the associated complex points. Then there exist $n = n_{D,r}$ and $s = s_{D,r}$, depending on D and r but not on ℓ and j, for which the points $P_r'(\ell, j)$ belong to $A(H_\ell)$ and satisfy the following:

(1) If ℓ is inert in K, then $\mathrm{Gal}(H_\ell/K)$ acts transitively on the set

$$\{P'_{A,r}(\ell, j), \ \ j \in \mathbf{P}_1(\mathbb{F}_\ell)\}$$

of Chow–Heegner points of level ℓ.

(2) If $\ell = \lambda\bar{\lambda}$ is split in K, then there exist $j_1, j_2 \in \mathbf{P}_1(\mathbb{F}_\ell)$ for which

$$P'_{A,r}(\ell, j_1) = \varepsilon_K(\lambda)\lambda^r P'_{A,r}, \quad P'_{A,r}(\ell, j_2) = \varepsilon_K(\bar{\lambda})\bar{\lambda}^r P'_{A,r},$$

and $\mathrm{Gal}(H_\ell/K)$ acts transitively on the remaining set

$$\{P'_{A,r}(\ell, j), \ \ j \in \mathbf{P}_1(\mathbb{F}_\ell) - \{j_1, j_2\}\}$$

of Chow–Heegner points of level ℓ. □

We now describe a few sample calculations that lend support to Conjecture 4.11.
1. *The case* $D = 7$. Consistent with the fact that the elliptic curve A has rank 0 over $\mathbb{Q}$ (and hence over K as well), the point $P_{A,r}$ appears to be a torsion point in $A(\mathbb{C})$, for all $1 \leq r \leq 31$. For example, the invariant J_1 agrees with $(\omega_1 + \omega_2)/8$ to the 200 decimal digits of accuracy that were calculated. When $\ell = 2$, it also appears that the quantities $J_1(2, j)$ belong to $\frac{1}{8}\Lambda_7$. There is no reason, however, to expect the Chow–Heegner points $P_{A,r}(\ell, j)$ to be torsion for larger values of ℓ. Experiments suggest that the constants in

Conjecture 4.11 are

$$n_{7,1} = 4, \quad s_{7,1} = 0.$$

For example, when $\ell = 3$, the ring class field of conductor ℓ is a cyclic quartic extension of K containing $K(\sqrt{21})$ as its quadratic subfield. In that case, the points $P'_{A,1}(3, j)$ satisfy

$$P'_{A,1}(3,0) = P'_{A,1}(3,1) = -P'_{A,1}(3,2) = -P'_{A,1}(3,\infty),$$

and agree to 600 digits of accuracy with a global point in $A(\mathbb{Q}(\sqrt{21}))$ of relatively small height, with x-coordinate given by

$$x = \frac{2594759111751009269208353605822093882 59}{413955894918450159522952049099986560 04}.$$

2. *The case* $D = 19$. To compute the Chow–Heegner points of conductor 3 in the case $D = 19$ and $r = 1$, it appears that one can take

$$n_{19,1} = 1, \quad s_{19,1} = 1.$$

Perhaps because of the small value of $n_{19,1}$, the points $P'_{A,1}(\ell, j)$ appear to be of relatively small height and can easily be recognized as global points, even for moderately large values of ℓ. For instance, the points $P'_{A,1}(3, j)$ seem to have x-coordinates of the form

$$x = \frac{-19 \pm 3\sqrt{57}}{2},$$

and their y-coordinates satisfying the degree 4 polynomial

$$x^4 + 2x^3 + 8124x^2 + 8123x - 217886,$$

whose splitting field is the ring class field H_3 of K of conductor 3.

When $\ell = 7$, which is split in $K/\mathbb{Q}$, the ring class field H_7 is a cyclic extension of K of degree 6. It appears that the points $P'_{A,1}(7,3)$ and $P'_{A,1}(7,5)$ belong to $A(K)$ and are given by

$$P'_{A,1}(7,3) = \frac{3+\sqrt{-19}}{2} P_A, \quad P'_{A,1}(7,5) = \frac{3-\sqrt{-19}}{2} P_A.$$

The six remaining points are grouped into three pairs of equal points,

$$P'_{A,1}(7,0) = P'_{A,1}(7,2), \quad P'_{A,1}(7,1) = P'_{A,1}(7,6), \quad P'_{A,1}(7,4) = P'_{A,1}(7,\infty),$$

whose x and y coordinates appear to satisfy the cubic polynomials

$$9x^3 + 95x^2 + 19x - 1444 \quad \text{and} \quad 27x^3 - 235x^2 + 557x + 1198,$$

respectively. The splitting field of both of these polynomials turns out to be the cubic subfield L of the ring class field of K of conductor 7. One obtains as a by-product of this calculation three independent points in $A(L)$ which are linearly independent over $\mathcal{O}_K$. We expect that these three points give a K-basis for $A(L) \otimes \mathbb{Q}$ (and therefore that $A(L)$ has rank 6) but have not checked this numerically.

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