Annals of Mathematics, **163** (2006), 301–346

# Elliptic units for real quadratic fields

By Henri Darmon and Samit Dasgupta

## Contents

1. A review of the classical setting
2. Elliptic units for real quadratic fields
   - 2.1. $p$-adic measures
   - 2.2. Double integrals
   - 2.3. Splitting a two-cocycle
   - 2.4. The main conjecture
   - 2.5. Modular symbols and Dedekind sums
   - 2.6. Measures and the Bruhat-Tits tree
   - 2.7. Indefinite integrals
   - 2.8. The action of complex conjugation and of $U_p$
3. Special values of zeta functions
   - 3.1. The zeta function
   - 3.2. Values at negative integers
   - 3.3. The $p$-adic valuation
   - 3.4. The Brumer-Stark conjecture
   - 3.5. Connection with the Gross-Stark conjecture
4. A Kronecker limit formula
   - 4.1. Measures associated to Eisenstein series
   - 4.2. Construction of the $p$-adic $L$-function
   - 4.3. An explicit splitting of a two-cocycle
   - 4.4. Generalized Dedekind sums
   - 4.5. Measures on $\mathbb{Z}_p \times \mathbb{Z}_p$
   - 4.6. A partial modular symbol of measures on $\mathbb{Z}_p \times \mathbb{Z}_p$
   - 4.7. From $\mathbb{Z}_p \times \mathbb{Z}_p$ to $\mathbb{X}$
   - 4.8. The measures $\mu$ and $\Gamma$-invariance

## Introduction

Elliptic units, which are obtained by evaluating modular units at quadratic imaginary arguments of the Poincaré upper half-plane, provide us with a rich source of arithmetic questions and insights. They allow the analytic construction of abelian extensions of imaginary quadratic fields, encode special values of zeta functions through the Kronecker limit formula, and are a prototype for Stark's conjectural construction of units in abelian extensions of number fields. Elliptic units have also played a key role in the study of elliptic curves with complex multiplication through the work of Coates and Wiles.

This article is motivated by the desire to transpose the theory of elliptic units to the context of real quadratic fields. The classical construction of elliptic units does not give units in abelian extensions of such fields.[1] Naively, one could try to evaluate modular units at real quadratic irrationalities; but these do not belong to the Poincaré upper half-plane $\mathcal{H}$. We are led to replace $\mathcal{H}$ by a $p$-adic analogue $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$, equipped with its structure of a rigid analytic space. Unlike its archimedean counterpart, $\mathcal{H}_p$ does contain real quadratic irrationalities, generating quadratic extensions in which the rational prime $p$ is either inert or ramified.

Fix such a real quadratic field $K \subset \mathbb{C}_p$, and denote by $K_p$ its completion at the unique prime above $p$. Chapter 2 describes an analytic recipe which to a modular unit $\alpha$ and to $\tau \in \mathcal{H}_p \cap K$ associates an element $u(\alpha, \tau) \in K_p^\times$, and conjectures that this element is a $p$-unit in a specific narrow ring class field of $K$ depending on $\tau$ and denoted $H_\tau$. The construction of $u(\alpha, \tau)$ is obtained by replacing, in the definition of "Stark-Heegner points" given in [Dar1], the weight-2 cusp form attached to a modular elliptic curve by the logarithmic derivative of $\alpha$, an Eisenstein series of weight 2. Conjecture 2.14 of Chapter 2, which formulates a Shimura reciprocity law for the $p$-units $u(\alpha, \tau)$, suggests that these elements display the same behavior as classical elliptic units in many key respects.

Assuming Conjecture 2.14, Chapter 3 relates the ideal factorization of the $p$-unit $u(\alpha, \tau)$ to the Brumer-Stickelberger element attached to $H_\tau/K$. Thanks to this relation, Conjecture 2.14 is shown to imply the prime-to-2 part of the Brumer-Stark conjectures for the abelian extension $H_\tau/K$—an implication which lends some evidence for Conjecture 2.14 and leads to the conclusion that the $p$-units $u(\alpha, \tau)$ are (essentially) the $p$-adic Gross-Stark units which enter in Gross's $p$-adic variant [Gr1] of the Stark conjectures, in the context of ring class fields of real quadratic fields.

Motivated by Gross's conjecture, Chapter 4 evaluates the $p$-adic logarithm of the norm from $K_p$ to $\mathbb{Q}_p$ of $u(\alpha, \tau)$ and relates this quantity to the first derivative of a partial $p$-adic zeta function attached to $K$ at $s = 0$. The resulting formula, stated in Theorem 4.1, can be viewed as an analogue of the Kronecker limit formula for real quadratic fields. In contrast with the analogue given in Ch. II, §3 of [Sie1] (see also [Za]), Theorem 4.1 involves non-archimedean in-

[1]Except when the extension in question is contained in a ring class field of an auxiliary imaginary quadratic field, an exception which is the basis for Kronecker's solution to Pell's equation in terms of values of the Dedekind $\eta$-function.

tegration and $p$-adic rather than complex zeta-values. Yet in some ways it is closer to the spirit of the original Kronecker limit formula because it involves the logarithm of an expression which belongs, at least conjecturally, to an abelian extension of $K$. Theorem 4.1 makes it possible to deduce Gross's $p$-adic analogue of the Stark conjectures for $H_\tau/K$ from Conjecture 2.14.

It should be stressed that Conjecture 2.14 leads to a genuine strengthening of the Gross-Stark conjectures of [Gr1] in the setting of ring class fields of real quadratic fields, and also of the refinement of these conjectures proposed in [Gr2]. Indeed, the latter exploits the special values at $s = 0$ of abelian $L$-series attached to $K$, as well as derivatives of the corresponding $p$-adic zeta functions, to recover the images of Gross-Stark units in $K_p^\times/\bar{\mathcal{O}}_K^\times$, where $\bar{\mathcal{O}}_K^\times$ is the topological closure in $K_p^\times$ of the unit group of $K$. Conjecture 2.14 of Chapter 2 proposes an explicit formula for the Gross-Stark units themselves. It would be interesting to see whether other instances of the Stark conjectures (both classical, and $p$-adic) are susceptible to similar refinements.[2]

## 1. A review of the classical setting

Let $\mathcal{H}$ be the Poincaré upper half-plane, and let $\Gamma_0(N)$ denote the standard Hecke congruence group acting on $\mathcal{H}$ by Möbius transformations. Write $Y_0(N)$ and $X_0(N)$ for the modular curves over $\mathbb{Q}$ whose complex points are identified with $\mathcal{H}/\Gamma_0(N)$ and $\mathcal{H}^*/\Gamma_0(N)$ respectively, where $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}_1(\mathbb{Q})$ is the extended upper half-plane.

A *modular unit* is a holomorphic nowhere-vanishing function on $\mathcal{H}/\Gamma_0(N)$ which extends to a meromorphic function on the compact Riemann surface $X_0(N)(\mathbb{C})$. A typical example of such a unit is the modular function $\Delta(\tau)/\Delta(N\tau)$. More generally, let $D_N$ be the free $\mathbb{Z}$-module generated by the formal $\mathbb{Z}$-linear combinations of the positive divisors of $N$, and let $D_N^0$ be the submodule of linear combinations of degree 0. We associate to each element $\delta = \sum n_d[d] \in D_N^0$ the modular unit

$$\Delta_\delta(\tau) = \prod_{d|N} \Delta(d\tau)^{n_d}. \tag{1}$$

Fix such a modular unit $\alpha = \Delta_\delta$ on $\Gamma_0(N)$. Its level $N$ will remain fixed from now on.

Let $M_0(N) \subset M_2(\mathbb{Z})$ denote the ring of integral $2 \times 2$ matrices which are upper-triangular modulo $N$. Given $\tau \in \mathcal{H}$, its *associated order* in $M_0(N)$, denoted $\mathcal{O}_\tau$, is the set of matrices in $M_0(N)$ which fix $\tau$ under Möbius trans-

[2]In a purely archimedean context, recent work of Ren and Sczech on the Stark conjectures for a complex cubic field suggests that the answer to this question should be "yes".

formations:

$$\mathcal{O}_\tau := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_0(N) \quad \text{such that} \quad a\tau + b = c\tau^2 + d\tau \right\}. \tag{2}$$

This set of matrices is identified with a discrete subring of $\mathbb{C}$ by sending the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the complex number $c\tau + d$. Hence $\mathcal{O}_\tau$ is identified either with $\mathbb{Z}$ or with an order in an imaginary quadratic field $K$.

Let $\mathcal{O}$ be such an order of discriminant $-D$, relatively prime to $N$. Define

$$\mathcal{H}^{\mathcal{O}} := \{\tau \in \mathcal{H} \text{ such that } \mathcal{O}_\tau \simeq \mathcal{O}\}.$$

This set is preserved under the action of $\Gamma_0(N)$ by Möbius transformations, and the quotient $\mathcal{H}^{\mathcal{O}}/\Gamma_0(N)$ is finite.

If $\tau = u + iv$ belongs to $\mathcal{H}^{\mathcal{O}}$, then the binary quadratic form

$$\tilde{Q}_\tau(x, y) = v^{-1}(x - y\tau)(x - y\bar{\tau})$$

of discriminant $-4$ is proportional to a unique *primitive* integral quadratic form denoted

$$Q_\tau(x, y) = Ax^2 + Bxy + Cy^2, \quad \text{with } A > 0. \tag{3}$$

Since $D$ is relatively prime to $N$, we have $N|A$ and $B^2 - 4AC = -D$. We introduce the invariant

$$u(\alpha, \tau) := \alpha(\tau). \tag{4}$$

The theory of complex multiplication (cf. [KL, Ch. 9, Lemma 1.1 and Ch. 11, Th. 1.2]) implies that $u(\alpha, \tau)$ belongs to an abelian extension of the imaginary quadratic field $K = \mathbb{Q}(\tau)$. More precisely, class field theory identifies $\mathrm{Pic}(\mathcal{O})$ with the Galois group of an abelian extension $H$ of $K$, the so-called *ring class field* attached to $\mathcal{O}$. Let $\mathcal{O}_H$ denote the ring of integers of $H$. If $\tau$ belongs to $\mathcal{H}^{\mathcal{O}}$, then

$$u(\alpha, \tau) \text{ belongs to } \mathcal{O}_H[1/N]^\times, \tag{5}$$

and

$$(\sigma - 1)u(\alpha, \tau) \text{ belongs to } \mathcal{O}_H^\times, \quad \text{for all } \sigma \in \mathrm{Gal}(H/K). \tag{6}$$

Let

$$\mathrm{rec} : \mathrm{Pic}(\mathcal{O}) \longrightarrow \mathrm{Gal}(H/K) \tag{7}$$

denote the reciprocity law map of global class field theory, which for all prime ideals $\mathfrak{p} \nmid D$ of $K$, sends the class of $\mathfrak{p} \cap \mathcal{O}$ to the inverse of the Frobenius element at $\mathfrak{p}$ in $\mathrm{Gal}(H/K)$. One disposes of an explicit description of the action of $\mathrm{Gal}(H/K)$ on the $u(\alpha, \tau)$ in terms of (7). To formulate this description, known as the *Shimura reciprocity law*, it is convenient to denote by $\Omega_N$ the set of homothety classes of pairs $(\Lambda_1, \Lambda_2)$ of lattices in $\mathbb{C}$ satisfying

$$\Lambda_1 \supset \Lambda_2, \quad \text{and} \quad \Lambda_1/\Lambda_2 \simeq \mathbb{Z}/N\mathbb{Z}. \tag{8}$$

Let $x \mapsto x'$ denote the nontrivial automorphism of $\mathrm{Gal}(K/\mathbb{Q})$. There is a natural bijection $\underline{\tau}$ from $\Omega_N$ to $\mathcal{H}/\Gamma_0(N)$, defined by sending $x = (\Lambda_1, \Lambda_2) \in \Omega_N$ to the complex number

$$\underline{\tau}(x) = \omega_1/\omega_2, \tag{9}$$

where $\langle \omega_1, \omega_2 \rangle$ is a basis of $\Lambda_1$ satisfying

$$\mathrm{Im}(\omega_1\omega_2' - \omega_1'\omega_2) > 0, \quad \text{and} \quad \Lambda_2 = \langle N\omega_1, \omega_2 \rangle. \tag{10}$$

A point $\tau \in \mathcal{H} \cap K$ belongs to $\underline{\tau}(\Omega_N(K))$, where

$$\Omega_N(K) := \{(\Lambda_1, \Lambda_2) \in \Omega_N \text{ with } \Lambda_1, \Lambda_2 \subset K\}/K^\times. \tag{11}$$

Given an order $\mathcal{O}$ of $K$, denote by $\Omega_N(\mathcal{O})$ the set of $(\Lambda_1, \Lambda_2) \in \Omega_N(K)$ such that $\mathcal{O}$ is the largest order preserving both $\Lambda_1$ and $\Lambda_2$. Note that

$$\underline{\tau}(\Omega_N(\mathcal{O})) = \mathcal{H}^{\mathcal{O}}/\Gamma_0(N).$$

Any element $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$ acts naturally on $\Omega_N(\mathcal{O})$ by translation:

$$\mathfrak{a} \star (\Lambda_1, \Lambda_2) := (\mathfrak{a}\Lambda_1, \mathfrak{a}\Lambda_2),$$

and hence also on $\mathcal{H}^{\mathcal{O}}/\Gamma_0(N)$. Denote this latter action by

$$(\mathfrak{a}, \tau) \mapsto \mathfrak{a} \star \tau, \qquad \text{for } \mathfrak{a} \in \mathrm{Pic}(\mathcal{O}), \quad \tau \in \mathcal{H}^{\mathcal{O}}/\Gamma_0(N). \tag{12}$$

Implicit in the definition of this action is the choice of a level $N$, which is usually fixed and therefore suppressed from the notation.

Fix a complex embedding $H \longrightarrow \mathbb{C}$. The following theorem is the main statement that we wish to generalize to real quadratic fields.

THEOREM 1.1. *If $\tau$ belongs to $\mathcal{H}^{\mathcal{O}}/\Gamma_0(N)$, then $u(\alpha, \tau)$ belongs to $H^\times$, and $(\sigma - 1)u(\alpha, \tau)$ belongs to $\mathcal{O}_H^\times$, for all $\sigma \in \mathrm{Gal}(H/K)$. Furthermore,*

$$u(\alpha, \mathfrak{a} \star \tau) = \mathrm{rec}(\mathfrak{a})^{-1} u(\alpha, \tau), \tag{13}$$

*for all $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$.*

Let $\log : \mathbb{R}^{>0} \longrightarrow \mathbb{R}$ denote the usual logarithm. The Kronecker limit formula expresses $\log |u(\alpha, \tau)|^2$ in terms of derivatives of certain zeta functions. The remainder of this chapter is devoted to describing this formula in the shape in which it will be generalized in Chapter 4.

To any positive-definite binary quadratic form $Q$ is associated the zeta function

$$\zeta_Q(s) = \sum_{m,n=-\infty}^{\infty}{}' \; Q(m,n)^{-s}, \tag{14}$$

where the prime on the summation symbol indicates that the sum is taken over pairs of integers $(m,n)$ different from $(0,0)$.

If $\tau$ belongs to $\mathcal{H}^{\mathcal{O}}$, define

$$\zeta_\tau(s) := \zeta_{Q_\tau}(s), \qquad \zeta(\alpha,\tau,s) := \sum_{d|N} n_d d^{-s}\zeta_{d\tau}(s). \tag{15}$$

Note that, for any $d|N$,

$$Q_{d\tau}(x,y) = \frac{A}{d}x^2 + Bxy + dCy^2,$$

so that the terms in the definition of $\zeta(\alpha,\tau,s)$ are zeta functions attached to integral quadratic forms of the same discriminant $-D$. Note also that $\zeta(\alpha,\tau,s)$ depends only on the $\Gamma_0(N)$-orbit of $\tau$.

The Kronecker limit formula can be stated as follows.

THEOREM 1.2. *Suppose that $\tau$ belongs to $\mathcal{H}^{\mathcal{O}}$. The function $\zeta(\alpha,\tau,s)$ is holomorphic except for a simple pole at $s=1$. It vanishes at $s=0$, and*

$$\zeta'(\alpha,\tau,0) = -\frac{1}{12}\log \mathrm{Norm}_{\mathbb{C}/\mathbb{R}}(u(\alpha,\tau)). \tag{16}$$

*Proof.* The function $\zeta_\tau(s)$ is known to be holomorphic everywhere except for a simple pole at $s=1$. Furthermore, the first Kronecker limit formula (cf. [Sie1], Theorem 1 of Ch. I, §1) states that, for all $\tau = u+iv \in \mathcal{H}^{\mathcal{O}}$, the function $\zeta_\tau(s)$ admits the following expansion near $s=1$:

(17)
$$\zeta_\tau(s) = \frac{2\pi}{\sqrt{D}}(s-1)^{-1} + \frac{4\pi}{\sqrt{D}}\left(C - \frac{1}{2}\log(2\sqrt{D}v) - \log(|\eta(\tau)|^2)\right) + O(s-1),$$

where

$$C = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots \frac{1}{n} - \log n)$$

is Euler's constant, and

$$\eta(\tau) = e^{\pi i\tau/12}\prod_{m=1}^{\infty}(1 - e^{2\pi i m\tau})$$

is the Dedekind $\eta$-function satisfying

$$\eta(\tau)^{24} = \Delta(\tau).$$

(The reader should note that Theorem I of Ch. I of [Sie1] is only written down for $D=4$; the case for general $D$ given in (17) is readily deduced from this.) The functional equation satisfied by $\zeta_\tau(s)$ allows us to write its expansion at $s=0$ as

$$\zeta_\tau(s) = -1 - \left(\kappa + 2\log(\sqrt{v}|\eta(\tau)|^2)\right)s + O(s^2),$$

where $\kappa$ is a constant which is unchanged when $\tau \in \mathcal{H}^{\mathcal{O}}$ is replaced by $d\tau$ with $d$ dividing $N$. It follows that $\zeta(\alpha, \tau, 0) = 0$, and a direct calculation shows that $\zeta'(\alpha, \tau, 0)$ is given by (16). $\square$

## 2. Elliptic units for real quadratic fields

Let $K$ be a real quadratic field, and fix an embedding $K \subset \mathbb{R}$. Also fix a prime $p$ which is inert in $K$ and does not divide $N$, as well as an embedding $K \subset \mathbb{C}_p$. Let

$$\mathcal{H}_p = \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$$

denote the $p$-adic upper half-plane which is endowed with an action of the group $\Gamma_0(N)$ and of the larger $\{p\}$-arithmetic group $\Gamma$ defined by

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}[1/p]) \text{ such that } N|c \right\}. \tag{18}$$

Given $\tau \in \mathcal{H}_p \cap K$, the *associated order* of $\tau$ in $M_0(N)[1/p]$, denoted $\mathcal{O}_\tau$, is defined by analogy with (2) as the set of matrices in $M_0(N)[1/p]$ which fix $\tau$ under Möbius transformations, i.e.,

$$\mathcal{O}_\tau := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_0(N)[1/p] \quad \text{such that} \quad a\tau + b = c\tau^2 + d\tau \right\}. \tag{19}$$

This set is identified with a $\mathbb{Z}[1/p]$-order in $K$—i.e., a subring of $K$ which is a free $\mathbb{Z}[1/p]$-module of rank 2.

Conversely, let $D > 0$ be a positive discriminant which is prime to $Np$, and let $\mathcal{O}$ be the $\mathbb{Z}[1/p]$-order of discriminant $D$. Set

$$\mathcal{H}_p^{\mathcal{O}} := \{\tau \in \mathcal{H}_p \text{ such that } \mathcal{O}_\tau = \mathcal{O}\}.$$

This set is preserved under the action of $\Gamma$ by Möbius transformations, and the quotient $\mathcal{H}_p^{\mathcal{O}}/\Gamma$ is finite. Note that the simplifying assumption that $N$ is prime to $D$ implies that the $\mathbb{Z}[1/p]$-order $\mathcal{O}_\tau$ is in fact equal to the full order associated to $\tau$ in $M_2(\mathbb{Z}[1/p])$.

Our goal is to associate to the modular unit $\alpha$ and to each $\tau \in \mathcal{H}_p^{\mathcal{O}}$ (taken modulo the action of $\Gamma$) a canonical invariant $u(\alpha, \tau) \in K_p^\times$ behaving "just like" the elliptic units of the previous chapter, in a sense that is made precise in Conjecture 2.14. To begin, it will be essential to make the following restriction on $\alpha$.

*Assumption* 2.1. There is an element $\xi \in \mathbb{P}_1(\mathbb{Q})$ such that $\alpha$ has neither a zero nor a pole at any cusp which is $\Gamma$-equivalent to $\xi$.

Examples of such modular units are not hard to exhibit. For example, when $N = 4$ the modular unit

$$\alpha = \Delta(z)^2 \Delta(2z)^{-3} \Delta(4z) \tag{20}$$

satisfies assumption 2.1 with $\xi = \infty$. More generally, this is true of the unit $\Delta_\delta$ of equation (1), provided that $\delta$ satisfies

$$\sum_d n_d d = 0. \tag{21}$$

*Remark* 2.2. When $N$ is square-free, two cusps $\xi = \frac{u}{v}$ and $\xi' = \frac{u'}{v'}$ are $\Gamma_0(N)$-equivalent if and only if $\gcd(v, N) = \gcd(v', N)$. Because $p$ does not divide $N$, it follows that two cusps are $\Gamma$-equivalent if and only if they are $\Gamma_0(N)$-equivalent.

*Remark* 2.3. Note that as soon as $X_0(N)$ has at least three cusps, there is a power $\alpha^e$ of $\alpha$ which can be written as

$$\alpha^e = \alpha_0 \alpha_\infty,$$

where $\alpha_j$ satisfies Assumption 2.1 with $\xi = j$. This will make it possible to define the image of $u(\alpha, \tau)$ in $K_p^\times \otimes \mathbb{Q}$ by the rule

$$u(\alpha, \tau) = (u(\alpha_0, \tau) u(\alpha_\infty, \tau)) \otimes \frac{1}{e}.$$

From now on, we will assume that $\alpha = \Delta_\delta$ is of the form given in (1) with the $n_d$ satisfying (21). The construction of $u(\alpha, \tau)$ proceeds in three stages which are described in Sections 2.1, 2.2 and 2.3.

2.1. *p-adic measures.* Recall that a $\mathbb{Z}_p$-valued (resp. integral) $p$-adic measure on $\mathbb{P}_1(\mathbb{Q}_p)$ is a finitely additive function

$$\mu : \left\{ \begin{array}{l} \text{Compact open} \\ \text{subsets } U \subset \mathbb{P}_1(\mathbb{Q}_p) \end{array} \right\} \longrightarrow \mathbb{Z}_p \text{ (resp. } \mathbb{Z}).$$

Such a measure can be integrated against any continuous $\mathbb{C}_p$-valued function $h$ on $\mathbb{P}_1(\mathbb{Q}_p)$ by evaluating the limit of Riemann sums

$$\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t)\, \mathrm{d}\mu(t) := \lim_{\{t_j \in U_j\}} \sum_j h(t_j) \mu(U_j),$$

taken over increasingly fine covers of $\mathbb{P}_1(\mathbb{Q}_p)$ by mutually disjoint compact open subsets $U_j$. If $\mu$ is an *integral* measure, and $h$ is nowhere vanishing, one can define a "multiplicative" refinement of the above integral by setting

$$\times\!\!\!\!\!\!\int_{\mathbb{P}_1(\mathbb{Q}_p)} h(t)\, \mathrm{d}\mu(t) := \lim_{\{t_j \in U_j\}} \prod_j h(t_j)^{\mu(U_j)}. \tag{22}$$

A *ball* in $\mathbb{P}_1(\mathbb{Q}_p)$ is a translate under the action of $\mathrm{PGL}_2(\mathbb{Q}_p)$ of the basic compact open subset $\mathbb{Z}_p \subset \mathbb{P}_1(\mathbb{Q}_p)$. Let $\mathcal{B}$ denote the set of balls in $\mathbb{P}_1(\mathbb{Q}_p)$. The following basic facts about balls will be used freely.

1. A measure $\mu$ is completely determined by its values on the balls. This is because any compact open subset of $\mathbb{P}_1(\mathbb{Q}_p)$ can be written as a disjoint union of elements of $\mathcal{B}$.

2. Any ball $B = \gamma\mathbb{Z}_p$ can be expressed uniquely as a disjoint union of $p$ balls,

$$B = B_0 \cup B_1 \cup \cdots \cup B_{p-1}, \qquad \text{where } B_j = \gamma(j + p\mathbb{Z}_p). \tag{23}$$

The following gives a simple criterion for a function on $\mathcal{B}$ to arise from a measure on $\mathbb{P}_1(\mathbb{Q}_p)$.

LEMMA 2.4. *If $\mu$ is any $\mathbb{Z}_p$-valued function on $\mathcal{B}$ satisfying*

$$\mu(\mathbb{P}_1(\mathbb{Q}_p) - B) = -\mu(B), \quad \mu(B) = \mu(B_0) + \cdots + \mu(B_{p-1}) \text{ for all } B \in \mathcal{B},$$

*then $\mu$ extends (uniquely) to a measure on $\mathbb{P}_1(\mathbb{Q}_p)$ with total measure* 0.

*Remark* 2.5. The proof of Lemma 2.4 can be made transparent by using the dictionary between measures on $\mathbb{P}_1(\mathbb{Q}_p)$ and *harmonic cocycles* on the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$, as explained in Section 2.6.

Let $\alpha^*(z)$ denote the modular unit on $\Gamma_0(Np)$ defined by

$$\alpha^*(z) := \alpha(z)/\alpha(pz).$$

Note that

$$\prod_{j=0}^{p-1} \alpha^*\left(\frac{z-j}{p}\right) = \frac{\prod_{j=0}^{p-1} \alpha\left(\frac{z-j}{p}\right)}{\alpha(z)^p}$$

$$= \frac{\alpha(pz)\prod_{j=0}^{p-1} \alpha\left(\frac{z-j}{p}\right)}{\alpha(pz)\alpha(z)^p} \tag{24}$$

$$= \frac{\alpha(z)^{p+1}}{\alpha(pz)\alpha(z)^p} = \alpha^*(z), \tag{25}$$

where (25) follows from the fact that the weight-two Eisenstein series $\mathrm{dlog}\,\alpha$ on $\Gamma_0(N)$ (whose $q$-expansion is given by (59) and (63) below) is an eigenvector of $T_p$ with eigenvalue $p+1$.

The following proposition is a key ingredient in the definition of $u(\alpha, \tau)$.

PROPOSITION 2.6. *There is a unique collection of integral p-adic measures on $\mathbb{P}_1(\mathbb{Q}_p)$, indexed by pairs $(r, s) \in \Gamma\xi \times \Gamma\xi$ and denoted $\mu_\alpha\{r \to s\}$, satisfying the following axioms for all $r, s \in \Gamma\xi$:*

1. $\mu_\alpha\{r \to s\}(\mathbb{P}_1(\mathbb{Q}_p)) = 0$.

2. $\mu_\alpha\{r \to s\}(\mathbb{Z}_p) = \frac{1}{2\pi i}\int_r^s \mathrm{dlog}\,\alpha^*(z)$.

3. ($\Gamma$-*equivariance*). *For all* $\gamma \in \Gamma$ *and all compact open* $U \subset \mathbb{P}_1(\mathbb{Q}_p)$,

$$\mu_\alpha\{\gamma r \to \gamma s\}(\gamma U) = \mu_\alpha\{r \to s\}(U).$$

*Proof.* The key point is that the group $\Gamma$ acts *almost transitively* on $\mathcal{B}$. There are two distinct $\Gamma$-orbits for this action, one consisting of the orbit of $\mathbb{Z}_p$ and the other of its complement $\mathbb{P}_1(\mathbb{Q}_p) - \mathbb{Z}_p$. To construct the system of measures $\mu_\alpha\{r \to s\}$ satisfying properties (1)–(3) above, we first define them as functions on $\mathcal{B}$. If $B$ is any ball then it can be expressed without loss of generality (after possibly replacing it by its complement) as

$$B = \gamma\mathbb{Z}_p, \qquad \text{for some } \gamma \in \Gamma. \tag{26}$$

Then properties (2) and (3) force the definition

$$\mu_\alpha\{r \to s\}(B) := \frac{1}{2\pi i}\int_{\gamma^{-1}r}^{\gamma^{-1}s} \mathrm{dlog}\,\alpha^*(z). \tag{27}$$

The line integral in (27) converges, since both endpoints belong to the set $\Gamma\xi = \Gamma_0(N)\xi$—this is the crucial stage where assumption 2.1 is used—and it is an integer by the residue theorem. Note also that the right-hand side of (27) does not depend on the expression of $B$ chosen in (26). This is because the element $\gamma$ that appears in (26) is well-defined up to multiplication on the right by an element of $\Gamma_0(Np) = \mathrm{Stab}_\Gamma(\mathbb{Z}_p)$. Since the integrand $\mathrm{dlog}\,\alpha^*$ is invariant under this group, (27) yields a well-defined rule. The function $\mu_\alpha\{r \to s\}$ thus defined on $\mathcal{B}$ extends by additivity to an integral measure on $\mathbb{P}_1(\mathbb{Q}_p)$. To see this let

$$B = B_0 \cup \cdots \cup B_{p-1} = \bigcup_{j=0}^{p-1} \gamma \begin{pmatrix} p & j \\ 0 & 1 \end{pmatrix} \mathbb{Z}_p$$

be the decomposition appearing in (23). Setting $r' = \gamma^{-1}r$ and $s' = \gamma^{-1}s$,

$$2\pi i \sum_{j=0}^{p-1} \mu_\alpha\{r \to s\}(B_j) = \sum_{j=0}^{p-1} \int_{\frac{r'-j}{p}}^{\frac{s'-j}{p}} \mathrm{dlog}\,\alpha^*(z) = \int_{r'}^{s'} U_p\,\mathrm{dlog}\,\alpha^*(z).$$

By (25), the differential form $\mathrm{dlog}\,\alpha^*(z)$ is invariant under $U_p$, and it follows that

$$\mu_\alpha\{r \to s\}(B_0) + \cdots + \mu_\alpha\{r \to s\}(B_{p-1}) = \mu_\alpha\{r \to s\}(B).$$

Proposition 2.6 now follows from Lemma 2.4. □

*Remark* 2.7. It follows from property 2 in Proposition 2.6 that

$$\mu_\alpha\{r \to s\} + \mu_\alpha\{s \to t\} = \mu_\alpha\{r \to t\},$$

for all $r, s, t \in \Gamma\xi$. In the terminology introduced in Section 2.5, $\mu_\alpha$ can thus be viewed as a *partial modular symbol* with values in the $\Gamma$-module of measures on $\mathbb{P}_1(\mathbb{Q}_p)$.

2.2. *Double integrals.* Let

$$\mathrm{ord}_p : \mathbb{C}_p^\times \longrightarrow \mathbb{Q} \subset K_p, \qquad \log_p : \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p$$

be the $p$-adic ordinal and Iwasawa's $p$-adic logarithm respectively, satisfying $\log_p(p) = 0$. Motivated by Definition 1.9 of [Dar1], we set

$$\int_{\tau_1}^{\tau_2}\int_r^s \mathrm{dlog}\,\alpha := \int_{\mathbb{P}_1(\mathbb{Q}_p)} \log_p\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_\alpha\{r \to s\}(t) \tag{28}$$

for $\tau_1, \tau_2 \in \mathcal{H}_p$ and $r, s \in \Gamma\xi$. Note that this new integral—which is $\mathbb{C}_p$-valued—is completely different from the complex line integral of $\mathrm{dlog}\,\alpha$ of equation (40) and so there is some abuse of notation in designating the integrand in the same way. However this notation is suggestive, and should result in no confusion since double integral signs are always used to describe the integral of (28).

The expression defined by (28) is additive in both variables of integration. Properties 1 and 3 of Proposition 2.6 imply that it is also $\Gamma$-invariant, i.e.,

$$\int_{\gamma\tau_1}^{\gamma\tau_2}\int_{\gamma r}^{\gamma s} \mathrm{dlog}\,\alpha = \int_{\tau_1}^{\tau_2}\int_r^s \mathrm{dlog}\,\alpha, \qquad \text{for all } \gamma \in \Gamma.$$

Note that the measures $\mu_\alpha\{r \to s\}$ involved in the definition of the double integral in (28) are actually $\mathbb{Z}$-valued; it is possible to perform the same multiplicative refinement as in equation (71) of [Dar1] to define the $K_p^\times$-valued multiplicative integral:

$$⨰_{\tau_1}^{\tau_2}\int_r^s \mathrm{dlog}\,\alpha = ⨰_{\mathbb{P}_1(\mathbb{Q}_p)} \left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_\alpha\{r \to s\}(t), \tag{29}$$

for $\tau_1, \tau_2 \in \mathcal{H}_p \cap K_p$ and $r, s \in \Gamma\xi$.

2.3. *Splitting a two-cocycle.* Using the double multiplicative integral of equation (29), we may associate to any $\tau \in \mathcal{H}_p \cap K_p$ and to any choice of base point $x \in \Gamma\xi$ a $K_p^\times$-valued two-cocycle

$$\kappa_\tau \in Z^2(\Gamma, K_p^\times)$$

by the rule

$$\kappa_\tau(\gamma_1, \gamma_2) = ⨰_{\gamma_1\gamma_2\tau}^{\gamma_1\tau}\int_x^{\gamma_1 x} \mathrm{dlog}\,\alpha.$$

It is instructive to compare the following proposition with Conjecture 5 of [Dar1].

Proposition 2.8. *The two-cocycles*

$$\mathrm{ord}_p(\kappa_\tau), \quad \log_p(\kappa_\tau) \in Z^2(\Gamma, K_p)$$

*are two-coboundaries. Their image in* $H^2(\Gamma, K_p)$ *does not depend on* $\tau$ *or* $x$.

An explicit splitting of $\mathrm{ord}_p(\kappa_\tau)$ will be given in Section 3 (Proposition 3.4), and of $\log_p(\kappa_\tau)$ in Section 4 (Proposition 4.7); see Section 2.7 for the connection between the indefinite integrals appearing in those propositions and the two-cocycle $\kappa_\tau$.

Given any integer $e > 0$, let $K_p^\times[e]$ denote the $e$-torsion subgroup of $K_p^\times$. Proposition 2.8 implies the existence of an element $\rho_\tau \in C^1(\Gamma, K_p^\times)$ satisfying

$$\kappa_\tau = d\rho_\tau \pmod{K_p^\times[e_\alpha]} \tag{30}$$

for some $e_\alpha$ dividing $p^2 - 1$. The minimal such integer $e_\alpha$ depends only on $\alpha$ and not on $\tau$. It is natural to expect that

$$e_\alpha \overset{?}{=} 1,$$

but we have not attempted to show this. One strategy to do so would be to apply the techniques of Section 4.7 in a "mod $p-1$ refined" context, as in the work of deShalit ([deS1], [deS2]).

*Remark* 2.9. Let $\mu_{p-1}$ denote the group of $(p-1)$st roots of unity in $\mathbb{C}_p^\times$. In many cases, one can give a direct proof that the natural image of $\kappa_\tau$ in $H^2(\Gamma, \mathbb{C}_p^\times/\mu_{p-1})$ vanishes. An element of $H^2(\Gamma, \mathbb{C}_p^\times)$ corresponds to a homomorphism

$$\phi_\kappa : H_2(\Gamma, \mathbb{Z}) \to \mathbb{C}_p^\times.$$

By the independence of the cohomology class $\kappa$ on $\tau$, this homomorphism takes values in $\mathbb{Q}_p^\times$. Up to 2 and 3 torsion,

$$H_2(\Gamma, \mathbb{Z}) = H_1(X_0(Np)(\mathbb{C}) - \Gamma\xi, \mathbb{Z})^{\mathrm{p-new}},$$

where the space on the right is the $p$-new subspace of the singular homology group of the modular curve $X_0(Np)$ with the cusps in $\Gamma\xi$ removed. The homomorphism $\phi_\kappa$ is Hecke equivariant, where the Hecke action on $\mathbb{Q}_p^\times$ is given by the eigenvalues of $\mathrm{dlog}\,\alpha^*$. Thus if there are no $p$-new modular units of level $Np$, regular on $\Gamma\xi$ and with the same eigenvalues as this Eisenstein series—for example, if $N$ is squarefree, or if $N = 4$—then it follows that the image of $\phi_\kappa$ lies in the torsion subgroup of $\mathbb{Q}_p^\times$.

The one-cochain $\rho_\tau$ which splits $\kappa_\tau$ is uniquely defined up to elements in $Z^1(\Gamma, K_p^\times) = \mathrm{Hom}(\Gamma, K_p^\times)$. Fortunately, we have:

Lemma 2.10. *The abelianization of* $\Gamma$ *is finite.*

*Proof.* See Theorem 2 of [Me] or Theorem 3 of [Se2]. □

Let $e_\Gamma$ denote the exponent of the abelianization of $\Gamma$, and let

$$e = \mathrm{lcm}(e_\alpha, e_\Gamma), \quad U = K_p^\times[e].$$

The image of $\rho_\tau$ in $C^1(\Gamma, K_p^\times/U)$ depends only on $\alpha$, $\tau$, and on the base point $x$, not on the choice of one-cochain $\rho_\tau$ satisfying (30).

Assume now that $\tau \in \mathcal{H}_p \cap K$. Let $\Gamma_\tau$ be the stabilizer of $\tau$ in $\Gamma$.

LEMMA 2.11. *The rank of* $\Gamma_\tau$ *is equal to one.*

*Proof.* The group $\Gamma_\tau$ is identified with the group $(\mathcal{O}_\tau)_1^\times$ of elements of norm 1 in the order $\mathcal{O}_\tau$ associated to $\tau$. By the Dirichlet unit theorem this group has rank one, and in fact the quotient $\Gamma_\tau/\langle\pm 1\rangle$ is isomorphic to $\mathbb{Z}$. ☐

LEMMA 2.12. *The restriction of* $\rho_\tau$ *to* $\Gamma_\tau$ *depends only on* $\alpha$ *and* $\tau$, *not on the choice of base point* $x \in \Gamma\xi$ *that was made to define* $\kappa_\tau$.

*Proof.* Write $\kappa_{\tau,x}$ and $\rho_{\tau,x}$ for $\kappa_\tau$ and $\rho_\tau$, respectively, to emphasize the dependence of these invariants on the choice of base point $x \in \Gamma\xi$. A direct computation (cf. for example Lemma 8.4 of [Dar2]) shows that if $y$ is another choice of base point, then

$$\kappa_{\tau,x} - \kappa_{\tau,y} = d\rho_{x,y},$$

where the one-cochain $\rho_{x,y} \in C^1(\Gamma, K_p^\times)$ vanishes on $\Gamma_\tau$. The lemma follows. ☐

Let $\varepsilon$ be a fundamental unit of $(\mathcal{O}_\tau)_1^\times \subset K^\times$, chosen to be greater than 1 or less than 1 according to whether $\tau > \tau'$ or $\tau < \tau'$, respectively, where $\tau'$ is the Galois conjugate of $\tau$. The unit $\varepsilon$ is independent of the choice of real embedding of $K$. Let $\gamma_\tau$ be the unique element of $\Gamma_\tau$ satisfying

$$\gamma_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \varepsilon \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

We define $u(\alpha, \tau)$ by setting

$$u(\alpha, \tau) := \rho_\tau(\gamma_\tau) \in K_p^\times/U. \tag{31}$$

Note that $u(\alpha, \tau)$ depends only on the $\Gamma$-orbit of $\tau$.

*Remark* 2.13. It may not be apparent to the reader why the somewhat intricate construction of $u(\alpha, \tau)$ given above is analogous to the construction of Section 1 leading to elliptic units. Some further explanation of the analogy (in the context of the Stark-Heegner points of [Dar1]) can be found in Sections 4 and 5 of [BDG], and in the uniformization theory developed in [Das1], [Das3].

2.4. *The main conjecture.* The elements $u(\alpha,\tau) \in K_p^\times/U$ are expected to behave exactly like the elliptic units $u(\alpha,\tau)$ of Chapter 1. To make this statement more precise we now formulate a conjectural Shimura reciprocity law for these elements.

A $\mathbb{Z}[1/p]$-lattice in $K$ is a $\mathbb{Z}[1/p]$-submodule of $K$ which is free of rank 2. Let $K_+^\times$ denote the multiplicative group of elements of $K$ of positive norm. By analogy with (11) we then set

$$\Omega_N(K) = \left\{ (\Lambda_1, \Lambda_2), \text{ with } \begin{array}{l} \Lambda_j \text{ a } \mathbb{Z}[1/p]\text{-lattice in } K, \\ \Lambda_1/\Lambda_2 \simeq \mathbb{Z}/N\mathbb{Z}. \end{array} \right\} /K_+^\times. \tag{32}$$

(In this definition it is important to take equivalence classes under multiplication by $K_+^\times$ rather than $K^\times$; see also Remark 2.19 of Section 2.8.) As in Chapter 1, there is a natural bijective map $\underline{\tau}$ from $\Omega_N(K)$ to $(\mathcal{H}_p \cap K)/\Gamma$, which to $x = (\Lambda_1, \Lambda_2)$ assigns

$$\underline{\tau}(x) = \omega_1/\omega_2, \tag{33}$$

where $\langle \omega_1, \omega_2 \rangle$ is a $\mathbb{Z}[1/p]$-basis of $\Lambda_1$ satisfying

$$\begin{array}{rl} \omega_1\omega_2' - \omega_1'\omega_2 & > 0, \\ \mathrm{ord}_p(\omega_1\omega_2' - \omega_1'\omega_2) & \equiv 0 \pmod 2, \end{array} \qquad \text{and } \Lambda_2 = \langle N\omega_1, \omega_2 \rangle. \tag{34}$$

Recall that $\mathcal{O}$ is a $\mathbb{Z}[1/p]$-order of $K$ of discriminant prime to $N$ (and $p$, by convention). As before denote by $\Omega_N(\mathcal{O})$ the set of pairs $(\Lambda_1, \Lambda_2) \in \Omega_N(K)$ such that $\mathcal{O}$ is the maximal $\mathbb{Z}[1/p]$-order of $K$ preserving both $\Lambda_1$ and $\Lambda_2$. Note that $\underline{\tau}(\Omega_N(\mathcal{O})) = \mathcal{H}_p^{\mathcal{O}}/\Gamma$.

Let $\mathrm{Pic}^+(\mathcal{O})$ denote the *narrow Picard group* of $\mathcal{O}$, defined as the group of projective $\mathcal{O}$-submodules of $K$ modulo homothety by $K_+^\times$. Class field theory identifies $\mathrm{Pic}^+(\mathcal{O})$ with the Galois group of an abelian extension $H$ of $K$, the *narrow ring class field* attached to $\mathcal{O}$. Let

$$\mathrm{rec} : \mathrm{Pic}^+(\mathcal{O}) \longrightarrow \mathrm{Gal}(H/K) \tag{35}$$

denote the reciprocity law map of global class field theory. The group $\mathrm{Pic}^+(\mathcal{O})$ acts naturally on $\Omega_N(\mathcal{O})$ by translation, and hence it also acts on $\underline{\tau}(\Omega_N(\mathcal{O})) = \mathcal{H}_p^{\mathcal{O}}/\Gamma$. Adopting the same notation as in equation (12) of Chapter 1, denote this latter action by

$$(\mathfrak{a}, \tau) \mapsto \mathfrak{a} \star \tau, \qquad \text{for } \mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O}), \quad \tau \in \mathcal{H}_p^{\mathcal{O}}/\Gamma. \tag{36}$$

The following conjecture can be viewed as a natural generalization of Theorem 1.1 for real quadratic fields.

CONJECTURE 2.14. *If $\tau$ belongs to $\mathcal{H}_p^{\mathcal{O}}/\Gamma$, then $u(\alpha,\tau)$ belongs to $\mathcal{O}_H[1/p]^\times/U$, and in fact,*

$$u(\alpha, \mathfrak{a} \star \tau) = \mathrm{rec}(\mathfrak{a})^{-1} u(\alpha, \tau) \pmod U, \tag{37}$$

*for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$.*

In spite of its strong analogy with Theorem 1.1, Conjecture 2.14 appears to lie deeper: its proof would yield an explicit class field theory for real quadratic fields.

Chapter 11 of [Das1] (cf. also [Das2]) describes efficient algorithms for calculating $u(\alpha,\tau)$ and uses these algorithms to obtain numerical evidence for Conjecture 2.14.

Evidence of a more theoretical nature will be given in Chapters 3 and 4 by relating the analytically defined elements $u(\alpha,\tau)$ to special values of zeta functions, in the spirit of Theorem 1.2.

The remainder of this chapter contains some preliminaries of a more technical nature which the reader may wish to skip on a first reading.

2.5. *Modular symbols and Dedekind sums.* We discuss the notion of *partial modular symbols* and the associated Dedekind sums that will be useful for the calculation of the $u(\alpha,\tau)$—both from a computational and a theoretical point of view.

*Partial modular symbols.* Let $\mathcal{M}_\xi$ denote the module of $\mathbb{Z}$-valued functions $m$ on $\Gamma\xi \times \Gamma\xi$, denoted $(r,s) \mapsto m\{r \to s\}$, and satisfying

$$m\{r \to s\} + m\{s \to t\} = m\{r \to t\}, \tag{38}$$

for all $r,s,t \in \Gamma\xi$. Functions of this sort will be called *partial modular symbols* with respect to $\xi$, and $\Gamma$. (This terminology is adopted because $m$ satisfies all the properties of a modular symbol except that it is not defined on all of $\mathbb{P}_1(\mathbb{Q})$ but only on a $\Gamma$-invariant subset of it.) More generally, if $M$ is any $\Gamma$-module, write $\mathcal{M}_\xi(M)$ for the group of $M$-valued partial modular symbols, equipped with the natural $\Gamma$-module structure

$$(\gamma m)\{r \to s\} := \gamma\left(m\{\gamma^{-1}r \to \gamma^{-1}s\}\right). \tag{39}$$

To the modular unit $\alpha$ is associated the $\mathbb{Z}$-valued $\Gamma_0(N)$-invariant partial modular symbol

$$m_\alpha\{r \to s\} := \frac{1}{2\pi i}\int_r^s \operatorname{dlog}\alpha. \tag{40}$$

*Dedekind sums.* The line integrals in (40) defining the modular symbol $m_\alpha$ can be expressed in terms of classical *Dedekind sums*

$$D\left(\frac{a}{m}\right) := \sum_{x=1}^{m} B_1\left(\frac{x}{m}\right) B_1\left(\frac{ax}{m}\right), \qquad \text{for } \gcd(a,m)=1, \quad m>0,$$

where

$$B_1(x) = \{x\} - 1/2 = x - [x] - 1/2$$

is the first Bernoulli polynomial made periodic. Corresponding to the element $\delta$ used to define $\alpha = \Delta_\delta$ in (1), one defines the modified Dedekind sum

$$D^\delta(x) := \sum_{d|N} n_d D(dx).$$

Following [Maz, II §2], we introduce the modified Dedekind-Rademacher homomorphism on $\Gamma_0(N)$:

$$\Phi_\delta \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} := \begin{cases} 0 & \text{if } c = 0; \\ 12\,\mathrm{sign}(c) D^\delta\left(\frac{a}{N|c|}\right) & \text{otherwise,} \end{cases} \tag{41}$$

as well as the corresponding homomorphism of $\Gamma_0(Np)$:

$$\Phi_\delta^* \begin{pmatrix} a & b \\ Npc & d \end{pmatrix} := \begin{cases} 0 & \text{if } c = 0; \\ 12\,\mathrm{sign}(c)\left(D^\delta\left(\frac{a}{pN|c|}\right) - D^\delta\left(\frac{a}{N|c|}\right)\right) & \text{otherwise.} \end{cases}$$

Note that the assumption (21) that was made on $\delta$ created a simplification in the behaviour of the Dedekind-Rademacher homomorphism, making it vanish on the upper-triangular matrices and eliminating the extra terms appearing in Equation (2.1) of [Maz] when $\delta = [N] - [1]$. In particular it is clear that $\Phi_\delta$ and $\Phi_\delta^*$ take integral values.

The modified Dedekind-Rademacher homomorphisms $\Phi_\delta$ and $\Phi_\delta^*$ attached to $\delta$ encode the periods of $\mathrm{dlog}\,\alpha$ and $\mathrm{dlog}\,\alpha^*$ respectively. For any choice of base points $x \in \mathcal{H} \cup \Gamma\xi$ and $\tau \in \mathcal{H}$, we have

$$\begin{aligned} -\Phi_\delta(\gamma) &= \frac{1}{2\pi i}\int_x^{\gamma x} \mathrm{dlog}\,\alpha = \frac{1}{2\pi i}(\log\alpha(\gamma\tau) - \log\alpha(\tau)), \\ -\Phi_\delta^*(\gamma) &= \frac{1}{2\pi i}\int_x^{\gamma x} \mathrm{dlog}\,\alpha^* = \frac{1}{2\pi i}(\log\alpha^*(\gamma\tau) - \log\alpha^*(\tau)), \end{aligned} \tag{42}$$

for all $\gamma$ in $\Gamma_0(N)$ and $\Gamma_0(Np)$ respectively. In particular, if $r, s$ belong to $\Gamma\xi$, we may evaluate the partial modular symbol $m_\alpha\{r \to s\}$ by choosing $\gamma \in \Gamma_0(N)$ such that $s = \gamma r$, and noting that

$$\frac{1}{2\pi i}\int_r^s \mathrm{dlog}\,\alpha = -\Phi_\delta(\gamma). \tag{43}$$

2.6. *Measures and the Bruhat-Tits tree.* Let $\mathcal{T}$ denote the *Bruhat-Tits tree* of $\mathrm{PGL}_2(\mathbb{Q}_p)$, whose set $\mathcal{V}(\mathcal{T})$ of vertices is in bijection with the $\mathbb{Q}_p^\times$-homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$, two vertices being joined by an edge if the corresponding classes admit representatives which are contained one in the other with index $p$. (See Chapter 5 of [Dar2] for a detailed discussion.) The group $\tilde{\Gamma}$ of matrices in $\mathrm{PGL}_2^+(\mathbb{Z}[1/p])$ which are upper-triangular modulo $N$ acts transitively on $\mathcal{V}(\mathcal{T})$ via its natural (left) action on $\mathbb{Q}_p^2$, and the group $\Gamma_0(N)$ is the stabilizer in $\tilde{\Gamma}$ of the basic vertex $v_0$ corresponding to the standard lattice $\mathbb{Z}_p^2$.

The *unramified upper half-plane* $\mathcal{H}_p^{\mathrm{nr}}$ is the set of $\tau \in \mathcal{H}_p$ such that $\mathbb{Q}_p(\tau)$ generates an unramified extension of $\mathbb{Q}_p$. The Bruhat-Tits tree can be viewed as a combinatorial "skeleton" of $\mathcal{H}_p$, and is the target of the *reduction map*

$$r : \mathcal{H}_p^{\mathrm{nr}} \longrightarrow \mathcal{V}(\mathcal{T}).$$

This map is compatible with the natural $\mathrm{PGL}_2(\mathbb{Q}_p)$-actions on both source and target, and its definition and main properties can be found, for example, in Chapter 5 of [Dar2].

To each $v \in \mathcal{V}(\mathcal{T})$ we associate a well-defined partial modular symbol $m_v\{r \to s\}$ by imposing the rules

$$m_{v_0}\{r \to s\} := m_\alpha\{r \to s\}, \quad m_{\gamma v}\{\gamma r \to \gamma s\} = m_v\{r \to s\},$$

for all $v \in \mathcal{V}(\mathcal{T})$, $\gamma \in \tilde{\Gamma}$, and $r, s \in \Gamma\xi$. In addition to the built-in $\Gamma$-equivariance relation satisfied by the collection $\{m_v\}$ of partial modular symbols, the assignment $v \mapsto m_v\{r \to s\}$ satisfies the following harmonicity property:

$$(44) \qquad \sum_{d(v',v)=1} m_{v'}\{r \to s\} = (p+1) m_v\{r \to s\}, \qquad \text{for all } v \in v(\mathcal{T}),$$

in which the sum on the left is taken over the $p+1$ vertices $v'$ which are adjacent to $v$. The relation (44) follows from the fact that $\mathrm{dlog}\,\alpha$ is a weight two Eisenstein series on $\Gamma_0(N)$ and hence an eigenvector for the Hecke operator $T_p$ with eigenvalue $p+1$.

Let $\mathcal{E}(\mathcal{T})$ denote the set of ordered edges of $\mathcal{T}$, i.e., the set of ordered pairs of adjacent vertices of $\mathcal{T}$. If $e = (v_s, v_t)$ is such an edge, it is convenient to write $s(e) := v_s$ and $t(e) = v_t$ for the source and target vertex of $e$ respectively, and $\bar{e} = (v_t, v_s)$ for the edge obtained from $e$ by reversing the orientation.

A ($\mathbb{Z}$-valued) harmonic cocycle on $\mathcal{T}$ is a function $f : \mathcal{E}(\mathcal{T}) \longrightarrow \mathbb{Z}$ satisfying

$$(45) \qquad \sum_{s(e)=v} f(e) = 0, \qquad \text{for all } v \in \mathcal{V}(\mathcal{T}),$$

as well as $f(\bar{e}) = -f(e)$, for all $e \in \mathcal{E}(\mathcal{T})$.

The collection of partial modular symbols $m_v$ gives rise to a system $m_e$ of partial modular symbols, indexed this time by the oriented edges of $\mathcal{T}$, by the rule

$$(46) \qquad m_e\{r \to s\} := m_{t(e)}\{r \to s\} - m_{s(e)}\{r \to s\}.$$

Note that, if $r$ and $s \in \Gamma\xi$ are fixed, the assignment $e \mapsto m_e\{r \to s\}$ is a $\mathbb{Z}$-valued harmonic cocycle on $\mathcal{T}$. This follows directly from (44).

As explained in Section 1.2 of [Dar1] or in Chapter 5 of [Dar2], to each ordered edge $e$ of $\mathcal{T}$ is attached a standard compact open subset of $\mathbb{P}_1(\mathbb{Q}_p)$, denoted $U_e$. Thanks to this assignment, the $\mathbb{Z}_p$-valued harmonic cocycles on

$\mathcal{T}$ are in natural bijection with the $\mathbb{Z}_p$-valued measures on $\mathbb{P}_1(\mathbb{Q}_p)$ by sending a cocycle $c$ to the measure $\mu$ satisfying

$$\mu(U_e) := c(e), \quad \text{for all } e \in \mathcal{E}(\mathcal{T}). \tag{47}$$

The harmonic cocycles $m_e\{r \to s\}$ of (46) give rise in this way to the $p$-adic measures $\mu_\alpha\{r \to s\}$ of Proposition 2.6, satisfying:

$$\mu_\alpha\{r \to s\}(U_e) = m_e\{r \to s\}. \tag{48}$$

2.7. *Indefinite integrals.* The double multiplicative integral of (29) can be used to associate to $\alpha$ and $\tau$ an $\mathcal{M}_\xi(K_p^\times)$-valued one-cocycle

$$\tilde{\kappa}_\tau \in Z^1(\Gamma, \mathcal{M}_\xi(K_p^\times)) \quad \text{defined by } \tilde{\kappa}_\tau(\gamma)\{r \to s\} = \times\!\!\!\!\!\!\int_\tau^{\gamma\tau}\int_r^s \operatorname{dlog}\alpha.$$

Let $\mathcal{F}_\xi(K_p^\times)$ denote the space of $K_p^\times$-valued functions on $\Gamma\xi$, and denote by

$$d : \mathcal{F}_\xi(K_p^\times) \longrightarrow \mathcal{M}_\xi(K_p^\times)$$

the $\Gamma$-module homomorphism defined by the rule

$$(df)\{r \to s\} := f(s)/f(r).$$

Finally, denote by

$$\delta : H^1(\Gamma, \mathcal{M}_\xi(K_p^\times)) \longrightarrow H^2(\Gamma, K_p^\times)$$

the connecting homomorphism arising from the $\Gamma$-cohomology of the exact sequence

$$0 \longrightarrow K_p^\times \longrightarrow \mathcal{F}_\xi(K_p^\times) \longrightarrow \mathcal{M}_\xi(K_p^\times) \longrightarrow 0.$$

One can see (cf. the discussion in Section 9.6 of [Dar2]) that

$$(\delta\tilde{\kappa}_\tau)(\gamma_2^{-1}, \gamma_1^{-1}) = \kappa_\tau(\gamma_1, \gamma_2).$$

Proposition 2.8 is a consequence of the following more precise statement whose proof will be given in Chapters 3 and 4.

Proposition 2.15. *The one-cocycles* $\operatorname{ord}_p(\tilde{\kappa}_\tau)$ *and* $\log_p(\tilde{\kappa}_\tau)$ *are one-coboundaries.*

As in the discussion following the statement of Proposition 2.8, Proposition 2.15 implies the existence of a $U \subset (K_p)^\times_{\text{tors}}$ such that

$$\tilde{\kappa}_\tau = d\tilde{\rho}_\tau \pmod{U}, \qquad \text{for some } \tilde{\rho}_\tau \in \mathcal{M}_\xi(K_p^\times), \tag{49}$$

and the image of $\tilde{\rho}_\tau$ in $\mathcal{M}_\xi(K_p^\times/U)$ is unique.

Define the indefinite integral involving only one $p$-adic endpoint of integration by the rule

$$\times\!\!\!\!\!\!\int^\tau\int_r^s \operatorname{dlog}\alpha := \tilde{\rho}_\tau\{r \to s\} \in K_p^\times/U.$$

This indefinite integral is completely characterized by the following three properties:

$$\text{(50)}\qquad \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{r}^{s}\mathrm{dlog}\,\alpha \times \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{s}^{t}\mathrm{dlog}\,\alpha = \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{r}^{t}\mathrm{dlog}\,\alpha, \qquad \text{for all } r,s,t\in\Gamma\xi,$$

$$\text{(51)}\qquad \times\!\!\!\!\!\!\int^{\tau_1}\!\!\int_{r}^{s}\mathrm{dlog}\,\alpha \div \times\!\!\!\!\!\!\int^{\tau_2}\!\!\int_{r}^{s}\mathrm{dlog}\,\alpha = \times\!\!\!\!\!\!\int_{\tau_2}^{\tau_1}\!\!\int_{r}^{s}\mathrm{dlog}\,\alpha, \qquad \text{for all } \tau_1,\tau_2\in\mathcal{H}_p,$$

$$\text{(52)}\qquad \times\!\!\!\!\!\!\int^{\gamma\tau}\!\!\int_{\gamma r}^{\gamma s}\mathrm{dlog}\,\alpha = \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{r}^{s}\mathrm{dlog}\,\alpha, \qquad \text{for all } \gamma\in\Gamma.$$

Letting $x\in\Gamma\xi$ be the base point that was used to construct $\rho_\tau$, we have

$$\text{(53)}\qquad \rho_\tau(\gamma) = \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{x}^{\gamma x}\mathrm{dlog}\,\alpha \pmod{U}.$$

In particular,

LEMMA 2.16. *The following equality holds in* $K_p^\times/U$:

$$u(\alpha,\tau) = \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{x}^{\gamma_\tau x}\mathrm{dlog}\,\alpha,$$

*for any base point* $x\in\Gamma\xi$.

2.8. *The action of complex conjugation and of* $U_p$. The partial modular symbol $m_\alpha$ used to define $u(\alpha,\tau)$ is *odd* in the sense that

$$m_\alpha\{-x\to -y\} = -m_\alpha\{x\to y\}$$

for all $x,y\in\Gamma\xi$ (cf. [Maz, Ch. II, §3]).

The complex conjugation associated to either of the infinite places $\infty_1$ or $\infty_2$ of $K$ is the same in $\mathrm{Gal}(H/K)$ since $H$ is a ring class field of $K$. Let $\tau_\infty\in\mathrm{Gal}(H/K)$ denote this element. The parity of $m_\alpha$ implies the following behaviour of the elements $u(\alpha,\tau)$ under the action of $\tau_\infty$.

PROPOSITION 2.17. *Assume conjecture* 2.14. *For all* $\tau\in\mathcal{H}_p^{\mathcal{O}}$,

$$\tau_\infty u(\alpha,\tau) = u(\alpha,\tau)^{-1}.$$

*Proof.* The fact that the partial modular symbol $m_\alpha$ is odd implies that the sign denoted $w_\infty$ in Proposition 5.13 of [Dar1] satisfies

$$w_\infty = -1.$$

The proof of Proposition 2.17 is then identical to the proof of Proposition 5.13 of [Dar1]. □

*Remark* 2.18. In the context of a modular elliptic curve $E$ treated in [Dar1], the sign $w_\infty$ can be chosen to be either 1 or $-1$ by working with either the even or odd modular symbol of $E$, corresponding to the choice of the real or imaginary period attached to $E$ respectively. In the situation treated here, where $E$ is replaced by the multiplicative group, only the odd modular symbol $m_\alpha$ remains available, in harmony with the fact that the multiplicative group has a single period, $2\pi i$, which is *purely imaginary.*

*Remark* 2.19. Suppose that $\mathcal{O}$ has a fundamental unit of negative norm. Then equivalence of ideals in the strict and usual sense coincide, so that the narrow ring class field $H$ associated to $\mathcal{O}$ is equal to the ring class field taken in the nonstrict sense, which is totally real. Conjecture 2.14 predicts that $\tau_\infty$ should act trivially on $u(\alpha,\tau)$ in this case, and that the $p$-units $u(\alpha,\tau)$ should be trivial. In fact it can be shown, independently of any conjectures, that

$$u(\alpha,\tau)=1, \qquad \text{for all } \tau\in\mathcal{H}_p^{\mathcal{O}}.$$

This suggests that interesting elements of $H^\times$ are obtained only when $H$ is a totally complex extension of $K$. This explains why it is so essential to work with equivalence of ideals in the narrow sense and with narrow ring class fields to obtain useful invariants.

Similarly to the proof of Proposition 2.17, the fact that the Eisenstein series $\operatorname{dlog}\alpha^*$ is fixed by the $U_p$ operator implies that the sign denoted $w$ in Proposition 5.13 of [Dar1] equals 1. Thus the invariance of the indefinite integral given in (52) holds for all $\gamma\in\tilde{\Gamma}\supset\Gamma/\langle\pm1\rangle$. In particular, the element $u(\alpha,\tau)$ depends only on the $\tilde{\Gamma}$ orbit of $\tau$.

## 3. Special values of zeta functions

It will be assumed for simplicity in this section that $p$ is inert (and not ramified) in $K/\mathbb{Q}$. Recall the $p$-adic ordinal

$$\operatorname{ord}_p: K_p^\times \longrightarrow \mathbb{Z}$$

mentioned in Section 2.3. The goal of this section is to give a precise formula for $\operatorname{ord}_p(u(\alpha,\tau))$ when $\tau\in\mathcal{H}_p\cap K$, in terms of the special values of certain zeta functions.

3.1. *The zeta function.* Given $\tau\in\mathcal{H}_p^{\mathcal{O}}$, the primitive integral binary quadratic form $Q_\tau$ associated to $\tau$ can be defined as in (3). This time, $Q_\tau$ is *non*-definite. Its discriminant is positive and is of the form $Dp^k$ for some integer $k\geq 0$, where $D$ is the discriminant of the $\mathbb{Z}[1/p]$-order $\mathcal{O}$. (By convention, the integer $D$ is taken to be prime to $p$.) By replacing $\tau$ by a $\tilde{\Gamma}$-translate, we may assume without loss of generality that

$$\text{the discriminant of } Q_\tau \text{ is equal to } D. \tag{54}$$

We will make this assumption from now on. In that case the generator $\gamma_\tau$ of $\Gamma_\tau/\langle\pm1\rangle$ belongs to $\Gamma_0(N)$. Note that the matrix $\gamma_\tau$ fixes the quadratic form

$Q_\tau$ under the usual action of $\mathrm{SL}_2(\mathbb{Z})$ on the set of binary quadratic forms. Furthermore, the simplifying assumption that $\gcd(D,N)=1$ implies that $\gamma_\tau = \tilde{\gamma}_\tau$, where the latter matrix is taken to be the generator of the stabilizer of the form $Q_\tau$ in $\mathrm{SL}_2(\mathbb{Z})$.

Given any nondefinite binary quadratic form $Q$ whose discriminant is not a perfect square, let $\gamma_Q$ be a generator of its stabilizer in $\mathrm{SL}_2(\mathbb{Z})$. Note that $Q$ takes on both positive and negative integral values, and that each value in the range of $Q$ is taken on infinitely often, since $Q$ is constant on the $\gamma_Q$-orbits in $\mathbb{Z}^2$. The definition of $\zeta_Q(s)$ given in (14) needs to be modified accordingly, by setting

$$\mathcal{W} := (\mathbb{Z}^2 - \{0\})/\langle\gamma_Q\rangle,$$

and letting

$$\zeta_Q(s) = \sum_{(m,n)\in\mathcal{W}} \mathrm{sign}(Q(m,n))|Q(m,n)|^{-s}, \tag{55}$$

where $\mathrm{sign}(x)=\pm 1$ denotes the sign of a nonzero real number $x$.

Equivalence classes of binary quadratic forms of discriminant $D$ are in natural bijection with narrow ideal classes of $\mathcal{O}\cap\mathcal{O}_K$-ideals, by associating to such an ideal class the suitably scaled norm form attached to a representative ideal. The partial zeta function attached to the narrow ideal class $\mathcal{A}$ is defined in the usual way by the rule

$$\zeta(s,\mathcal{A}) := \sum_{I\in\mathcal{A}} \mathrm{Norm}(I)^{-s}.$$

If $\mathcal{A}$ is a narrow ideal class, let $\mathcal{A}^*$ be the ideal class corresponding to $\alpha\mathcal{A}$ for some $\alpha\in K^\times$ of negative norm, and let $Q$ be a quadratic form of discriminant $D$ associated to $\mathcal{A}$. A standard calculation (cf. the beginning of Section 2 of [Za], for example) shows that

$$\zeta_Q(s) = \zeta(s,\mathcal{A}) - \zeta(s,\mathcal{A}^*). \tag{56}$$

Note in particular that $\zeta_Q(s)=0$ if $\mathcal{O}$ contains a unit of negative norm, since $\mathcal{A}=\mathcal{A}^*$ in that case.

We mimic the definitions of equation (15) and define

$$\zeta_\tau(s) := \zeta_{Q_\tau}(s), \qquad \zeta(\alpha,\tau,s) := \sum_{d|N} n_d d^s \zeta_{d\tau}(s). \tag{57}$$

(Observe that $s$ rather than $-s$ appears as the exponent of $d$ in the definition of $\zeta(\alpha,\tau,s)$.) As in (15), the function $\zeta(\alpha,\tau,s)$ is a simple linear combination of zeta functions atttached to integral quadratic forms of the same (positive) discriminant $D$. Note that $\zeta(\alpha,\tau,s)$ depends only on the $\Gamma_0(N)$-orbit of the element $\tau\in\mathcal{H}_p^{\mathcal{O}}$ normalized to satisfy (54).

Let $\mathbb{A}_K$ denote the ring of adeles of $K$. A finite order idele class character

$$\chi = \prod_v \chi_v : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

is called a *ring class character* if it is trivial on $\mathbb{A}_{\mathbb{Q}}^\times$. If $\chi$ is such a character, then its two archimedean components $\chi_{\infty_1}$ and $\chi_{\infty_2}$ attached to the two real places of $K$ are either both trivial, or both equal to the sign character. In the former case $\chi$ is called *even* and in the latter, it is said to be *odd*. Any ring class character can be interpreted as a character on the narrow Picard group $G_{\mathcal{O}} := \mathrm{Pic}^+(\mathcal{O})$ of narrow ideal classes attached to a fixed order $\mathcal{O}$ of $K$ whose conductor is equal to the conductor of $\chi$.

Formula (56) shows that the zeta functions $\zeta_\tau(s)$ with $\tau \in \mathcal{H}_p^{\mathcal{O}}$ can be interpreted in terms of partial zeta functions encoding the zeta function of $K$ twisted by *odd* ring class characters of $G_{\mathcal{O}}$. More precisely, letting $\tau_0$ be any element of $\mathcal{H}_p^{\mathcal{O}}$ which is equivalent to $\sqrt{D}$ under the action of $\mathrm{SL}_2(\mathbb{Z})$, we have:

$$\sum_{\sigma \in G_{\mathcal{O}}} \chi(\sigma) \zeta_{\sigma * \tau_0}(s) = \begin{cases} 0 & \text{if } \chi \text{ is even;} \\ L(K, \chi, s) & \text{if } \chi \text{ is odd.} \end{cases} \tag{58}$$

The main formula of this chapter is

THEOREM 3.1. *Suppose that $\tau$ belongs to $\mathcal{H}_p^{\mathcal{O}}$, and is normalized by the action of $\tilde{\Gamma}$ to satisfy* (54). *Then*

$$\zeta(\alpha, \tau, 0) = \frac{1}{12} \cdot \mathrm{ord}_p(u(\alpha, \tau)).$$

3.2. *Values at negative integers.* In this section we give a formula for the value of $\zeta(\alpha, \tau, 0)$ in terms of complex periods of $\mathrm{dlog}\, \alpha$. This formula is a special case of a more general one expressing $\zeta(\alpha, \tau, 1-r)$ in terms of periods of certain Eisenstein series of weight $2r$, for odd $r \geq 1$. The logarithmic derivatives $\mathrm{dlog}\, \alpha$ and $\mathrm{dlog}\, \alpha^*$ can be written as

$$\mathrm{dlog}\, \alpha(z) = 2\pi i F_2(z)\, \mathrm{d}z, \qquad \mathrm{dlog}\, \alpha^*(z) = 2\pi i F_2^*(z)\, \mathrm{d}z, \tag{59}$$

where $F_2(z)$ and $F_2^*(z)$ are the weight two Eisenstein series on $\Gamma_0(N)$ and $\Gamma_0(Np)$, respectively, given by the formulae

$$F_2(z) = -24 \sum_{d|N} d n_d E_2(dz), \qquad F_2^*(z) = F_2(z) - pF_2(pz), \tag{60}$$

and $E_2(z)$ is the standard Eisenstein series of weight 2

$$\begin{aligned} E_2(z) &= \frac{1}{(2\pi i)^2} \left( \zeta(2) + \frac{1}{2} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^2} \right) \\ &= -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n, \quad q = e^{2\pi i \tau}. \end{aligned} \tag{61}$$

(We remark that the double series used to define $E_2$ is not absolutely convergent and the resulting expression is not invariant under $\mathrm{SL}_2(\mathbb{Z})$. For a discussion of the weight two Eisenstein series, see Section 3.10 of [Ap] for example.)

The Eisenstein series of (61) and (60) are part of a natural *family* of Eisenstein series of varying weights. For even $k \geq 2$, consider the standard Eisenstein series of weight $k$:

$$E_k(z) = \frac{2(k-1)!}{(2\pi i)^k} \sum_{m,n=-\infty}^{\infty}{}' \frac{1}{(mz+n)^k} = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n. \tag{62}$$

Define likewise, as a function of the element $\delta = \sum_d n_d d$ used to define the modular unit $\alpha$, the higher weight Eisenstein series

$$\begin{aligned} F_k(z) &= -24 \sum_{d|N} n_d \cdot d \cdot E_k(dz) \\ &= -\frac{48(k-1)!}{(2\pi i)^k} \sum_{m,n=-\infty}^{\infty}{}' \left( \frac{1}{(mz+n)^k} \sum_{d|(N,m)} n_d d \right) \\ &= -24 \sum_{n=1}^{\infty} \sigma_{k-1}(n) \sum_{d|N} n_d d q^{nd}. \end{aligned} \tag{63}$$

The $F_k$ are modular forms of weight $k$ on $\Gamma_0(N)$, holomorphic on the upper half-plane. Note that these Eisenstein series have no constant term and hence are holomorphic at the cusp $i\infty$. We also define, for the purposes of $p$-adic interpolation, the function

$$F_k^*(z) = F_k(z) - p^{k-1} F_k(pz).$$

We extend the definition of $E_k(z)$ and $F_k(z)$ to all $k \geq 2$ by letting $E_k = F_k = 0$ for $k$ odd.

Recall the standard right action of $\mathrm{GL}_2^+(\mathbb{R})$ on the space of modular forms of weight $k$, given by

$$F|_\gamma(z) = \frac{\det(\gamma)}{(cz+d)^k} F(\gamma z) \text{ when } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Now, the definition of $F_k^*$ can be written

$$F_k^* = F_k - p^{k-2} F_k|_P(z), \quad \text{where } P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

The following proposition expresses $\zeta(\alpha, \tau, 1-r)$ in terms of periods of $F_{2r}$.

Proposition 3.2. *For all odd integers $r > 0$,*

$$12 \cdot \zeta(\alpha, \tau, 1-r) = \int_{\xi}^{\gamma_\tau \xi} Q_\tau(z,1)^{r-1} F_{2r}(z)\, \mathrm{d}z.$$

*Proof.* Let $k \geq 2$ be a positive integer and let $\widetilde{E}_k$ denote the weight $k$ Eisenstein series

$$\widetilde{E}_k = \frac{(2\pi i)^k}{2(k-1)!} E_k(z) \quad \left( = {\sum_{m,n}}' \frac{1}{(mz+n)^k} \quad \text{if } k > 2 \right).$$

By Hilfsatz 1 of [Sie2], letting $z_0 \in \mathcal{H}$ be an arbitrary base point, the following identity holds for all integers $r > 1$:

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} \widetilde{E}_{2r}(z)\, \mathrm{d}z = (-1)^{r-1} \frac{(r-1)!^2}{(2r-1)!} D^{r-\frac{1}{2}} \sum_{\mathcal{W}} Q_\tau(m,n)^{-r}. \tag{64}$$

Suppose that $r > 1$ is an odd integer. Then

$$\sum_{\mathcal{W}} Q_\tau(m,n)^{-r} = \sum_{\mathcal{W}} \operatorname{sign}(Q_\tau(m,n)) |Q_\tau(m,n)|^{-r} = \zeta_\tau(r),$$

so that

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} \widetilde{E}_{2r}(z)\, \mathrm{d}z = \frac{(r-1)!^2}{(2r-1)!} D^{r-\frac{1}{2}} \zeta_\tau(r). \tag{65}$$

On the other hand, it follows from the relation (58) and from the functional equation for $L(K,\chi,s)$ for odd characters (cf. [La, Cor. 1 after Th. 14 of §8, Ch. XIV]) that $\zeta_\tau(s)$ satisfies the functional equation

$$\zeta_\tau(1-s) = \frac{D^{s-\frac{1}{2}}}{\pi^{2s-1}} \Gamma\left(\frac{s+1}{2}\right)^2 \Gamma\left(\frac{2-s}{2}\right)^{-2} \zeta_\tau(s). \tag{66}$$

Hence if $r \geq 2$ is an even positive integer,

$$\zeta_\tau(1-r) = 0,$$

while if $r \geq 1$ is odd,

$$\zeta_\tau(1-r) = \frac{4D^{r-\frac{1}{2}}}{(2\pi)^{2r}} (r-1)!^2 \zeta_\tau(r). \tag{67}$$

Combining this functional equation with (65), we obtain

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} \widetilde{E}_{2r}(z)\, \mathrm{d}z = \frac{(2\pi)^{2r}}{4(2r-1)!} \zeta_\tau(1-r).$$

Since

$$E_k(z) = \frac{2(k-1)!}{(2\pi i)^k} \widetilde{E}_k(z),$$

it follows that

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} E_{2r}(z)\, \mathrm{d}z = -\frac{1}{2} \zeta_\tau(1-r). \tag{68}$$

From the definition of $F_k$,

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} F_{2r}(z)\,\mathrm{d}z = -24 \sum_{d|N} n_d \cdot d \cdot \int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} E_{2r}(dz)\,\mathrm{d}z.$$

Making the change of variables $w = d \cdot z$, we obtain

$$d \cdot \int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} E_{2r}(dz)\,\mathrm{d}z = \int_{d \cdot z_0}^{d\cdot\gamma_\tau z_0} Q_\tau\left(\frac{w}{d}, 1\right)^{r-1} E_{2r}(w)\,\mathrm{d}w.$$

Recall that $\tilde{\gamma}_{d\tau}$ denotes the generator of the stabilizer of $d\tau$ in $\mathrm{SL}_2(\mathbb{Z})$ (chosen in such a way that $\gamma_{d\tau}$ is a positive power of $\tilde{\gamma}_{d\tau}$). Note that

$$d\gamma_\tau z_0 = \tilde{\gamma}_{d\tau}(dz_0), \qquad Q_\tau\left(\frac{w}{d}, 1\right) = \frac{1}{d} Q_{d\tau}(w, 1).$$

Hence

$$d \cdot \int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} E_{2r}(dz)\,\mathrm{d}z = \int_{dz_0}^{\tilde{\gamma}_{d\tau}(dz_0)} \frac{1}{d^{r-1}} Q_{d\tau}(w)^{r-1} E_{2r}(w)\,\mathrm{d}w.$$

The expression on the right is equal to $-\frac{d^{1-r}}{2}\zeta_{d\tau}(1-r)$, by (68). It follows that, for all odd $r > 1$,

$$\int_{z_0}^{\gamma_\tau z_0} Q_\tau^{r-1} F_{2r}(z)\,\mathrm{d}z = 12 \sum_{d|N} n_d d^{1-r} \zeta_{d\tau}(1-r) = 12 \cdot \zeta(\alpha, \tau, 1-r). \tag{69}$$

The integrand in the left-hand expression involves an Eisenstein series which is holomorphic at $\infty$, hence we may replace the base point $z_0 \in \mathcal{H}$ by the cusp $\infty$ (or any other cusp which belongs to the same $\Gamma_0(N)$-orbit).

In the case where $r = 1$, using (59), (42), and (41), we see that the expression on the left of (69) is equal to

$$\int_{z_0}^{\gamma_\tau z_0} F_2(z)\,\mathrm{d}z = \frac{1}{2\pi i} \int_{z_0}^{\gamma_\tau z_0} \mathrm{dlog}\,\alpha = \Phi_\delta(\gamma_\tau) = 12\,\mathrm{sign}(c) D^\delta\left(\frac{a}{N|c|}\right), \tag{70}$$

where

$$\gamma_\tau = \begin{pmatrix} a & b \\ Nc & * \end{pmatrix}$$

and $D^\delta$ is the modified Dedekind sum introduced previously. Meyer's formula expressing the special values of partial zeta functions attached to real quadratic fields at $s = 0$ in terms of Dedekind sums can be used to derive the identity

$$12\zeta(\alpha, \tau, 0) = -\Phi_\delta(\gamma_\tau). \tag{71}$$

(Cf. [Za, Eq. (4.1)] for a statement of Meyer's formula in the case where $D$ is fundamental; the general case can be derived from equation (18) in §5 of [CS] for example.) It follows that Proposition 3.2 holds for $r = 1$ as well, in light of the fact that $\mathrm{dlog}\,\alpha$ is holomorphic at $\xi$ so that the base point $z_0$ can be replaced by the cusp $\xi$ in the expression on the left of (70). $\square$

The evaluation of the right-hand side in Theorem 3.1 is taken up in the next section.

3.3. *The $p$-adic valuation.* To compute $\mathrm{ord}_p(u(\alpha,\tau))$, it will be useful to have at our disposal a formula for the $p$-adic valuation of a $p$-adic (multiplicative) line integral. We describe such a formula in the case where the $p$-adic endpoints of integration belong to the unramified upper half-plane $\mathcal{H}_p^{\mathrm{nr}}$, in terms of the reduction map from $\mathcal{H}_p^{\mathrm{nr}}$ to $\mathcal{V}(\mathcal{T})$ introduced in Section 2.6.

LEMMA 3.3. *For all $\tau_1, \tau_2 \in \mathcal{H}_p^{\mathrm{nr}}$ and for all $r, s \in \Gamma\xi$,*

$$\mathrm{ord}_p\left(\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2}\int_r^s \mathrm{dlog}\,\alpha\right) = \sum_{e:r(\tau_1)\longrightarrow r(\tau_2)} m_e\{r\to s\},$$

*where the sum on the right is taken over the ordered edges in the path of $\mathcal{T}$ joining $r(\tau_1)$ to $r(\tau_2)$.*

A complete proof of this formula is given, for example, in Lemma 2.5 of [BDG]. $\square$

PROPOSITION 3.4. *Let $v = r(\tau)$. Then*

$$\mathrm{ord}_p\left(\times\!\!\!\!\!\!\int^{\tau}\int_r^s \mathrm{dlog}\,\alpha\right) = m_v\{r\to s\}.$$

*Proof.* By Lemma 3.3, for all $\gamma\in\Gamma$,

$$\mathrm{ord}_p\left(\times\!\!\!\!\!\!\int_{\tau}^{\gamma\tau}\int_r^s \mathrm{dlog}\,\alpha\right) = \sum_{e:v\to\gamma v} m_e\{r\to s\}, \tag{72}$$

where the sum on the right is taken over the ordered edges in the path joining $v$ to $\gamma v$. By (46), this sum is equal to the telescoping sum

$$\begin{aligned}\sum_{e:v\to\gamma v} m_{t(e)}\{r\to s\} - m_{s(e)}\{r\to s\} &= m_{\gamma v}\{r\to s\} - m_v\{r\to s\}\\ &= m_v\{\gamma^{-1}r\to\gamma^{-1}s\} - m_v\{r\to s\}\\ &= (dm_v)(\gamma)\{r\to s\},\end{aligned}$$

so that $\mathrm{ord}_p(\tilde\kappa_\tau) = dm_v$. It follows from the defining equation (49) for $\tilde\rho_\tau$ and from the fact that $\mathcal{M}_\xi(\mathbb{Z})^\Gamma = 0$ that

$$\mathrm{ord}_p(\tilde\rho_\tau) = m_v. \tag{73}$$

The lemma follows. $\square$

We may assume without loss of generality that $\tau$ has been normalized to satisfy (54), so that $r(\tau) = v_0$, where $v_0$ is the vertex of $\mathcal{T}$ corresponding to the standard lattice $\mathbb{Z}_p^2$. In this case the matrix $\gamma_\tau$ belongs to $\Gamma_0(N)$ and generates the stabilizer of $\tau$ in that group; furthermore we have $m_{v_0} = m_\alpha$.

COROLLARY 3.5. *Let $x$ be any base point in $\Gamma\xi$. Then*

$$\mathrm{ord}_p(u(\alpha,\tau)) = \frac{1}{2\pi i}\int_x^{\gamma_\tau x} \mathrm{dlog}\,\alpha = -\Phi_\delta(\gamma_\tau).$$

*Proof.* By Lemma 2.16 and Proposition 3.4,

$$\mathrm{ord}_p(u(\alpha,\tau)) = \mathrm{ord}_p\left(\times\!\!\!\!\!\!\int^{\tau}\int_x^{\gamma_\tau x} \mathrm{dlog}\,\alpha\right) = m_\alpha\{x \to \gamma_\tau x\}. \tag{74}$$

The lemma follows from the definition of $m_\alpha$ given in (40) and from (42). $\square$

The proof of Theorem 3.1 now follows by combination of (71) and Corollary 3.5.

3.4. *The Brumer-Stark conjecture.* Given $\tau \in \mathcal{H}_p^{\mathcal{O}}$, let $\mathrm{BS}_\tau$ denote the *Brumer-Stickelberger* element in the integral group ring of $G_{\mathcal{O}} = \mathrm{Pic}^+(\mathcal{O})$, defined by

$$\mathrm{BS}_\tau := \sum_{\sigma\in G_{\mathcal{O}}} \zeta_{\sigma*\tau}(0)\cdot\sigma^{-1}.$$

This element is independent of the choice of $\tau \in \mathcal{H}_p^{\mathcal{O}}$, up to multiplication by an element of $G_{\mathcal{O}}$. Relation (58) implies that $\mathrm{BS}_\tau$ agrees with the usual Brumer-Stickelberger element attached to the extension $H/K$.

To any modular unit $\alpha$ and with $\tau \in \mathcal{H}_p^{\mathcal{O}}$ we may also associate the *modified Brumer-Stickelberger element* by setting

$$\mathrm{BS}(\alpha,\tau) = \sum_{\sigma\in G_{\mathcal{O}}} \zeta(\alpha,\sigma*\tau,0)\sigma^{-1}. \tag{75}$$

Let $\mathrm{Cl}(H)$ denote the class group of $H$, viewed as a $\mathbb{Z}[G_{\mathcal{O}}]$-module in a natural way. Let $I$ denote the augmentation ideal of $\mathbb{Z}[G_{\mathcal{O}}]$. The following conjecture is a reformulation of the usual Brumer-Stark conjecture for $H/K$, generalising Stickelberger's theorem on class groups of abelian extensions of $\mathbb{Q}$.

CONJECTURE 3.6. *The element $\mathrm{BS}_\tau$ annihilates $I\,\mathrm{Cl}(H)\otimes\mathbb{Z}[1/2]$.*

Conjecture 3.6 is proved in this case thanks to the work of Wiles [Wi]. We give a more direct proof which is *conditional* on Conjecture 2.14, in the spirit of Stickelberger's original proof in the abelian case. Because it is only conditional, this result is more notable for what it says about Conjecture 2.14 than about the Brumer-Stark conjectures.

PROPOSITION 3.7. *Assume Conjecture 2.14. Then the Brumer-Stickelberger element $\mathrm{BS}_\tau$ annihilates $I\,\mathrm{Cl}(H)\otimes\mathbb{Z}[1/2]$.*

*Proof.* For any modular unit $\alpha$, we have the relation

$$\mathrm{BS}(\alpha,\tau) = J_\alpha\cdot\mathrm{BS}_\tau,$$

where $J_\alpha \in I$ is an element which depends on $\alpha$ and $\tau$ and is defined as follows. The integral quadratic form $Q_\tau = Ax^2 + Bxy + Cy^2$ attached to $\tau \in \mathcal{H}_p^{\mathcal{O}}$ determines an $\mathcal{O}$-ideal of norm $d$, for each $d|N$, by the rule

$$\mathfrak{a}_d = \langle d, B - \sqrt{D} \rangle.$$

Then

$$J_\alpha = \sum_{d|N} n_d \cdot \mathrm{rec}(\mathfrak{a}_d).$$

By the Chebotarev density theorem, the elements $J_\alpha$ generate $I$ as $\alpha$ ranges over the possible modular units. Hence it is enough to show that

$$\mathrm{BS}(\alpha, \tau) \text{ annihilates } \mathrm{Cl}(H) \otimes \mathbb{Z}[1/2].$$

Let $\tilde{H}$ denote the maximal subfield of the Hilbert class field of $H$ which is of odd degree over $H$. Note that $\tilde{H}$ is Galois over $K$, and even over $\mathbb{Q}$. Class field theory identifies $\mathrm{Gal}(\tilde{H}/H)$ with

$$M := \mathrm{Cl}(H) \otimes \mathbb{Z}[1/2]$$

as modules over $\mathbb{Z}[G_{\mathcal{O}}][1/2]$. Since $\mathrm{Gal}(H/\mathbb{Q})$ is a generalized dihedral group, the generator of $\mathrm{Gal}(K/\mathbb{Q})$ lifts to an involution $\iota \in \mathrm{Gal}(H/\mathbb{Q})$. Lift $\iota$ further to an involution in $\mathrm{Gal}(\tilde{H}/\mathbb{Q})$. (This can be done since $\tilde{H}$ is of odd degree over $H$.) Choose any $\sigma \in M$. By the Chebotarev density theorem, there exists a rational prime $p$ such that

$$\mathrm{Frob}_p(\tilde{H}/\mathbb{Q}) = \sigma\iota.$$

In particular, $p$ is inert in $K$. Note that $p$, as a prime ideal of $K$, splits completely in $H/K$. Choosing a prime $\mathfrak{p}$ of $H$ above $p$, we have

$$\mathrm{Frob}_p(\tilde{H}/K) = \mathrm{Frob}_{\mathfrak{p}}(\tilde{H}/H) = \sigma\iota\sigma\iota = \sigma\sigma^\iota.$$

The factorization of $u(\alpha, \tau)$ and its conjugates given by Theorem 3.1 implies that

$$\mathrm{BS}(\alpha, \tau) \text{ annihilates } \mathrm{Frob}_{\mathfrak{p}}(\tilde{H}/H) = \sigma^{1+\iota}.$$

Since $\sigma$ was chosen arbitrarily, it follows that

$$\mathrm{BS}(\alpha, \tau) \text{ annihilates } (1 + \iota)M. \tag{76}$$

Let $c_\infty \in \mathrm{Gal}(H/K)$ denote complex conjugation. Since $\iota$ was an arbitrary lift of the generator of $\mathrm{Gal}(K/\mathbb{Q})$, we could have replaced it by $\iota c_\infty$ in the preceding argument, yielding

$$\mathrm{BS}(\alpha, \tau) \text{ annihilates } (1 + \iota c_\infty)M. \tag{77}$$

Note furthermore that by definition $(1 + c_\infty)\,\mathrm{BS}(\alpha, \tau) = 0$, so *a fortiori*

$$\mathrm{BS}(\alpha, \tau) \text{ annihilates } (1 + c_\infty)M. \tag{78}$$

Since the module $M$ has odd order, it decomposes as a direct sum of simultaneous eigenspaces for the action of the commuting involutions $\iota$ and $c_\infty$. Each eigenspace belongs to at least one of the subspaces in (76), (77), or (78). The result follows. $\square$

3.5. *Connection with the Gross-Stark conjecture.* A general result of Deligne and Ribet (cf. the discussion in [Gr1, §2]) implies the existence of a $p$-adic meromorphic function $\zeta_p(\alpha, \tau, s)$ of the variable $s \in \mathbb{Z}_p$ characterized by its values on a dense set of negative integers:

(79)
$$\zeta_p(\alpha, \tau, n) = (1 - p^{-2n})\zeta(\alpha, \tau, n), \quad \text{for all } n \leq 0, \quad n \equiv 0 \pmod{2(p-1)}.$$

Let $U_{H,p}$ denote the group of $p$-units of $H$ defined by Gross in Proposition 3.8 of [Gr1]:

$$U_{H,p} := \{\epsilon \in H^\times : ||\epsilon||_{\mathfrak{D}} = 1 \text{ for all places } \mathfrak{D} \text{ which do not divide } p\}.$$

Since the places $\mathfrak{D}$ involved in the definition of $U_{H,p}$ include all the archimedean ones, it follows that $U_{H,p}$ is infinite only when $H$ has no real embeddings, and that images of the elements of $U_{H,p}$ under all the complex embeddings of $H$ lie on the unit circle.

Proposition 2.17 implies that the $p$-unit $u(\alpha, \tau)$ belongs to $U_{H,p}$ (assuming, of course, conjecture 2.14). Since

$$\mathrm{ord}_p(u(\alpha, \tau)) = 12 \cdot \zeta(\alpha, \tau, 0),$$

Conjecture (2.12) of [Gr1] (cf. the formulation given in Proposition 3.8 of [Gr1]) suggests that one should have

$$\log_p \mathrm{Norm}_{K_p/\mathbb{Q}_p}(u(\alpha, \tau)) = -12 \cdot \zeta_p'(\alpha, \tau, 0). \tag{80}$$

In fact, the relation (80) is essentially *equivalent* (by varying $\alpha$ appropriately) to the Gross-Stark conjecture for $H/K$, assuming Conjecture 2.14. The next chapter is devoted to the explicit construction of $\zeta_p(\alpha, \tau, s)$ and to a proof of (80).

## 4. A Kronecker limit formula

The first three sections of this chapter give an explicit construction of the $p$-adic zeta function $\zeta_p(\alpha, \tau, s)$ satisfying the interpolation property (79). The following theorem is then proved.

THEOREM 4.1. *Suppose that $\tau$ belongs to $\mathcal{H}_p^{\mathcal{O}}$, and is normalized by the action of $\tilde{\Gamma}$ to satisfy* (54). *Then*

$$\zeta_p'(\alpha, \tau, 0) = -\frac{1}{12} \cdot \log_p \mathrm{Norm}_{K_p/\mathbb{Q}_p}(u(\alpha, \tau)).$$

Note the clear analogy between this formula and the classical Kronecker limit formula stated in Theorem 1.2. Theorem 4.1 allows us to deduce the Gross-Stark conjecture for $H/K$ from Conjecture 2.14. It should be pointed out that Conjecture 2.14 is stronger and more precise than Gross's conjecture in that setting, since it gives a formula for the Gross-Stark unit $u(\alpha, \tau)$ itself, and not just its norm to $\mathbb{Q}_p$.

4.1. *Measures associated to Eisenstein series .* Let

$$\mathbb{X} := (\mathbb{Z}_p \times \mathbb{Z}_p)' \subset (\mathbb{Q}_p \times \mathbb{Q}_p - \{0\}),$$

considered as column vectors, where $(\mathbb{Z}_p \times \mathbb{Z}_p)'$ denotes the set of *primitive vectors* $(x, y) \in \mathbb{Z}_p^2$ satisfying $\gcd(x, y) = 1$. The space $\mathbb{Q}_p^2 - \{0\}$ is endowed with a natural action of $\Gamma$ by left multiplication. There is a $\mathbb{Z}_p^\times$-bundle map

$$\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p) \quad \text{given by} \quad (x, y) \mapsto x/y.$$

The crucial technical ingredient in the construction of $\zeta_p(\alpha, \tau, s)$ and in the proof of Theorem 4.1 is the following result, which can be viewed as an extension of Proposition 2.6 to the family of Eisenstein series introduced in the previous section.

THEOREM 4.2. *Fix $\alpha$ and $\xi$ as before. There is a unique collection of $p$-adic measures on the space $\mathbb{Q}_p^2 - \{0\}$, indexed by pairs $(r, s) \in \Gamma\xi \times \Gamma\xi$ and denoted $\mu\{r \to s\}$, satisfying the following properties*:

1. *For every homogeneous polynomial $h(x, y) \in \mathbb{Z}[x, y]$ of degree $k - 2$,*

$$\int_{\mathbb{X}} h(x, y)\, \mathrm{d}\mu\{r \to s\}(x, y) = \mathrm{Re}\left( (1 - p^{k-2}) \int_r^s h(z, 1) F_k(z)\, \mathrm{d}z \right). \tag{81}$$

2. ($\Gamma$-*equivariance*) *For all $\gamma \in \Gamma$ and all compact, open $U \subset \mathbb{Q}_p^2 - \{0\}$,*

$$\mu\{\gamma r \to \gamma s\}(\gamma U) = \mu\{r \to s\}(U).$$

3. (*Invariance under multiplication by $p$*).

$$\mu\{r \to s\}(pU) = \mu\{r \to s\}(U).$$

*Furthermore this measure satisfies*:

4. *For every homogeneous polynomial $h(x, y) \in \mathbb{Z}[x, y]$ of degree $k - 2$,*

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} h(x, y)\, \mathrm{d}\mu\{r \to s\}(x, y) = \mathrm{Re}\left( \int_r^s h(z, 1) F_k^*(z)\, \mathrm{d}z \right).$$

*Remark* 4.3. The function $(r, s) \mapsto \mu\{r \to s\}$ defines a partial modular symbol with values in the space of measures on $\mathbb{Q}_p^2 - \{0\}$. Objects of this type appear in Glenn Stevens' study of two-variable $p$-adic $L$-functions attached to Hida (and Coleman) families of eigenforms. More precisely, when $\mathrm{dlog}\, \alpha$ is replaced by a weight-two cuspidal eigenform $f$ which is *ordinary* at $p$, Stevens attaches to $f$ a measure-valued modular symbol via Hida's theory of families of eigenforms, and uses it to define the two-variable $p$-adic $L$-function attached to this family. There is also a theory in the nonordinary setting, where it becomes necessary to replace $p$-adic measures by *locally analytic distributions* in the sense of Stevens.

The proof of Theorem 4.2 is postponed to the end of the paper (beginning with Section 4.4). The following lemma shows how the measures $\mu\{r \to s\}$ are related to the measures $\mu_\alpha\{r \to s\}$ of the previous section.

LEMMA 4.4. *For all compact open $U \subset \mathbb{P}^1(\mathbb{Q}_p)$,*

$$\mu\{r \to s\}(\pi^{-1}U) = \mu_\alpha\{r \to s\}(U). \tag{82}$$

Recall that $\pi^{-1}(U) \subset \mathbb{X}$ by definition.

*Proof.* Define a collection of measures $\pi_*\mu\{r \to s\}$ on $\mathbb{P}_1(\mathbb{Q}_p)$ by the rule

$$\pi_*\mu\{r \to s\}(U) := \mu\{r \to s\}(\pi^{-1}(U)).$$

Theorem 4.2 implies that the collection of measures $\pi_*\mu\{r \to s\}$ satisfies all the properties of $\mu_\alpha$ spelled out in Proposition 2.6. To see that $\pi_*\mu\{r \to s\}$ satisfies the required $\Gamma$-invariance property, note that

$$\pi_*\mu\{\gamma r \to \gamma s\}(\gamma U) = \mu\{\gamma r \to \gamma s\}(\pi^{-1}(\gamma U)) = \mu\{\gamma r \to \gamma s\}(\gamma\pi^{-1}(U)),$$

where the last equality follows from the fact that both $\pi^{-1}(\gamma U)$ and $\gamma\pi^{-1}(U)$ are fundamental regions for the action of $\langle p\rangle$ on the inverse image of $\gamma U$ in $\mathbb{Q}_p^2 - \{0\}$. Hence

$$\pi_*\mu\{\gamma r \to \gamma s\}(\gamma U) = \mu\{r \to s\}(\pi^{-1}(U)) = \pi_*\mu\{r \to s\}(U).$$

Lemma 4.4 follows from the uniqueness in Proposition 2.6. □

4.2. *Construction of the $p$-adic $L$-function.* The special values of $\zeta(\alpha, \tau, s)$ at certain, even negative, integers can be expressed in terms of the measure $\mu$ described in Section 4.1.

LEMMA 4.5. *For all odd integers $r > 0$*

$$12(1 - p^{2r-2}) \cdot \zeta(\alpha, \tau, 1 - r) = \int_{\mathbb{X}} Q_\tau(x, y)^{r-1}\, d\mu\{\xi \to \gamma_\tau\xi\}(x, y).$$

*Proof.* This follows directly from Proposition 3.2 in light of the properties of the measure $\mu$ spelled out in Theorem 4.2. □

Suppose that the integer $r$ (in addition to being odd) is congruent to 1 modulo $p-1$. Then by Lemma 4.5,

$$12(1-p^{2r-2})\cdot\zeta(\alpha,\tau,1-r)=\int_{\mathbb{X}}\langle Q_\tau(x,y)\rangle^{r-1}\,\mathrm{d}\mu\{\xi\to\gamma_\tau\xi\}(x,y), \tag{83}$$

where for $x\in\mathbb{Z}_p^\times$, the expression $\langle x\rangle$ denotes the unique element in $1+p\mathbb{Z}_p$ which differs from $x$ by a $(p-1)$st root of unity. The advantage of the expression (83) is that it interpolates $p$-adically, expressing $\zeta(\alpha,\tau,1-r)$ with its Euler factor at $p$ removed, as a function of the $p$-adic variable $r$. This leads us to define

$$\zeta_p(\alpha,\tau,s)=\frac{1}{12}\int_{\mathbb{X}}\langle Q_\tau(x,y)\rangle^{-s}\,\mathrm{d}\mu\{\xi\to\gamma_\tau\xi\}(x,y)$$

for all $s\in\mathbb{Z}_p$. Note that one recovers the $p$-adic $L$-function introduced in Section 3.5 which is uniquely characterized by the interpolation property (79).

In terms of this explicit definition of $\zeta_p(\alpha,\tau,s)$, we have

LEMMA 4.6. *The derivative $\zeta_p'(\alpha,\tau,s)$ at $s=0$ is given by*

$$\zeta_p'(\alpha,\tau,0)=-\frac{1}{12}\int_{\mathbb{X}}\log_p\left(Q_\tau(x,y)\right)\mathrm{d}\mu\{\xi\to\gamma_\tau\xi\}(x,y).$$

*Proof.* This is a direct consequence of the definition. $\square$

4.3. *An explicit splitting of a two-cocycle.* We now turn to the calculation of the one-cochain $\rho_\tau$, or, equivalently, of the expression

$$\times\!\!\!\!\!\!\int^{\tau}\int_r^s \mathrm{dlog}\,\alpha.$$

A formula for this indefinite integral can be given in terms of the system of $p$-adic measures $\mu$ of Theorem 4.2.

PROPOSITION 4.7. *Let $\mu$ be as in Theorem* 4.2. *Then*

$$\log_p\left(\times\!\!\!\!\!\!\int^{\tau}\int_r^s \mathrm{dlog}\,\alpha\right)=\int_{\mathbb{X}}\log_p(x-\tau y)\,\mathrm{d}\mu\{r\to s\}(x,y).$$

*Proof.* If we *define*

$$\int^{\tau}\int_r^s \mathrm{dlog}\,\alpha^? :=\int_{\mathbb{X}}\log_p(x-\tau y)\,\mathrm{d}\mu\{r\to s\}(x,y),$$

then a direct calculation shows that the resulting expression satisfies

$$\int^{\tau}\int_r^s \mathrm{dlog}\,\alpha^? + \int^{\tau}\int_s^t \mathrm{dlog}\,\alpha^? = \int^{\tau}\int_r^t \mathrm{dlog}\,\alpha^?, \qquad \text{for all } r,s,t\in\Gamma\xi, \tag{84}$$

$$\int^{\tau_1}\int_r^s \mathrm{dlog}\,\alpha^? - \int^{\tau_2}\int_r^s \mathrm{dlog}\,\alpha^? = \int_{\tau_2}^{\tau_1}\int_r^s \mathrm{dlog}\,\alpha, \tag{85}$$

as well as

$$\int^{\gamma\tau}\!\!\int_{\gamma r}^{\gamma s} \operatorname{dlog}\alpha^? = \int^{\tau}\!\!\int_{r}^{s} \operatorname{dlog}\alpha^?, \qquad \text{for all } \gamma \in \Gamma. \tag{86}$$

These properties are the additive counterparts of equations (50), (51) and (52) of Section 2.7, which uniquely determine the $p$-adic indefinite multiplicative integral attached to $\operatorname{dlog}\alpha$. It follows that

$$\int^{\tau}\!\!\int_{r}^{s} \operatorname{dlog}\alpha^? = \log_p \left( \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{r}^{s} \operatorname{dlog}\alpha \right),$$

as was to be shown. □

We can now prove Theorem 4.1:

*Proof of Theorem* 4.1. By Lemma 2.16, we have

$$\log_p \operatorname{Norm}_{K_p/\mathbb{Q}_p}(u(\alpha,\tau)) = \log_p \left( \times\!\!\!\!\!\!\int^{\tau}\!\!\int_{r}^{\gamma_\tau r} \operatorname{dlog}\alpha \times \times\!\!\!\!\!\!\int^{\tau'}\!\!\int_{r}^{\gamma_\tau r} \operatorname{dlog}\alpha \right),$$

for any $r \in \Gamma\xi$. By Proposition 4.7 and the fact that $Q_\tau(x,y)$ is proportional to $(x-\tau y)(x-\tau' y)$ and that $\mu(\mathbb{X}) = 0$, the expression on the right is equal to

$$\int_{\mathbb{X}} \log_p Q_\tau(x,y)\, \mathrm{d}\mu\{r \to \gamma_\tau r\}(x,y).$$

The result now follows from Lemma 4.6. □

The remainder of the paper is devoted to the proof of Theorem 4.2.

4.4. *Generalized Dedekind sums.* In this section we evaluate the integrals appearing in the right side of (81) in Theorem 4.2, which characterize the partial modular symbol of measures $\mu$. The computations of this section are not new, but we include them for completeness and notational consistency. Let $f$ denote a modular form and let $a_f(0)$ denote the constant term of its $q$-expansion at $\infty$. For any relatively prime integers $a$ and $c$ with $c \geq 1$, the function

$$A_f(s;a,c) = e^{\pi i s/2} c^{s-1} \int_0^\infty (f(it + a/c) - a_f(0))\, t^{s-1}\, \mathrm{d}t$$

is well defined for $\mathrm{Re}(s)$ large enough, and has a meromorphic continuation to all of $\mathbb{C}$. For the Eisenstein series $E_{2k}$ with $k > 1$, this is given by

(87)

$$A_{E_{2k}}(s;a,c) = e^{\pi i s/2} c^{2k-2} \frac{\Gamma(s)}{(2\pi)^s} \sum_{h=1}^{c} \left[ \zeta(s+1-2k, h/c) \sum_{m=1}^{\infty} \frac{1}{m^s} e^{2\pi i m h a/c} \right],$$

where $\zeta(s,b)$ denotes the Hurwitz zeta function. This is a relatively standard computation, carried out for example in Proposition 3.1 of [Fuk]. Let us calculate the real part of this expression for $s$ an integer, $1 \le s \le 2k-1$. Note that when $s=1$, the term for $h=c$ in (87) is taken to be

$$\lim_{s\to 1} \zeta(s+1-2k)\zeta(s) \in \mathbb{R}.$$

The Hurwitz zeta function has the well known value $\zeta(1-n,b) = -B_n(b)/n$, where the Bernoulli polynomials $B_n$ are defined by the power series

$$\frac{e^{tb}}{e^t-1} = \sum_{n=0}^{\infty} \frac{B_n(b)}{n!} t^{n-1}.$$

Furthermore, for any real number $x$,

$$\begin{aligned}\operatorname{Re}\left(\frac{i^s(s-1)!}{(2\pi)^s}\sum_{m=1}^{\infty}\frac{1}{m^s}e^{2\pi imx}\right) &= \frac{i^s(s-1)!}{2(2\pi)^s}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1}{m^s}e^{2\pi imx}\\ &= \frac{(-1)^{s+1}}{2}\cdot\frac{\tilde{B}_s(x)}{s},\end{aligned}$$

where

$$\tilde{B}_s(x) := \begin{cases} 0 & \text{if } s=1 \text{ and } x\in\mathbb{Z} \\ B_s(\{x\}) = B_s(x-[x]) & \text{otherwise.}\end{cases}$$

(See Section II of [Hal] for this last equation.) Hence we obtain

$$\operatorname{Re}\left(A_{E_{2k}}(s;a,c)\right) = c^{2k-2}\frac{(-1)^s}{2}\sum_{h=1}^{c}\frac{B_{2k-s}(h/c)}{2k-s}\cdot\frac{\tilde{B}_s(ha/c)}{s}. \tag{88}$$

We would like to replace the term $B_{2k-s}(h/c)$ by $\tilde{B}_{2k-s}(h/c)$ in the sum above. Only the term for $h=c$, which we now consider, may cause difficulty. If $s$ is even, then $B_{2k-s}(1) = B_{2k-s}(0)$ since in general one has

$$B_n(1-x) = (-1)^n B_n(x).$$

If $s$ is odd then the other term in the product is $\tilde{B}_s(ha/c) = 0$. Thus in either case we may replace the term $B_{2k-s}(h/c)$ by $\tilde{B}_{2k-s}(h/c)$. This motivates the following definition.

*Definition* 4.8. Let $s,t \ge 0$. For $a$ and $c$ relatively prime and $c>0$, the *generalized Dedekind sum* $\tilde{D}_{s,t}(a/c)$ is defined by

$$\tilde{D}_{s,t}(a/c) := c^{s-1}\sum_{h=1}^{c}\tilde{B}_s(h/c)\tilde{B}_t(ha/c).$$

Note that the sum may be taken over any complete set of representatives $h$ mod $c$. For $s,t\ge 1$, define

$$D_{s,t}(a/c) := \frac{\tilde{D}_{s,t}(a/c)}{st}.$$

*Remark* 4.9. When $s = t = 1$,

$$D_{1,1}(a/c) = \tilde{D}_{1,1}(a/c) = D(a/c) - \frac{1}{4}.$$

Equation (88) may be written in terms of the generalized Dedekind sums as

$$\mathrm{Re}\left(A_{E_k}(s; a, c)\right) = c^{s-1}\frac{(-1)^s}{2} D_{k-s,s}(a/c). \tag{89}$$

This formula continues to hold when $k$ is odd, since then the Dedekind sum $D_{k-s,s}(a/c)$ vanishes (using the relation

$$\tilde{B}_s(-x) = (-1)^s \tilde{B}_s(x)).$$

From the definition of $F_k$, we find

$$A_{F_k}(s; a, Nc) = -24 \sum_{d|N} n_d A_{E_k}(s; a, Nc/d). \tag{90}$$

We are now ready to evaluate the integrals appearing in (81). Let $0 \leq n \leq k-2$. Using the change of variables $z = it + a/(Nc)$, we find

(91)

$$\begin{aligned}\int_{a/Nc}^{i\infty} z^n F_k(z)\, \mathrm{d}z &= \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc}\right)^{n-\ell} (Nc)^{-\ell} A_{F_k}(\ell + 1; a, Nc) \\ &= -24 \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc}\right)^{n-\ell} (Nc)^{-\ell} \sum_{d|N} n_d A_{E_k}\left(\ell + 1; a, \frac{Nc}{d}\right).\end{aligned}$$

In view of (89), the real part of (91) is equal to

$$12 \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc}\right)^{n-\ell} (-1)^\ell \sum_{d|N} n_d d^{-\ell} D_{k-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right). \tag{92}$$

As we now check, equation (92) remains valid for $k = 2$. In this case the desired formula simplifies to

$$\int_{a/Nc}^{i\infty} F_2(z)\, \mathrm{d}z = 12 \sum_{d|N} n_d D_{1,1}\left(\frac{a}{Nc/d}\right) = 12 D^\delta\left(\frac{a}{Nc}\right),$$

which is nothing but equation (43).

4.5. *Measures on $\mathbb{Z}_p \times \mathbb{Z}_p$.* Let $\xi = \frac{a}{Nc} \in \Gamma\infty$, and assume that $p$ does not divide $c$. In this section we prove the following crucial lemma.

Lemma 4.10. *Let $\xi \in \Gamma\infty$ have denominator not divisible by $p$. There exists a unique $\mathbb{Z}_p$-valued measure $\nu_\xi$ on $\mathbb{Z}_p \times \mathbb{Z}_p$ such that*

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} h(x, y)\, \mathrm{d}\nu_\xi(x, y) = \mathrm{Re}\left((1 - p^{k-2}) \int_{\xi}^{i\infty} h(z, 1) F_k(z)\, \mathrm{d}z\right) \tag{93}$$

*for every homogeneous polynomial $h(x, y) \in \mathbb{Z}[x, y]$ of degree $k - 2$.*

Equation (93) is equivalent to the statement that

(94)
$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} x^n y^m \,\mathrm{d}\nu_\xi(x,y) = \mathrm{Re}\left((1-p^{n+m})\int_{a/Nc}^{i\infty} z^n F_{n+m+2}(z)\,\mathrm{d}z\right)$$
$$= 12(1-p^{n+m})\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\frac{a}{Nc}\right)^{n-\ell}(-1)^\ell\sum_{d|N} n_d d^{-\ell} D_{n+m-\ell+1,\ell+1}\left(\frac{a}{Nc/d}\right)$$

for all integers $n, m \geq 0$. Denote the last expression appearing in equation (94) by $I_{n,m}\in\mathbb{Q}$. Our key tool in showing the existence and uniqueness of $\nu_\xi$ is the following result, which is the two-variable version of a classical theorem of Mahler (see Theorem 3.3.1 of [Hida]).

Lemma 4.11. *Let $b_{n,m}\in\mathbb{Z}_p$ be constants indexed by integers $n, m\geq 0$. There exists a unique measure $\nu$ on $\mathbb{Z}_p\times\mathbb{Z}_p$ such that*
$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p}\binom{x}{n}\binom{y}{m}\mathrm{d}\nu(x,y) = b_{n,m}.$$

Thus to prove Lemma 4.10, we must show that the rational numbers
$$J_{n,m} := \sum_{i=0}^{n}\sum_{j=0}^{m} c_{n,i}c_{m,j}I_{i,j}$$
lie in $\mathbb{Z}_p$, where the rational numbers $c_{n,i}$ are defined by the equation
$$\binom{x}{n} = \sum_{i=0}^{n} c_{n,i}x^i.$$
Our proof of this fact will follow the proof of the existence of $p$-adic Dirichlet $L$-functions, as in Section 3.4 of [Hida].

Consider the rightmost term appearing in the definition (94) of $I_{n,m}$ (here $k = n+m+2$):

(95)
$$d^{-\ell}D_{k-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right) = d^{-\ell}\left(\frac{Nc}{d}\right)^{k-\ell-2}\sum_{h=1}^{Nc/d}\frac{\tilde{B}_{k-\ell-1}(\frac{h}{Nc/d})}{k-\ell-1}\frac{\tilde{B}_{\ell+1}(\frac{ha}{Nc/d})}{\ell+1}$$
$$= \left(\frac{Nc}{d}\right)^{k-\ell-2}\sum_{h=1}^{Nc}\frac{\tilde{B}_{k-\ell-1}(\frac{h}{Nc/d})}{k-\ell-1}\frac{\tilde{B}_{\ell+1}(\frac{ha}{Nc})}{\ell+1},$$

where (95) follows from the distribution relation for Bernoulli numbers. For each $h = 1,\ldots,Nc$, write $\theta = \{ha/Nc\}$. Let $x$ be a formal variable and write $u = e^x$. Then the Bernoulli numbers are given by the power series

(96)
$$\frac{u^\theta}{u-1} - \frac{1}{x} + F_h = \sum_{s=0}^{\infty}\frac{\tilde{B}_{s+1}(\frac{ha}{Nc})}{(s+1)!}x^s,$$

where $F_h = 1/2$ when $h = Nc$ and $F_h = 0$ otherwise (the error term $F_h$ deals with the discrepancy between $\tilde{B}_1(0)$ and $B_1(0)$). Similarly, write $\beta_d = \{hd/Nc\}$, let $y$ be a formal variable and write $v = e^y$; we then have

$$\sum_{d|N} n_d \frac{v^{\beta_d/d}}{v^{1/d}-1} + G_h = \sum_{t=0}^{\infty} \sum_{d|N} n_d \frac{\tilde{B}_{t+1}(\frac{hd}{Nc})}{(t+1)!} \left(\frac{y}{d}\right)^t, \tag{97}$$

where $G_h$ is a constant in $\frac{1}{2}\mathbb{Z}$. Multiplying (96) and (97), and summing over all $h$, we obtain

$$H(u,v) := \sum_{h=1}^{Nc} \left( \sum_{d|N} n_d \frac{v^{\beta_d/d}}{v^{1/d}-1} + G_h \right) \left( \frac{u^\theta}{u-1} + F_h \right) \tag{98}$$

$$= \sum_{s,t=0}^{\infty} \sum_{h=1}^{Nc} \sum_{d|N} n_d \frac{\tilde{B}_{s+1}(\frac{ha}{Nc})}{(s+1)!} \frac{\tilde{B}_{t+1}(\frac{hd}{Nc})}{(t+1)!} x^s \left(\frac{y}{d}\right)^t. \tag{99}$$

Note that the $-1/x$ terms from (96) have dropped out in (98) since summing (97) over all $h$ gives the value 0. By the same reasoning, we may replace $\frac{u^\theta}{u-1}$ in equation (98) defining $H(u,v)$ by $\frac{u^\theta-1}{u-1}$ (this will be useful in later computations). Recalling that $u = e^x$ and $v = e^y$, we define the commuting differential operators

$$D_u = u\frac{\partial}{\partial u} = \frac{\partial}{\partial x} \text{ and } D_v = v\frac{\partial}{\partial v} = \frac{\partial}{\partial y}.$$

Using (95) and (99), we then have

$$(1-p^{n+m}) \sum_{d|N} n_d d^{-\ell} \frac{D_{n+m-\ell+1,\ell+1}\left(\frac{a}{Nc/d}\right)}{(n+m-\ell+1)(\ell+1)}$$
$$= (Nc)^{n+m-\ell} (D_u^\ell D_v^{n+m-\ell} H^*(u,v))|_{(u,v)=(1,1)},$$

where

$$H^*(u,v) := H(u,v) - H(u^p, v^p).$$

We thus find that

(100)
$$\begin{aligned} I_{n,m} &= 12 \sum_{\ell=0}^{n} \binom{n}{\ell} \left(\frac{a}{Nc}\right)^{n-\ell} (-1)^\ell (Nc)^{n+m-\ell} (D_u^\ell D_v^{n+m-\ell} H^*(u,v))|_{(1,1)} \\ &= (NcD_v)^m (aD_v - D_u)^n (12H^*(u,v))|_{(1,1)}. \end{aligned}$$

If we define a change of variables $(u,v) = (z^{-1}, w^{Nc}z^a)$, then $D_w = NcD_u$ and $D_z = aD_u - D_v$. Hence we obtain

$$J_{n,m} = \binom{D_w}{m} \binom{D_z}{n} (12H^*(u,v))|_{(1,1)}.$$

The following lemma will allow us to prove that these rational numbers lie in $\mathbb{Z}_p$.

LEMMA 4.12. *Consider the subset $R$ of $\mathbb{Z}_p(u^{1/Nc}, v^{1/Nc})$ defined by*

$$R := \left\{ \frac{P}{Q} \text{ where } P, Q \in \mathbb{Z}_p[u^{1/Nc}, v^{1/Nc}] \text{ and } Q(1,1) \in \mathbb{Z}_p^\times \right\}.$$

*Now $R$ is a ring stable under the operators $\binom{D_w}{m}$ and $\binom{D_z}{n}$.*

*Proof.* The proof of this proposition follows exactly as in Lemma 3.4.2 of [Hida], except for the subtlety that we must check that $\mathbb{Z}_p[u^{1/Nc}, v^{1/Nc}]$ is stable under the given differential operators; for this it suffices to check that for example

$$\binom{D_z}{n}(v^{1/Nc}) = \frac{z^n}{n!}\frac{\partial^n}{\partial z^n}(wz^{a/Nc}) = \binom{a/Nc}{n} wz^{a/Nc},$$

which lies in $\mathbb{Z}_p[u^{1/Nc}, v^{1/Nc}]$ because $p$ does not divide $Nc$; similarly for the other cases. □

Thus to prove that $J_{n,m} \in \mathbb{Z}_p$, it suffices to prove that $H^*(u,v)$ is an element of $R$, and for this it suffices to prove that $H(u,v) \in R$. Writing

$$\Psi_d(v) := 1 + v^{1/d} + \cdots + v^{(d-1)/d},$$

we have

$$\sum_{d|N} n_d \frac{v^{\beta_d/d}}{v^{1/d}-1} = \frac{1}{v-1}\sum_{d|N} n_d v^{\beta_d/d}\Psi_d(v) = \frac{1}{\Psi_{Nc}(v)} \cdot \frac{\sum_{d|N} n_d v^{\beta_d/d}\Psi_d(v)}{v^{1/Nc}-1}. \tag{101}$$

Since the numerator of the rightmost term in (101) is a polynomial in $v^{1/Nc}$ which vanishes when $v^{1/Nc} = 1$, the rightmost term itself is a polynomial in $v^{1/Nc}$. Since we are assuming that $p$ does not divide $Nc$, equation (101) then implies that

$$\sum_{d|N} n_d \frac{v^{\beta_d/d}}{v^{1/d}-1} \in R.$$

Similarly one shows that $\frac{u^\theta - 1}{u-1} \in R$, and it follows that $H(u,v) \in R$. This concludes the proof of Lemma 4.10.

4.6. *A partial modular symbol of measures on $\mathbb{Z}_p \times \mathbb{Z}_p$.* In this section, we use the measures $\nu_\xi$ to construct a partial modular symbol of measures on $\mathbb{Z}_p \times \mathbb{Z}_p$ encoding the periods of $F_k$. Note that $\mathbb{Z}_p \times \mathbb{Z}_p$ is stable under the action of $\Gamma_0(N)$.

LEMMA 4.13. *There exists a unique* $\Gamma_0(N)$*-invariant partial modular symbol* $\nu$ *of* $\mathbb{Z}_p$*-valued measures on* $\mathbb{Z}_p \times \mathbb{Z}_p$ *such that*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(x,y)\,\mathrm{d}\nu\{r\to s\}(x,y) = \mathrm{Re}\left((1-p^{k-2})\int_r^s h(z,1)F_k(z)\,\mathrm{d}z\right) \tag{102}$$

*for* $r,s\in\Gamma\infty$, *and every homogeneous polynomial* $h(x,y)\in\mathbb{Z}[x,y]$ *of degree* $k-2$.

*Proof.* Uniqueness follows from Lemma 4.11; we must show existence. Let $M$ denote the $\Gamma$-module of degree zero divisors on the set $\Gamma\infty$. Let $M'\subset M$ be the set of divisors $m$ for which there exists a $\mathbb{Z}_p$-valued measure $\nu\{m\}$ on $\mathbb{Z}_p\times\mathbb{Z}_p$ such that

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(x,y)\,\mathrm{d}\nu\{m\}(x,y) = \mathrm{Re}\left((1-p^{k-2})\int_m h(z,1)F_k(z)\,\mathrm{d}z\right).$$

(Here $\int_m$ is defined by $\int_{[x]-[y]} := \int_x^y$, and extended by linearity.) We must show that $M'=M$.

It is clear that $M'$ is a subgroup of $M$. We will show that $M'$ is a $\Gamma_0(N)$-stable submodule. Let $m\in M'$ and $\gamma=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\Gamma_0(N)$ ; for compact open $U\subset\mathbb{Z}_p\times\mathbb{Z}_p$ define

$$\nu\{\gamma m\}(U) := \nu\{m\}(\gamma^{-1}U).$$

Define a right action of $\Gamma_0(N)$ on the space of polynomials in two variables by

$$h|_\gamma(x,y) = h(Ax+By, Cx+Dy).$$

We calculate

$$\begin{aligned}\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(u,v)\,\mathrm{d}\nu\{\gamma m\}(u,v) &= \int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(u,v)\,\mathrm{d}\nu\{m\}(\gamma^{-1}(u,v))\\ &= \int_{\mathbb{Z}_p\times\mathbb{Z}_p} h|_\gamma(x,y)\,\mathrm{d}\nu\{m\}(x,y)\\ &= \mathrm{Re}\left((1-p^{k-2})\int_m h|_\gamma(z,1)F_k(z)\,\mathrm{d}z\right)\\ &= \mathrm{Re}\left((1-p^{k-2})\int_{\gamma m} h(u,1)F_k(u)\,\mathrm{d}u\right),\end{aligned} \tag{103}$$

where equation (103) uses the change of variables $u=\gamma z$ and the fact that $F_k|_{\gamma^{-1}} = F_k$. Therefore, $M'$ is a $\Gamma_0(N)$-stable submodule of $M$. Lemma 4.10 shows that $[a/Nc]-[\infty]\in M'$ when $p$ does not divide $c$. Since the $\Gamma_0(N)$-module generated by these elements is all of $M$, we indeed have

$M' = M$. Furthermore, the $\Gamma_0(N)$-invariance of $\nu$ follows from uniqueness and the calculation of (103) above. □

4.7. *From $\mathbb{Z}_p \times \mathbb{Z}_p$ to $\mathbb{X}$.* In this section we show that the measures $\nu\{x \to y\}$ of Lemma 4.13 are supported on the set $\mathbb{X} \subset \mathbb{Z}_p \times \mathbb{Z}_p$ of primitive vectors.

LEMMA 4.14. *Let $r, s \in \Gamma\infty$. Then,*

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} h(x,y)\,\mathrm{d}\nu\{r \to s\}(x,y) = \mathrm{Re}\left(\int_r^s h(z,1)F_k^*(z)\,\mathrm{d}z\right)$$

*for every homogeneous polynomial $h(x,y) \in \mathbb{Z}[x,y]$ of degree $k-2$.*

*Proof.* The characteristic function of the open set $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ is $\lim_{j\to\infty} y^{(p-1)p^j}$. For notational simplicity, let $g = (p-1)p^j$ throughout the remainder of this section. Then for $n, m \geq 0$ and $k = n+m+2$, we have

(104)

$$\begin{aligned}
&\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} x^n y^m \,\mathrm{d}\nu\left\{\frac{a}{Nc} \to \infty\right\}(x,y) \\
&\quad = \lim_{j\to\infty} \int_{\mathbb{Z}_p \times \mathbb{Z}_p} x^n y^{m+g}\,\mathrm{d}\nu\left\{\frac{a}{Nc} \to \infty\right\}(x,y) \\
&\quad = \lim_{j\to\infty} 12(1-p^{k+g-2}) \sum_{\ell=0}^{n} \binom{n}{\ell}\left(\frac{a}{Nc}\right)^{n-\ell}(-1)^\ell \\
&\qquad \times \sum_{d|N} \frac{n_d}{d^\ell} D_{k+g-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right) \\
&\quad = 12\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\frac{a}{Nc}\right)^{n-\ell}(-1)^\ell \sum_{d|N}\frac{n_d}{d^\ell}\lim_{j\to\infty} D_{k+g-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right).
\end{aligned}$$

Meanwhile we calculate

(105)

$$\begin{aligned}
&\mathrm{Re}\left(\int_{\frac{a}{Nc}}^{i\infty} z^n F_k^*(z)\,\mathrm{d}z\right) \\
&\quad = \mathrm{Re}\left(\int_{\frac{a}{Nc}}^{i\infty} z^n F_k(z)\,\mathrm{d}z - p^{k-n-2}\int_{\frac{pa}{Nc}}^{i\infty} z^n F_k(z)\,\mathrm{d}z\right) \\
&\quad = 12\sum_{\ell=0}^{n}\binom{n}{\ell}\left(\frac{a}{Nc}\right)^{n-\ell}(-1)^\ell \\
&\qquad \times \sum_{d|N}\frac{n_d}{d^\ell}\left[D_{k-\ell-1,\ell+1}\left(\frac{a}{Nc/d}\right) - p^{k-\ell-2}D_{k-\ell-1,\ell+1}\left(\frac{pa}{Nc/d}\right)\right].
\end{aligned}$$

The following lemma implies that (104) and (105) are equal, and finishes the proof. □

LEMMA 4.15. *Let $s, t \geq 0$. For any rational number $x$, in $\mathbb{Q}_p$:*

$$\lim_{j\to\infty} \tilde{D}_{s+g,t}(x) = \tilde{D}_{s,t}(x) - p^{s-1}\tilde{D}_{s,t}(px). \tag{106}$$

*Proof.* This essentially follows from the generalized Kummer congruences for Bernoulli polynomials. Let $x = a/c$ and assume first that $p$ does not divide $c$. Let $b$ denote an integer such that $abp \equiv 1 \pmod{c}$. Note that

$$\tilde{D}_{s,t}(a/c) = c^{s-1}\sum_{\ell=1}^{c} \tilde{B}_s(\ell bp/c)\tilde{B}_t(\ell/c). \tag{107}$$

Similarly

$$\tilde{D}_{s+g,t}(a/c) = c^{s+g-1}\sum_{\ell=1}^{c} \tilde{B}_{s+g}(\ell bp/c)\tilde{B}_t(\ell/c)$$

and

$$\tilde{D}_{s,t}(pa/c) = c^{s-1}\sum_{\ell=1}^{c} \tilde{B}_s(\ell b/c)\tilde{B}_t(\ell/c).$$

Write $y = \{\ell bp/c\}$ and $y' = \{\ell b/c\}$. Since $c^g \to 1$, it suffices to prove that

$$\lim_{j\to\infty} B_{s+g}(y) = B_s(y) - p^{s-1}B_s(y').$$

For $s > 0$, this follows from the proof of Theorem 3.2 of [You], which applies for our purposes even in the case $s \equiv 0 \pmod{p-1}$. For $s = 0$, the desired equality follows from the fact that the $p$-adic $L$-function $L_p(s,\chi)$ for a Dirichlet character $\chi$ is analytic at $s = 1$ unless $\chi = 1$, in which case $L_p$ has a simple pole with residue $1 - 1/p$. This completes the proof for the case $x \in \mathbb{Z}_p$.

We now handle the case $x \notin \mathbb{Z}_p$. From equation (107), one sees that

$$\tilde{D}_{s,t}(a/c) = c^{s-t}\tilde{D}_{t,s}(bp/c).$$

Thus the result proved above is that

$$\lim_{j\to\infty} \tilde{D}_{t,s+g}(bp/c) = \tilde{D}_{t,s}(bp/c) - p^{s-1}\tilde{D}_{t,s}(b/c) \tag{108}$$

whenever $p \nmid c$. By switching indices in a similar fashion, equation (106) for $x = a/bp$ becomes

$$\lim_{j\to\infty} (bp)^{s+g-t}\tilde{D}_{t,s+g}(c/bp) = (bp)^{s-t}\tilde{D}_{t,s}(c/bp) - p^{s-1}b^{s-t}\tilde{D}_{t,s}(c/b) \tag{109}$$

where $ac \equiv 1 \pmod{bp}$. We will reduce equation (109) to equation (108) by means of the reciprocity law for these generalized Dedekind sums, given in

Theorem 2 of [Hal]. When $b > 0$, the reciprocity law states

$$b^{s-t}\tilde{D}_{t,s}(c/b) = \mathrm{sign}(c)\sum_{\ell=0}^{t}\frac{s}{s+\ell}\binom{t}{\ell}(-1)^{s+\ell}b^{-\ell}c^{s-t+\ell}\tilde{D}_{t-\ell,s+\ell}(b/c) \tag{110}$$

$$+\sum_{\sigma=0}^{s+t}\frac{\binom{s+t-\sigma-1}{t-1}\binom{s+t}{\sigma}}{\binom{s+t}{t}}(-1)^{\sigma}b^{\sigma-t}c^{s-\sigma}\tilde{D}_{t+s-\sigma,\sigma}(0) \tag{111}$$

$$+\begin{cases} -\mathrm{sign}(c)/4 & \text{if } s=t=1 \\ 0 & \text{otherwise.}\end{cases}$$

Note that the sum in (110) is taken to be 0 if $s = 0$. We will call the terms in the sum on line (110) "type I" terms and those on line (111) "type II" terms. Using the Dedekind reciprocity law on each of the terms in (109), one easily checks that the desired limit holds for the type I terms by (108). The same is true for each of the type II terms with $\sigma = 0, \ldots, s+t$. To conclude the proof, one checks that each of the type II terms for $\sigma = s+t+1, \ldots, s+t+g$ arising from the reciprocity law for $(bp)^{s+g-t}\tilde{D}_{t,s+g}(c/bp)$ has $\mathrm{ord}_p$ greater than $\mathrm{ord}_p(g)$ minus some constant depending only on $s$ and $t$. Thus in the limit, the sum of these terms vanishes. □

We can now prove:

Lemma 4.16. *The measures $\nu\{r \to s\}$ are supported on $\mathbb{X}$.*

*Proof.* Let $\gamma \in \Gamma_0(N)$. As in (103) above, we calculate for a homogeneous polynomial $h(x,y)$ of degree $k-2$,

$$\int_{\gamma(\mathbb{Z}_p\times\mathbb{Z}_p^\times)} h(x,y)\,\mathrm{d}\nu\{r\to s\}(x,y) = \mathrm{Re}\left(\int_r^s h(z,1)F_k^*|_{\gamma^{-1}}(z)\,\mathrm{d}z\right). \tag{112}$$

Let $\{\gamma_i\}_{i=1}^{p+1}$ be a set of left coset representatives for $\Gamma_0(N)/\Gamma_0(Np)$. Then

$$\bigcup_{i=1}^{p+1}\gamma_i(\mathbb{Z}_p\times\mathbb{Z}_p^\times)$$

is a degree $p$ cover of $\mathbb{X}$. Hence from (112) we find that

$$p\int_{\mathbb{X}} h(x,y)\,\mathrm{d}\nu\{r\to s\}(x,y) = \sum_{i=1}^{p+1}\mathrm{Re}\left(\int_r^s h(z,1)F_k^*|_{\gamma_i^{-1}}(z)\,\mathrm{d}z\right). \tag{113}$$

Now

$$\begin{aligned}\sum_{i=1}^{p+1}F_k^*|_{\gamma_i^{-1}} &= \sum_{i=1}^{p+1}\left(F_k|_{\gamma_i^{-1}} - p^{k-2}F_k|_{P\gamma_i^{-1}}\right)\\ &= (p+1)F_k - T_pF_k = (p-p^{k-1})F_k,\end{aligned}$$

since $F_k$ is evidently an eigenform for $T_p$ with eigenvalue $1+p^{k-1}$. Thus (113) becomes

$$\int_{\mathbb{X}} h(x,y)\,\mathrm{d}\nu\{r\to s\}(x,y) = \mathrm{Re}\left((1-p^{k-2})\int_r^s h(z,1)F_k(z)\,\mathrm{d}z\right).$$

Therefore, the integral on $\mathbb{X}$ of any polynomial $h(x,y)$ equals the integral on $\mathbb{Z}_p\times\mathbb{Z}_p$ of $h(x,y)$; this implies that the measure $\nu\{r\to s\}$ is supported on $\mathbb{X}$. □

4.8. *The measures $\mu$ and $\Gamma$-invariance.* The compact open set $\mathbb{X}$ is a fundamental domain for the action of multiplication by $p$ on $\mathbb{Q}_p^2-\{0\}$. Hence if we define for compact open $U\subset\mathbb{X}$:

$$\mu\{r\to s\}(U) := \nu\{r\to s\}(U),$$

then $\mu$ extends uniquely to a $\Gamma_0(N)$-invariant partial modular symbol of $\mathbb{Z}_p$-valued measures on $\mathbb{Q}_p^2-\{0\}$ which is invariant under the action of multiplication by $p$:

$$\mu\{r\to s\}(pU) = \mu\{r\to s\}(U)$$

for all compact open $U\subset\mathbb{Q}_p^2-\{0\}$. Lemmas 4.13, 4.14, and 4.16 show that $\mu$ satisfies properties (1) and (4) of Theorem 4.2. Furthermore, property (3) is satisfied by construction. Thus to complete the proof of Theorem 4.2, it remains to show that the partial modular symbol of measures $\mu$ is $\Gamma$-invariant.

Lemma 4.17. *The partial modular symbol $\mu$ is invariant under $\tilde{\Gamma}$.*

*Proof.* Since $\tilde{\Gamma}$ is generated by $\Gamma_0(N)$ and $P=\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}$, it suffices to show that $\mu$ is invariant for the action of $P$. For a homogeneous polynomial $h(x,y)$ of degree $k-2$, we have

(114)
$$\int_{\mathbb{X}} h(x,y)\,\mathrm{d}\mu\{P^{-1}r\to P^{-1}s\}(P^{-1}(x,y)) = \int_{P^{-1}\mathbb{X}} h(pu,v)\,\mathrm{d}\mu\left\{\frac{r}{p}\to\frac{s}{p}\right\}(u,v).$$

Writing $P^{-1}\mathbb{X}$ as a disjoint union

$$P^{-1}\mathbb{X} = (\mathbb{Z}_p\times\mathbb{Z}_p^\times)\bigsqcup\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix}^{-1}(\mathbb{Z}_p^\times\times p\mathbb{Z}_p)$$

and using the invariance of $\mu$ under multiplication by $p$, we see that (114) becomes

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(pu,v)\,\mathrm{d}\mu\left\{\frac{r}{p}\to\frac{s}{p}\right\}(u,v) + \int_{\mathbb{Z}_p^\times\times p\mathbb{Z}_p} h(u,v/p)\,\mathrm{d}\mu\left\{\frac{r}{p}\to\frac{s}{p}\right\}(u,v).$$

By the homogeneity of $h$, one simplifies the above expression:

$$\begin{aligned} p^{2-k} \int_{\mathbb{X}} h(pu,v)\,\mathrm{d}\mu\left\{\frac{r}{p}\to\frac{s}{p}\right\}(u,v) \\ +(1-p^{2-k}) \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(pu,v)\,\mathrm{d}\mu\left\{\frac{r}{p}\to\frac{s}{p}\right\}(u,v) \\ = \mathrm{Re}\Bigg(p^{2-k}(1-p^{k-2})\int_{\frac{r}{p}}^{\frac{s}{p}} h(pz,1)F_k(z)\,\mathrm{d}z \\ + (1-p^{2-k})\int_{\frac{r}{p}}^{\frac{s}{p}} h(pz,1)F_k^*(z)\,\mathrm{d}z\Bigg) \end{aligned}$$

$$= \mathrm{Re}\left((p^{2-k}-1)\int_{\frac{r}{p}}^{\frac{s}{p}} h(pz,1)p^{k-1}F_k(pz)\,\mathrm{d}z\right) \tag{115}$$

$$= \mathrm{Re}\left((1-p^{k-2})\int_r^s h(u,1)F_k(u)\,\mathrm{d}u\right), \tag{116}$$

where (115) uses the definition of $F_k^*$ and (116) uses the change of variables $u = pz$. Since this equals the integral over $\mathbb{X}$ of $h(x,y)$ against the measure $\mu\{r\to s\}$, we find that $\mu$ is indeed invariant for the action of $P$. □

McGill University, Montreal, Quebec, CA
*E-mail address*: darmon@math.mcgill.ca
*URL*: http://www.math.mcgill.ca/~darmon/

Harvard University, Cambridge, MA
*E-mail address*: dasgupta@math.harvard.edu
*URL*: http://www.math.harvard.edu/~dasgupta/

## References

[Ap] T. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Second edition, *Grad. Texts in Math.* **41**, Springer-Verlag, New York, 1990.

[BD1] M. Bertolini and H. Darmon, The $p$-adic $L$-functions of modular elliptic curves, in *Mathematics Unlimited*—2001 *and Beyond*, 109–170, Springer-Verlag, New York, 2001.

[BDG] M. Bertolini, H. Darmon, and P. Green, Periods and points attached to quadratic algebras, in *Heegner points and Rankin L-series* (H. Darmon and S. Zhang, eds.), *Math. Sci. Res. Inst. Publ.* **49**, 323–367, Cambridge Univ. Press, Cambridge, 2004.

[CS] J. Coates and W. Sinnott, On $p$-adic $L$-functions over real quadratic fields, *Invent. Math.* **25** (1974), 253–279.

[Dar1] H. Darmon, Integration on $\mathcal{H}_p\times\mathcal{H}$ and arithmetic applications, *Ann. of Math.* **154** (2001), 589–639.

[Dar2] ———, *Rational Points on Modular Elliptic Curves*, *CBMS Regional Conference Series in Mathematics* **101**, Published for the Conference Board of the Mathematical Sciences, Washington, D.C., A. M. S., Providence, RI, 2004.

[Das1] S. Dasgupta, Gross-Stark units, Stark-Heegner points, and class fields of real quadratic fields, Ph.D. thesis, Univ. of California-Berkeley, 2004.

[Das2] ———, Computations of elliptic units for real quadratic fields, *Canadian J. Math.*, to appear.

[Das3] ———, Stark-Heegner points on modular Jacobians, *Ann. Sci. École Norm. Sup.* **38** (2005), 427–469.

[deS1] E. de Shalit, On certain Galois representations related to the modular curve $X_1(p)$, *Compositio Math.* **95** (1995), 69–100.

[deS2] ———, $p$-adic periods and modular symbols of elliptic curves of prime conductor, *Invent. Math.* **121** (1995), 225–255.

[Fuk] S. Fukuhara, Generalized Dedekind symbols associated with the Eisenstein series, *Proc. Amer. Math. Soc.* **127** (1999), 2561–2568.

[Gr1] B. H. Gross, $p$-adic $L$-series at $s = 0$, *J. Fac. Sci. Univ. Tokyo Sect.* IA *Math.* **28** (1981), 979–994 (1982).

[Gr2] ———, On the values of abelian $L$-functions at $s = 0$, *J. Fac. Sci. Univ. Tokyo Sect.* IA *Math.* **35** (1988), 177–197.

[GS] R. Greenberg and G. Stevens, $p$-adic $L$-functions and $p$-adic periods of modular forms, *Invent. Math.* **111** (1993), 407–447.

[GvdP] L. Gerritzen and M. van der Put, in *Shottky Groups and Mumford Curves*, *Lecture Notes in Math.* **817**, Springer-Verlag, New York, 1980.

[Hal] U. Halbritter, Some new reciprocity formulas for generalized Dedekind sums, *Results Math.* **8** (1985), 21–46.

[Hida] H. Hida, *Elementary Theory of L-functions and Eisenstein Series*, *London Math. Society Student Texts* **26**, Cambridge Univ. Press, Cambridge, 1993.

[KL] D. S. Kubert and S. Lang, *Modular Units*, *Grundlehren der Mathematischen Wissenschaften* **244**, Springer-Verlag, New York, 1981.

[La] S. Lang, *Algebraic Number Theory*, Second edition, *Graduate Texts in Math.* **110** Springer-Verlag, New York, 1994.

[Man] J. I. Manin, Parabolic points and zeta functions of modular curves, *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 19–66.

[Maz] B. Mazur, On the arithmetic of special values of $L$ functions, *Invent. Math.* **55** (1979), 207–240.

[Me] J. Mennicke, On Ihara's modular group, *Invent. Math.* **4** (1967), 202–228.

[Se1] J-P. Serre, *Trees*, Springer-Verlag, New York, 1980.

[Se2] ———, Le problème des groupes de congruence pour $\mathrm{SL}_2$, *Ann. of Math.* **92** (1970), 489–527.

[Sie1] C. L. Siegel, *Advanced Analytic Number Theory*, Second edition, *Tata Institute of Fundamental Research Studies in Math.* **9**, Tata Institute of Fundamental Research, Bombay, 1980.

[Sie2] ———, Bernoullische Polynome und quadratische Zahlkörper, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl.* II **1968** (1968), 7–38.

[Ste] G. Stevens, The Eisenstein measure and real quadratic fields, in *Théorie des Nombres* (Quebec, PQ, 1987), 887–927, de Gruyter, Berlin, 1989.

[Wi] A. Wiles, On a conjecture of Brumer, *Ann. of Math.* **131** (1990), 555–565.

[You] P. T. Young, Kummer congruences for values of Bernoulli and Euler polynomials, *Acta Arith.* **48** (2001), 277–288.

[Za] D. Zagier, A Kronecker limit formula for real quadratic fields, *Math. Ann.* **213** (1975), 153–184.

(Received December 1, 2003)