Vol. 98, No. 2 DUKE MATHEMATICAL JOURNAL © 1999

p-ADIC PERIODS, p-ADIC L-FUNCTIONS, AND THE p-ADIC UNIFORMIZATION OF SHIMURA CURVES

MASSIMO BERTOLINI AND HENRI DARMON

Contents

1. Introduction. Let $E/\mathbb{Q}$ be a modular elliptic curve of conductor N, and let p be a prime of split multiplicative reduction for E. Write $\mathbb{C}_p$ for a fixed completion of an algebraic closure of $\mathbb{Q}_p$. Tate's theory of p-adic uniformization of elliptic curves yields a rigid-analytic, $\mathrm{Gal}(\mathbb{C}_p/\mathbb{Q}_p)$-equivariant uniformization of the $\mathbb{C}_p$-points of E:

$$0 \to q^{\mathbb{Z}} \to \mathbb{C}_p^\times \xrightarrow{\Phi_{\mathrm{Tate}}} E(\mathbb{C}_p) \to 0, \tag{1}$$

where $q \in p\mathbb{Z}_p$ is the p-adic period of E.

Mazur, Tate, and Teitelbaum conjectured in [MTT] that the cyclotomic p-adic L-function of $E/\mathbb{Q}$ vanishes at the central point to order one greater than that of its classical counterpart. Furthermore, they proposed a formula for the leading coefficient of such a p-adic L-function. In the special case where the analytic rank of $E(\mathbb{Q})$ is zero, they predicted that the ratio of the special value of the first derivative of the cyclotomic p-adic L-function and the algebraic part of the special value of the complex L-function of $E/\mathbb{Q}$ is equal to the quantity

$$\frac{\log_p(q)}{\mathrm{ord}_p(q)}$$

Received 9 June 1997. Revision received 20 February 1998.
1991 *Mathematics Subject Classification*. Primary 11G05, 11G18; Secondary 11G40.
Bertolini partially supported by Gruppo Nazionale per le Strutture Algebriche e Geometriche e loro Applicazioni (Consiglio Nazionale delle Ricerche); Ministero dell' Università e della Ricerca Scientifica e Tecnologica, progetto nazionale "Geometria algebrica"; and by Human Capital and Mobility Programme of the European Community, under contract ERBCH RXCT940557.
Darmon partially supported by grants from Fonds pour la Formation de Chercheurs et l'Aide à la Recherche and the Natural Sciences and Engineering Research Council, and by an Alfred P. Sloan research award.

(where $\log_p$ is Iwasawa's cyclotomic logarithm), which is defined purely in terms of the p-adic uniformization of E. Greenberg and Stevens [GS] gave a proof of this special case. See also the work of Boichut [Boi] in the case of analytic rank one.

The article [BD1] formulates an analogue of the conjectures of [MTT] in which the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ is replaced by the anticyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field K. When p is split in K and the sign of the functional equation of $L(E/K, s)$ is $+1$, this conjecture relates the first derivative of the anticyclotomic p-adic L-function of E to the anticyclotomic logarithm of the p-adic period of E. The present paper supplies a proof of this conjecture. Our proof is based on the theory of p-adic uniformization of Shimura curves.

More precisely, assume that K is an imaginary quadratic field with $(\mathrm{disc}(K), N) = 1$ such that

(i) p is split in K;
(ii) E is semistable at the rational primes that divide N and are inert in K;
(iii) the number of these rational primes is odd.

The complex L-function $L(E/K, s)$ of E over K has a functional equation and an analytic continuation to the whole complex plane. Under our assumptions, the sign of the functional equation of $L(E/K, s)$ is $+1$ (cf. [GZ, p. 71]), and hence $L(E/K, s)$ vanishes to even order at $s = 1$.

Fix a positive integer c prime to N, and let $\mathcal{O}$ be the order of K of conductor c. Let H_n be the ring class field of K of conductor cp^n, with $n \geq 0$, and let H_∞ be the union of the H_n. By class field theory, the Galois group $\mathrm{Gal}(H_\infty/H_0)$ is identified with $\mathcal{O}^\times \backslash (\mathcal{O}_K \otimes \mathbb{Z}_p)^\times / \mathbb{Z}_p^\times \simeq \mathbb{Z}_p \times \mathbb{Z}/((p-1)/u)\mathbb{Z}$, with $u := (1/2)\#\mathcal{O}^\times$. Moreover, $\mathrm{Gal}(H_0/K)$ is identified with the Picard group $\mathrm{Pic}(\mathcal{O})$. Set

$$\mathbf{G}_n := \mathrm{Gal}(H_n/K), \qquad \mathbf{G}_\infty := \mathrm{Gal}(H_\infty/K).$$

Thus, $\mathbf{G}_\infty$ is isomorphic to the product of $\mathbb{Z}_p$ by a finite abelian group. Choose a prime $\mathfrak{p}$ of K above p. Identify $K_\mathfrak{p}$ with $\mathbb{Q}_p$, and let

$$\mathrm{rec}_p : \mathbb{Q}_p^\times \to \mathbf{G}_\infty$$

be the reciprocity map of local class field theory. Define the integral completed group ring of $\mathbf{G}_\infty$ to be

$$\mathbb{Z}[[\mathbf{G}_\infty]] := \varprojlim_n \mathbb{Z}[\mathbf{G}_n],$$

where the inverse limit is taken with respect to the natural projections of group rings.

In Section 3, we recall the construction explained in [BD1, Sec. 2.7] of an element

$$\mathcal{L}_p(E/K) \in \mathbb{Z}[[\mathbf{G}_\infty]]$$

attached to $(E, H_\infty/K)$, which interpolates the special values $L(E/K, \chi, 1)$ of $L(E/K, s)$ twisted by finite-order characters of $\mathbf{G}_\infty$. The construction of this p-adic L-function is based on the ideas of Gross [Gr] and a generalization due to Daghigh [Dag]. We show that $\mathcal{L}_p(E/K)$ belongs to the augmentation ideal I of $\mathbb{Z}[[\mathbf{G}_\infty]]$. Let

$\mathscr{L}'_p(E/K)$ be the natural image of $\mathscr{L}_p(E/K)$ in $I/I^2 = \mathbf{G}_\infty$. The element $\mathscr{L}'_p(E/K)$ should be viewed as the first derivative of $\mathscr{L}_p(E/K)$ at the central point.

Let $f = \sum_{n\geq 1} a_n q^n$ be the newform attached to E, and let

$$\Omega_f := 4\pi^2 \iint_{\mathscr{H}/\Gamma_0(N)} |f(\tau)|^2 \, d\tau \wedge i\, d\bar{\tau}$$

be the Petersson inner product of f with itself. We assume that E is the strong Weil curve for the Shimura curve parametrization defined in Section 4. Set $d := \mathrm{disc}(\mathbb{O})$, and let n_f be the positive integer defined later in this introduction and specified further in Section 2. Our main result (stated in a special case: see Theorem 6.4 for the general statement) is the following.

THEOREM 1.1. *Suppose that $c = 1$. The equality (up to sign)*

$$\mathscr{L}'_p(E/K) = \frac{\mathrm{rec}_p(q)}{\mathrm{ord}_p(q)} \sqrt{L(E/K,1)\Omega_f^{-1} \cdot d^{1/2} u^2 n_f}$$

holds in $I/I^2 \otimes \mathbb{Q}$.

For the convenience of the reader, we now briefly sketch the strategy of the proof of Theorem 1.1.

Write the conductor N of E as pN^+N^-, where N^+ (respectively, N^-) is divisible only by primes that are split (respectively, inert) in K. Under our assumptions, N^- has an odd number of prime factors, and pN^- is squarefree. Denote by B the definite quaternion algebra over $\mathbb{Q}$ of discriminant N^-, and fix an Eichler order R of B of level N^+p. Let Γ be the subgroup of elements of $\mathbb{Q}_p^\times \backslash R[1/p]^\times$ whose norm has even p-adic valuation, and set $\mathcal{N} := \mathrm{Hom}(\Gamma, \mathbb{Z})$. The module $\mathcal{N}$ is a free abelian group and is equipped with the action of a Hecke algebra $\mathbb{T}$ attached to modular forms of level N that are new at N^-p. In Section 2, we also define a canonical free quotient $\mathcal{N}_{\mathrm{sp}}$ of $\mathcal{N}$, which is stable for the action of $\mathbb{T}$ and is such that the image of $\mathbb{T}$ in $\mathrm{End}(\mathcal{N}_{\mathrm{sp}})$ corresponds to modular forms that are split multiplicative at p. Let π_f be the idempotent of $\mathbb{T} \otimes \mathbb{Q}$ associated with f, and let n_f be a positive integer such that $\eta_f := n_f \pi_f$ belongs to $\mathbb{T}$. Denote by $\mathcal{N}^f$ the submodule of $\mathcal{N}$ on which $\mathbb{T}$ acts via the character

$$\phi_f : \mathbb{T} \to \mathbb{Z}, \quad T_n \mapsto a_n$$

defined by f. By the multiplicity-one theorem, the module $\mathcal{N}^f$ is isomorphic to $\mathbb{Z}$. The operator η_f yields a map (denoted in the same way by an abuse of notation) $\eta_f : \mathcal{N} \to \mathcal{N}^f$, which factors through $\mathcal{N}_{\mathrm{sp}}$. We define an element $\mathscr{L}_p(\mathcal{N}_{\mathrm{sp}}/K) \in \mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[[\mathbf{G}_\infty]]$, such that (up to sign)

$$(\eta_f \otimes \mathrm{id})\big(\mathscr{L}_p\big(\mathcal{N}_{\mathrm{sp}}/K\big)\big) = c_p \cdot \mathscr{L}_p(E/K),$$

where $c_p := \mathrm{ord}_p(q)$. We recall that the derivative $\mathscr{L}'_p(E/K)$ of $\mathscr{L}_p(E/K)$ belongs to $\mathcal{N}^f \otimes \mathbf{G}_\infty = \mathbf{G}_\infty$.

On the other hand, the module $\mathcal{N}$ is related to the theory of p-adic uniformization of Shimura curves. Let $\mathcal{B}$ be the indefinite quaternion algebra of discriminant pN^-, and let $\mathcal{R}$ be an Eichler order of $\mathcal{B}$ of level N^+. Write X for the Shimura curve over $\mathbb{Q}$ associated with $\mathcal{R}$ (see Section 4), and write J for the jacobian of X. A theorem of Cherednik (see [C]), combined with the theory of jacobians of Mumford curves (see [GvdP]), yields a rigid-analytic uniformization

$$0 \to \Lambda \to \mathcal{N} \otimes \mathbb{C}_p^\times \xrightarrow{\Phi} J(\mathbb{C}_p) \to 0, \tag{2}$$

where Λ is the lattice of p-adic periods of J. The Tate uniformization (1) is obtained from the sequence (2) by applying the operator η_f to the Hecke modules $\mathcal{N} \otimes \mathbb{C}_p^\times$ and $J(\mathbb{C}_p)$ of (2). In particular, the p-adic period q of E can be viewed as an element of the module $\mathcal{N}^f \otimes \mathbb{C}_p^\times$, and in fact one checks that it belongs to $\mathcal{N}^f \otimes \mathbb{Q}_p^\times = \mathbb{Q}_p^\times$. An explicit calculation of p-adic periods, combined with a formula for $L(E/K, 1)$ given in [Gr] and [Dag], proves Theorem 1.1.

A similar strategy was used in [BD2], when p is inert in K and the sign of the functional equation of $L(E/K, s)$ is -1, to obtain a p-adic analytic construction of a Heegner point in terms of the first derivative of an anticyclotomic p-adic L-function.

It is worth observing that an analogous strategy has not (yet) been proven to work in the case of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. The difficulty is that of relating in a natural way the construction of the cyclotomic p-adic L-function, which is defined in terms of modular symbols, to the p-adic uniformization of Shimura curves. Schneider [Sch] has proposed the definition of a p-adic L-function based on the notion, which stems directly from the theory of p-adic uniformization, of rigid-analytic modular symbol. Klingenberg [Kl] has proven an exceptional zero formula similar to Theorem 1.1 for this rigid-analytic p-adic L-function. However, the relation (if any) between Schneider's p-adic L-function and the cyclotomic p-adic L-function considered in [MTT] is at present mysterious.

The reader is also referred to Teitelbaum's paper [T], where the theory of p-adic uniformization of Shimura curves is used to formulate analogues of the conjectures of [MTT] for cyclotomic p-adic L-functions attached to modular forms of higher weight.

The proof by Greenberg and Stevens [GS] of the cyclotomic "exceptional zero" formula of [MTT] follows a completely different strategy from the one of this paper, and is based on Hida's theory of p-adic families of modular forms.

Finally, let us mention that Kato, Kurihara, and Tsuji [KKT] recently announced more general results on the conjectures of [MTT], which make use of an Euler system constructed by Kato from modular units in towers of modular function fields.

2. Definite quaternion algebras and graphs. We keep the notation and assumptions of the introduction. In particular, we recall that K is an imaginary quadratic field and B is a definite quaternion algebra of discriminant N^-. Given a rational prime ℓ, and orders O of K and S of B, set

$$K_\ell := K \otimes \mathbb{Z}_\ell, \qquad B_\ell := B \otimes \mathbb{Z}_\ell, \qquad O_\ell := O \otimes \mathbb{Z}_\ell, \qquad S_\ell := S \otimes \mathbb{Z}_\ell.$$

Denote by $\hat{\mathbb{Z}} = \prod \mathbb{Z}_\ell$ the profinite completion of $\mathbb{Z}$. Set

$$\hat{K} := K \otimes \hat{\mathbb{Z}}, \qquad \hat{B} := B \otimes \hat{\mathbb{Z}}, \qquad \hat{O} := O \otimes \hat{\mathbb{Z}} = \prod O_\ell, \qquad \hat{S} := S \otimes \hat{\mathbb{Z}} = \prod S_\ell.$$

Fix an Eichler order R of B of level N^+p. Equip R with an *orientation*, that is, a collection of algebra homomorphisms

$$\mathfrak{o}_\ell^+ : R \to \mathbb{Z}/\ell^n\mathbb{Z}, \qquad \ell^n \| N^+ p,$$

$$\mathfrak{o}_\ell^- : R \to \mathbb{F}_{\ell^2}, \qquad \ell \mid N^-.$$

The group $\hat{B}^\times$ acts transitively (on the right) on the set of Eichler orders of level N^+p by the rule

$$S * \hat{b} := \big(\hat{b}^{-1}\hat{S}\hat{b}\big) \cap B.$$

The orientation on R induces an orientation on $R * \hat{b}$, and the stabilizer of the oriented order R is equal to $\mathbb{Q}^\times \hat{R}^\times$. This sets up a bijection between the set of oriented Eichler orders of level N^+p and the coset space $\mathbb{Q}^\times \hat{R}^\times \backslash \hat{B}^\times$. Likewise, there is a bijection between the set of oriented Eicher orders of level N^+p modulo conjugation by $B^\times$ and the double coset space

$$\hat{R}^\times \backslash \hat{B}^\times / B^\times.$$

Set $\Gamma_+ := \mathbb{Q}_p^\times \backslash R[1/p]^\times$ and, as in the introduction, let Γ be the image in Γ_+ of the elements in $R[1/p]^\times$ whose reduced norm has even p-adic valuation.

LEMMA 2.1. *Γ has index 2 in Γ_+.*

Proof. See [BD2, Lemma 1.5].

Let $\mathcal{T}$ be the *Bruhat-Tits tree* associated with the local algebra B_p. The set of vertices $\mathcal{V}(\mathcal{T})$ of $\mathcal{T}$ is equal to the set of maximal orders in B_p. The set $\vec{\mathcal{E}}(\mathcal{T})$ of oriented edges of $\mathcal{T}$ is equal to the set of oriented Eichler orders of level p in B_p. Thus, $\vec{\mathcal{E}}(\mathcal{T})$ can be identified with the coset space $\mathbb{Q}_p^\times R_p^\times \backslash B_p^\times$, by mapping $b_p \in B_p^\times$ to $R_p * b_p = b_p^{-1} R_p b_p$. Similarly, if $\underline{R}_p$ is a maximal order in B_p containing R_p, we identify $\mathcal{V}(\mathcal{T})$ with the coset space $\mathbb{Q}_p^\times \underline{R}_p^\times \backslash B_p^\times$. Define the graphs

$$\mathcal{G} := \mathcal{T}/\Gamma, \qquad \mathcal{G}_+ := \mathcal{T}/\Gamma_+.$$

By strong approximation (see [Vi, p. 61]), there is an identification

$$\vec{\mathcal{E}}\big(\mathcal{G}_+\big) = \hat{R}^\times \backslash \hat{B}^\times / B^\times$$

of the set of oriented edges of $\mathcal{G}_+$ with the set of conjugacy classes of oriented Eichler orders of level N^+p.

Fixing a vertex v_0 of $\mathcal{T}$ gives rise to an orientation of $\mathcal{T}$ in the following way. A vertex of $\mathcal{T}$ is called *even* (respectively, *odd*) if it has even (respectively, odd) distance from v_0. The direction of an edge is said to be positive if it goes from the even to the odd vertex. Since Γ sends even vertices to even ones, and odd vertices to odd ones, the orientation of $\mathcal{T}$ induces an orientation of $\mathcal{G}$. Define a map

$$\kappa : \mathcal{E}(\mathcal{G}) \to \vec{\mathcal{E}}(\mathcal{G}_+)$$

from the set of edges of $\mathcal{G}$ to the set of oriented edges of $\mathcal{G}_+$, by mapping an edge $\{v, v'\} \pmod{\Gamma}$ of $\mathcal{G}$, where v and v' are vertices of $\mathcal{T}$ and we assume that v is even, to the oriented edge $(v, v') \pmod{\Gamma_+}$ of $\mathcal{G}_+$.

LEMMA 2.2. *The map κ is a bijection.*

Proof. Suppose that $(v, v') \pmod{\Gamma_+} = (u, u') \pmod{\Gamma_+}$. Thus, there is $\gamma \in \Gamma_+$ such that $\gamma v = u$ and $\gamma v' = u'$. If v and u are both even, γ must belong to Γ, and this proves the injectivity of κ. As for surjectivity, $(v, v') \pmod{\Gamma_+}$ is the image by κ of $\{v, v'\} \pmod{\Gamma}$ if v is even, and of $\{wv, wv'\} \pmod{\Gamma}$, where w is any element of $\Gamma_+ - \Gamma$, if v is odd.

Given two vertices v and v' of $\mathcal{T}$, write $\mathrm{path}(v, v')$ for the natural image in $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ of the unique geodesic on $\mathcal{T}$ joining v with v'. For example, if v and v' are even vertices joined by four consecutive edges e_1, e_2, e_3, e_4, by our convention for orienting the edges of $\mathcal{T}$, $\mathrm{path}(v, v')$ is the image in $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ of $e_1 - e_2 + e_3 - e_4$.

There is a coboundary map

$$\partial^* : \mathbb{Z}[\mathcal{V}(\mathcal{G})] \to \mathbb{Z}[\mathcal{E}(\mathcal{G})],$$

which maps the image in $\mathcal{V}(\mathcal{G})$ of an odd (respectively, even) vertex v of $\mathcal{T}$ to the image in $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ of the formal sum of the edges of $\mathcal{T}$ emanating from v (respectively, the opposite of this sum). There is also a boundary map

$$\partial_* : \mathbb{Z}[\mathcal{E}(\mathcal{G})] \to \mathbb{Z}[\mathcal{V}(\mathcal{G})],$$

which maps an edge e to the difference $v' - v$ of its vertices, where v is the even vertex and v' is the odd vertex of e. The integral homology (respectively, the integral cohomology) of the graph $\mathcal{G}$ is defined by $H_1(\mathcal{G}, \mathbb{Z}) = \ker(\partial_*)$ (respectively, $H^1(\mathcal{G}, \mathbb{Z}) = \mathrm{coker}(\partial^*)$).

Let

$$\langle\ ,\ \rangle : \mathbb{Z}[\mathcal{E}(\mathcal{G})] \times \mathbb{Z}[\mathcal{E}(\mathcal{G})] \to \mathbb{Z}$$

be the pairing on $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ defined by the rule $\langle e_i, e_j \rangle := \omega_{e_i} \delta_{ij}$, where the e_i are the elements of the standard basis of $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ and ω_{e_i} is the order of the stabilizer in Γ of a lift of e_i to $\mathcal{T}$. Likewise, let

$$\langle\langle\ ,\ \rangle\rangle : \mathbb{Z}[\mathcal{V}(\mathcal{G})] \times \mathbb{Z}[\mathcal{V}(\mathcal{G})] \to \mathbb{Z}$$

be the pairing on $\mathbb{Z}[\mathcal{V}(\mathcal{G})]$ defined by $\langle\langle v_i, v_j\rangle\rangle := \omega_{v_i}\delta_{ij}$, where the v_i are the elements of the standard basis of $\mathbb{Z}[\mathcal{V}(\mathcal{G})]$ and ω_{v_i} is the order of the stabilizer in Γ of a lift of v_i to $\mathcal{T}$.

We use the notation $\mathcal{M}$ to indicate the module $H^1(\mathcal{G}, \mathbb{Z})$. Let $\bar{\Gamma}$ be the maximal torsion-free abelian quotient of Γ. As in the introduction, write $\mathcal{N}$ for $\mathrm{Hom}(\bar{\Gamma}, \mathbb{Z})$. Given an element $\gamma \in \Gamma$, denote by $\bar{\gamma}$ the natural image of γ in $\bar{\Gamma}$.

LEMMA 2.3. (i) *The map from* $\bar{\Gamma}$ *to* $H_1(\mathcal{G}, \mathbb{Z})$ *that sends* $\bar{\gamma} \in \bar{\Gamma}$ *to the cycle* $\mathrm{path}(v_0, \gamma v_0)$*, where* v_0 *is any vertex of* $\mathcal{G}$ *and* γ *is any lift of* $\bar{\gamma}$ *to* Γ*, is an isomorphism.*

(ii) *The map from* $\mathcal{M}$ *to* $\mathcal{N}$ *that sends* $m \in \mathcal{M}$ *to the homomorphism*

$$\bar{\gamma} \mapsto \langle \mathrm{path}(v_0, \gamma v_0), m\rangle$$

is injective and has finite cokernel.

Proof (Sketch). Part (i) is proved in [Se]. Part (ii) follows from part (i) and from the fact that the maps ∂^* and ∂_* are adjoint with respect to the pairings defined above.

Write $\mathcal{M}_{\mathrm{sp}}$ for the maximal torsion-free quotient of $\mathcal{M}/(w+1)\mathcal{M}$, with $w \in \Gamma_+ - \Gamma$. By part (i) of Lemma 2.3, the action of $w \in \Gamma_+ - \Gamma$ on $H_1(\mathcal{G}, \mathbb{Z})$ induces an action of w on $\mathcal{N}$. Write $\mathcal{N}_{\mathrm{sp}}$ for the maximal torsion-free quotient of $\mathcal{N}/(w+1)\mathcal{N}$. We have an induced map from $\mathcal{M}_{\mathrm{sp}}$ to $\mathcal{N}_{\mathrm{sp}}$ that is injective and has finite cokernel.

The module $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ is equipped with the natural action of an algebra $\tilde{\mathbb{T}}$ generated over $\mathbb{Z}$ by the Hecke correspondences T_ℓ for $\ell \nmid N$ and U_ℓ for $\ell \mid N$, coming from its double coset description: see [BD1, Sec. 1.5]. The module $H_1(\mathcal{G}, \mathbb{Z})$ is stable under the action of $\tilde{\mathbb{T}}$. Hence, by part (i) of Lemma 2.3, the algebra $\tilde{\mathbb{T}}$ also acts on the modules $\mathcal{N}$ and $\mathcal{N}_{\mathrm{sp}}$. Let $\mathbb{T}$ and $\mathbb{T}_{\mathrm{sp}}$ denote the image of $\tilde{\mathbb{T}}$ in $\mathrm{End}(\mathcal{N})$ and $\mathrm{End}(\mathcal{N}_{\mathrm{sp}})$, respectively. Thus, there are natural surjections $\tilde{\mathbb{T}} \to \mathbb{T} \to \mathbb{T}_{\mathrm{sp}}$. By an abuse of notation, we denote by T_ℓ and U_ℓ also the natural images in $\mathbb{T}$ and $\mathbb{T}_{\mathrm{sp}}$ of T_ℓ and U_ℓ .

The next proposition clarifies the relation between the modules $\mathcal{N}$ and $\mathcal{N}_{\mathrm{sp}}$ and the theory of modular forms.

PROPOSITION 2.4. *Let* ϕ *be an algebra homomorphism from* $\mathbb{T}$ *(respectively,* $\mathbb{T}_{\mathrm{sp}}$*) to* $\mathbb{C}$*, and let* $a_n := \phi(T_n)$*. Then, the* a_n *are the Fourier coefficients of a normalized eigenform of level* N*, which is new at* N^-p *(respectively, is new at* N^-p *and is split multiplicative at* p*). Conversely, any such modular form arises as above from a character of* $\mathbb{T}$ *(respectively,* $\mathbb{T}_{\mathrm{sp}}$*).*

Proof. Eichler's trace formula identifies the Hecke-module $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ with a space of modular forms of level N that are new at N^-. Moreover, the algebra $\mathbb{T}$ can also be viewed as the Hecke algebra of the module $\mathcal{M}$ defined above, and Proposition 1.4 of [BD2] shows that $\mathcal{M}$ is equal to the "p-new" quotient of $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$. This proves the statement of Proposition 2.4 concerning characters of $\mathbb{T}$. The abelian variety associated to a p-new modular form f is split multiplicative at p if and only if

$U_p f = f$. Moreover, the Atkin-Lehner involution at p acts on a p-new modular form as $-U_p$, and acts on $\mathcal{M}$ as Γ_+/Γ. This concludes the proof of Proposition 2.4.

Modular parametrizations, I. We now make a specific choice of the operator η_f (where f is the newform of level N attached to E) considered in the introduction. It is used in formulating the results in the sequel of the paper.

As stated in Lemma 2.3, $\bar{\Gamma}$ can be identified with the homology group $H_1(\mathcal{G}, \mathbb{Z}) \subset \mathbb{Z}[\mathcal{E}(\mathcal{G})]$. Thus, when convenient, we tacitly view elements of $\bar{\Gamma}$ as contained in $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$. The restriction of the pairing on $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ defined above to $\bar{\Gamma}$ yields the *monodromy pairing* (denoted in the same way by an abuse of notation)

$$\langle\ ,\ \rangle : \bar{\Gamma} \times \bar{\Gamma} \to \mathbb{Z}.$$

Let $\mathbb{Z}[\mathcal{E}(\mathcal{G})]^f$ (respectively, $\bar{\Gamma}^f$) be the submodule of $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ (respectively, $\bar{\Gamma}$) on which $\tilde{\mathbb{T}}$ (respectively, $\mathbb{T}$) acts via the character associated with f. Note that the quotient of $\mathbb{Z}[\mathcal{E}(\mathcal{G})]$ by $\bar{\Gamma}$ is torsion free, and thus there is a canonical identification $\mathbb{Z}[\mathcal{E}(\mathcal{G})]^f = \bar{\Gamma}^f$. Let e^f be a generator of $\bar{\Gamma}^f \simeq \mathbb{Z}$.

Define the "modular parametrizations"

$$\pi_* : \bar{\Gamma} \to \bar{\Gamma}^f, \qquad \pi^* : \bar{\Gamma}^f \to \bar{\Gamma}$$

by $\pi_*(e) := \langle e, e^f \rangle e^f$ and $\pi^*(e^f) := e^f$. Since

$$\left(\pi^* \circ \pi_*\right)^2 = \left\langle e^f, e^f \right\rangle \left(\pi^* \circ \pi_*\right),$$

we obtain that $\pi^* \circ \pi_*$ is equal to $\langle e^f, e^f \rangle \pi_f$, where π_f is the idempotent of $\mathbb{T} \otimes \mathbb{Q}$ associated with f. From now on, we assume that the operator η_f is defined by

$$\eta_f := \pi^* \circ \pi_*,$$

so that the integer n_f is equal to $\langle e^f, e^f \rangle$.

As observed in the introduction, the operator η_f induces a map $\mathcal{N} \to \mathbb{Z}$, which is well defined up to sign. Since f has split multiplicative reduction at p, this map factors through a map $\mathcal{N}_{\mathrm{sp}} \to \mathbb{Z}$. By an abuse of notation, we indicate both of the above maps by η_f.

Remark 2.5. The module $\bar{\Gamma}$ can be identified with the character group associated with the reduction modulo p of $\mathrm{Pic}^0(X)$, where X is the Shimura curve considered in the introduction. As is explained in Section 4, the map $\pi^* \circ \pi_*$ on $\bar{\Gamma}$ is induced by functoriality from a modular parametrization $\mathrm{Pic}^0(X) \to E$.

3. The p-adic L-function. Let $\mathcal{O}_n$ denote the order of K of conductor cp^n, $n \geq 0$. (We usually write $\mathcal{O}$ instead of $\mathcal{O}_0$.) Equip the orders $\mathcal{O}_n$ with compatible orientations, that is, with compatible algebra homomorphisms

$$\mathfrak{o}_\ell^+ : \mathcal{O}_n \to \mathbb{Z}/\ell^m\mathbb{Z}, \qquad \ell^m \| N^+ p,$$

$$\mathfrak{d}_\ell^- : \mathbb{O}_n \to \mathbb{F}_{\ell^2}, \qquad \ell \mid N^-.$$

An algebra homomorphism of $\mathbb{O}_n$ into an oriented Eichler order S of level N^+p is called an *oriented optimal embedding* if it respects the orientation on $\mathbb{O}_n$ and on S, and does not extend to an embedding of a larger order into S. Consider pairs (R_ξ, ξ), where R_ξ is an oriented Eichler order of level N^+p and ξ is an element of $\mathrm{Hom}(K, B)$ that restricts to an oriented optimal embedding of $\mathbb{O}_n$ into R_ξ. A *Gross point of conductor cp^n* ($n \geq 0$) is a pair as above, taken modulo the action of $B^\times$.

By our previous remarks, a Gross point can be viewed naturally as an element of the double coset space

$$W := \left(\hat{R}^\times \backslash \hat{B}^\times \times \mathrm{Hom}(K, B)\right)/B^\times.$$

(See [Gr, Sec. 3] for more details.) Strong approximation gives the identification

$$W = \left(\vec{\mathscr{E}}(\mathcal{T}) \times \mathrm{Hom}(K, B)\right)/\,\Gamma_+.$$

By Lemma 2.2, there is a natural map of $\mathbb{Z}$-modules $\mathbb{Z}[W] \to \mathbb{Z}[\mathscr{E}(\mathscr{G})]$, where $\mathbb{Z}[W]$ is the module of finite formal $\mathbb{Z}$-linear combinations of elements of W. The Hecke algebra $\tilde{\mathbb{T}}$ of $\mathbb{Z}[\mathscr{E}(\mathscr{G})]$ acts naturally also on $\mathbb{Z}[W]$ (see [BD1, Sec. 1.5]), in such a way that the above map is $\tilde{\mathbb{T}}$-equivariant.

The group $\mathbf{G}_n = \mathrm{Pic}(\mathbb{O}_n) = \hat{\mathbb{O}}_n^\times \backslash \hat{K}^\times / K^\times$ acts simply transitively on the Gross points of conductor cp^n by the rule

$$\sigma\left(R_\xi, \xi\right) := \left(R_\xi * \hat{\xi}(\sigma)^{-1}, \xi\right),$$

where $\hat{\xi}$ denotes the extension of ξ to a map from $\hat{K}$ to $\hat{B}$.

Now, fix a Gross point $P_0 = (R_0, \xi_0) \pmod{B^\times}$ of conductor c. By the above identification, P_0 corresponds to a pair $(\vec{e}_0, \xi_0) \in \vec{\mathscr{E}}(\mathcal{T}) \times \mathrm{Hom}(K, B)$, modulo the action of Γ_+. As above, the origin v_0 of $\vec{e}_0$ determines an orientation of $\mathcal{T}$. Let $\vec{e}$ be one of the p oriented edges of $\mathcal{T}$ originating from $\vec{e}_0$. All the Gross points corresponding to pairs $(\vec{e}, \xi_0)$ as above have conductor cp, except for one, which has conductor c. Fix an end

$$\left(\vec{e}_0, \vec{e}_1, \ldots, \vec{e}_n, \ldots\right)$$

such that $(\vec{e}_1, \xi_0)$ defines a Gross point of conductor cp. Then, $(\vec{e}_n, \xi_0)$ defines a Gross point P_n of conductor cp^n, for all $n \geq 0$.

Denote by $\mathrm{Norm}_{H_{n+1}/H_n}$ the norm operator $\sum_{g \in \mathrm{Gal}(H_{n+1}/H_n)} g$.

LEMMA 3.1. (1) *Let $u = (1/2)\#\mathbb{O}^\times$. The equality*

$$U_p P_0 = u \,\mathrm{Norm}_{H_1/H_0} P_1 + \sigma_{\mathfrak{p}} P_0$$

holds in $\mathbb{Z}[W]$ for a prime $\mathfrak{p}$ above p, where $\sigma_{\mathfrak{p}} \in \mathrm{Gal}(H_0/K)$ denotes the image of $\mathfrak{p}$ by the Artin map.

(2) *For $n \geq 1$,*

$$U_p P_n = \mathrm{Norm}_{H_{n+1}/H_n} P_{n+1}.$$

Proof. The proof follows from the definition of the operator U_p (see [BD1, Sec. 1.5]) and the action of $\mathrm{Pic}(\mathbb{O}_n)$ on the Gross points.

Figure 1, drawn in the case where $p = 2$, illustrates geometrically the relation between the Galois action and the action of the Hecke correspondence U_p.

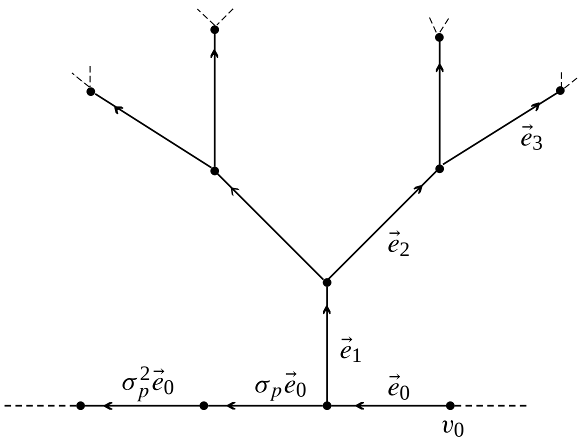

FIGURE 1

By Lemma 2.3, the natural map from $\mathbb{Z}[W]$ to $\mathbb{Z}[\mathscr{E}(\mathscr{G})]$ induces maps from $\mathbb{Z}[W]$ to the modules $\mathcal{N}$ and $\mathcal{N}_{\mathrm{sp}}$. These maps are Hecke-equivariant.

The Gross points P_n give rise to a p-adic distribution on $\mathbf{G}_\infty$ with values in the module $\mathcal{N}_{\mathrm{sp}}$ as follows. Given $g \in \mathbf{G}_n$, denote by e_n^g the natural image of P_n^g in $\mathcal{N}_{\mathrm{sp}}$. For $n \geq 0$, define the truncated p-adic L-function

$$\mathscr{L}_{p,n}\big(\mathcal{N}_{\mathrm{sp}}/K\big) := \sum_{g \in \mathbf{G}_n} e_n^g \cdot g^{-1} \in \mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[\mathbf{G}_n].$$

Note that $\mathscr{L}_{p,n}(\mathcal{N}_{\mathrm{sp}}/K)$ is well defined up to multiplication by elements of $\mathbf{G}_n$.

For $n \geq 1$, let $\nu_n : \mathbb{Z}[\mathbf{G}_n] \to \mathbb{Z}[\mathbf{G}_{n-1}]$ be the natural projection of groups rings.

LEMMA 3.2. (1) *The equality*

$$\nu_1\big(\mathscr{L}_{p,1}\big(\mathcal{N}_{\mathrm{sp}}/K\big)\big) = u^{-1}\big(1-\sigma_{\mathfrak{p}}\big)\mathscr{L}_{p,0}\big(\mathcal{N}_{\mathrm{sp}}/K\big)$$

holds in $\mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[\mathbf{G}_0]$.

(2) *For* $n \geq 2$, *the equality*

$$\nu_n\big(\mathscr{L}_{p,n}\big(\mathcal{N}_{\mathrm{sp}}/K\big)\big) = \mathscr{L}_{p,n-1}\big(\mathcal{N}_{\mathrm{sp}}/K\big)$$

holds in $\mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[\mathbf{G}_{n-1}]$.

Proof. By Proposition 2.4, the operator U_p acts as $+1$ on $\mathcal{N}_{\mathrm{sp}}$. The claim follows from Lemma 3.1 and the fact that $\mathcal{N}_{\mathrm{sp}}$ is torsion free.

Define the p-adic L-function attached to $\mathcal{N}_{\mathrm{sp}}$ to be

$$\mathcal{L}_p\big(\mathcal{N}_{\mathrm{sp}}/K\big) := \varprojlim_{n} \mathcal{L}_{p,n}\big(\mathcal{N}_{\mathrm{sp}}/K\big) \in \mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[[\mathbf{G}_\infty]].$$

We now define the p-adic L-function attached to E. Observe that the maximal quotient $\bar{\Gamma}_f$ of $\bar{\Gamma}$ on which $\mathbb{T}$ acts via the character associated with f is isomorphic to $\mathbb{Z}$. Let e_f be a generator of $\bar{\Gamma}_f$. The monodromy pairing on $\bar{\Gamma}$ induces a $\mathbb{Z}$-valued pairing on $\bar{\Gamma}^f \times \bar{\Gamma}_f$. Write $\hat{c}_p$ for the positive integer $|\langle e^f, e_f\rangle|$.

LEMMA 3.3. *The element* $(\eta_f \otimes \mathrm{id})(\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K)) \in \mathbb{Z}[[\mathbf{G}_\infty]]$ *is divisible by* $\hat{c}_p$.

Proof. Consider the maps

$$\tilde{\pi}_* : \mathbb{Z}[\mathscr{E}(\mathscr{G})] \to \mathbb{Z}[\mathscr{E}(\mathscr{G})]^f, \qquad \tilde{\pi}^* : \mathbb{Z}[\mathscr{E}(\mathscr{G})]^f \to \mathbb{Z}[\mathscr{E}(\mathscr{G})]$$

defined by $\tilde{\pi}_*(e) := \langle e, e^f\rangle e^f$ and $\tilde{\pi}^*(e^f) := e^f$. (The modular parametrizations π_* and π^* introduced in Section 2 are obtained from these maps by restriction.) Hence, $\tilde{\eta}_f := \tilde{\pi}^* \circ \tilde{\pi}_*$ is an element of $\tilde{\mathbb{T}}$, equal to $\langle e^f, e^f\rangle \tilde{\pi}_f$, where $\tilde{\pi}_f$ is the idempotent in $\tilde{\mathbb{T}} \otimes \mathbb{Q}$ associated with f. We have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[\mathscr{E}(\mathscr{G})] & \longrightarrow & \mathcal{N} \\ {\scriptstyle \tilde{\eta}_f}\big\downarrow & & \big\downarrow{\scriptstyle \eta_f} \\ \mathbb{Z}[\mathscr{E}(\mathscr{G})]^f & \longrightarrow & \mathcal{N}^f, \end{array}$$

where the upper horizontal map is defined in Lemma 2.3, and the lower horizontal map is the restriction of the upper one. Note that $\mathcal{N}^f$ is equal to $\mathrm{Hom}(\bar{\Gamma}_f, \mathbb{Z})$ and therefore is generated by the homomorphism $e_f \mapsto 1$. With our choices of generators for $\mathbb{Z}[\mathscr{E}(\mathscr{G})]^f$ and $\mathcal{N}^f$, the lower map of the above diagram is described as multiplication by the integer $\langle e^f, e_f\rangle$. The proof of Lemma 3.2 also shows that mapping the Gross points of conductor cp^n to $\mathbb{Z}[\mathscr{E}(\mathscr{G})]^f$ by the map $\tilde{\eta}_f$ yields a p-adic distribution in $\mathbb{Z}[\mathscr{E}(\mathscr{G})]^f \otimes \mathbb{Z}[[\mathbf{G}_\infty]]$. By the above diagram, the image of this distribution in $\mathcal{N}^f \otimes \mathbb{Z}[[\mathbf{G}_\infty]]$ is equal to $(\eta_f \otimes \mathrm{id})(\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K))$. This proves the lemma.

Remark 3.4. In Section 4, we show that the integers $\hat{c}_p$ and c_p are equal.

Define the p-adic L-function attached to E to be

$$\mathcal{L}_p(E/K) = \hat{c}_p^{-1}(\eta_f \otimes \mathrm{id})\big(\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K)\big) \in \mathbb{Z}[[\mathbf{G}_\infty]].$$

Observe that $\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K)$ and $\mathcal{L}_p(E/K)$ are well defined up to multiplication by elements of $\mathbf{G}_\infty$.

Recall the quantities Ω_f and d defined in the introduction.

THEOREM 3.5. *Let $\chi : \mathbf{G}_\infty \to \mathbb{C}^\times$ be a finite-order character of conductor cp^n, with $n \geq 1$. Then the equality*

$$\left|\chi\left(\mathcal{L}_p(E/K)\right)\right|^2 = \frac{L(E/K,\chi,1)}{\Omega_f}\sqrt{d}\cdot(n_f u)^2$$

holds.

Proof. See [Gr], [Dag], and [BD1, Sec. 2.10].

Remark 3.6. (1) Theorem 3.5 suggests that $\mathcal{L}_p(E/K)$ should really be viewed as the square root of a p-adic L-function, and hence we should define the anticyclotomic p-adic L-function of E to be $\mathcal{L}_p(E/K)\otimes\mathcal{L}_p(E/K)^*$, where $*$ denotes the involution of $\mathbb{Z}[[\mathbf{G}_\infty]]$ given on grouplike elements by $g \mapsto g^{-1}$. See Section 2.7 of [BD1] for more details.

(2) More generally, the p-adic L-function $\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K)$ interpolates special values of the complex L-series attached to the modular forms on $\mathbb{T}_{\mathrm{sp}}$ (described in Proposition 2.4).

Let $\sigma_{\mathfrak{p}}$ be as in Lemma 3.1. Denote by H the subextension of H_0 that is fixed by $\sigma_{\mathfrak{p}}$, and set

$$G_n := \mathrm{Gal}(H_n/H), \qquad G_\infty := \mathrm{Gal}(H_\infty/H),$$

$$\Sigma := \mathrm{Gal}(H_0/H) = G_0, \qquad \Delta := \mathrm{Gal}(H/K).$$

Note the exact sequences of Galois groups

$$0 \to G_n \to \mathbf{G}_n \to \Delta \to 0,$$

$$0 \to G_\infty \to \mathbf{G}_\infty \to \Delta \to 0.$$

The group Δ is naturally identified with the Picard group $\mathrm{Pic}(\mathcal{O}[1/p])$, and G_∞ is equal to the image of the reciprocity map $\mathrm{rec}_p : \mathbb{Q}_p^\times \to \mathbf{G}_\infty$ (where we identified $\mathbb{Q}_p^\times$ with $K_{\mathfrak{p}}^\times$). Let I be the kernel of the augmentation map $\mathbb{Z}[[\mathbf{G}_\infty]] \to \mathbb{Z}$, and let I_Δ be the kernel of the augmentation map $\mathbb{Z}[[\mathbf{G}_\infty]] \to \mathbb{Z}[\Delta]$.

LEMMA 3.7. (i) *$\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K)$ belongs to $\mathcal{N}_{\mathrm{sp}} \otimes I_\Delta$.*
(ii) *$\mathcal{L}_p(E/K)$ belongs to I_Δ.*

Proof. There are canonical isomorphisms

$$\mathbb{Z}[[\mathbf{G}_\infty]]/I_\Delta = \mathbb{Z}[\mathbf{G}_n]/I_{\Delta,n} = \mathbb{Z}[\Delta],$$

where $I_{\Delta,n}$ is the natural image of I_Δ in $\mathbb{Z}[\mathbf{G}_n]$. By Lemma 3.2, the image of $\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K)$ in $\mathcal{N}_{\mathrm{sp}} \otimes (\mathbb{Z}[[\mathbf{G}_\infty]]/I_\Delta)$ is equal to the image of $\mathcal{L}_{p,1}(\mathcal{N}_{\mathrm{sp}}/K)$ in $\mathcal{N}_{\mathrm{sp}} \otimes (\mathbb{Z}[\mathbf{G}_1]/I_{\Delta,1}) = \mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[\Delta]$. The first part of the lemma now follows from Lemma 3.2(1). The second part follows directly from the first.

Since I_Δ is contained in I, the element $\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K)$ belongs to $\mathcal{N}_{\mathrm{sp}} \otimes I$ and $\mathcal{L}_p(E/K)$ belongs to I. Denote by

$$\mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/K\big), \qquad \mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/H\big)$$

the natural image of $\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/K)$ in $\mathcal{N}_{\mathrm{sp}} \otimes I/I^2 = \mathcal{N}_{\mathrm{sp}} \otimes \mathbf{G}_\infty$ and $\mathcal{N}_{\mathrm{sp}} \otimes I_\Delta/I_\Delta^2 = \mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[\Delta] \otimes G_\infty$, respectively. Likewise, let

$$\mathcal{L}'_p(E/K), \qquad \mathcal{L}'_p(E/H)$$

be the natural image of $\mathcal{L}_p(E/K)$ in $I/I^2 = \mathbf{G}_\infty$ and $I_\Delta/I_\Delta^2 = \mathbb{Z}[\Delta] \otimes G_\infty$, respectively. The above elements should be viewed as derivatives of p-adic L-functions at the central point.

In order to carry out the calculations of the next sections, it is useful to observe that the derivatives $\mathcal{L}'_p(\mathcal{N}_{\mathrm{sp}}/K)$ and $\mathcal{L}'_p(\mathcal{N}_{\mathrm{sp}}/H)$ can be expressed in terms of the derivatives of certain partial p-adic L-functions. Set $h := \#(\Delta)$. Fix Gross points of conductor c,

$$P_0 = P_0^1, \dots, P_0^h,$$

corresponding to pairs (R_0^i, ξ_0^i), $i = 1, \dots, h$, which are representatives for the Σ-orbits of the Gross points of conductor c. Writing $[P_0^i]$ for the Σ-orbit of P_0^i, let δ_i be the element of Δ such that

$$\big[\delta_i P_0^1\big] = \big[P_0^i\big].$$

Suppose that P_0^i corresponds to a pair $(\vec{e}_0(i), \xi_0^i) \in \vec{\mathcal{E}}(\mathcal{T}) \times \mathrm{Hom}(K, B)$, modulo the action of Γ_+. Fix ends

$$\big(\vec{e}_0(i), \vec{e}_1(i), \dots, \vec{e}_n(i), \dots\big)$$

such that $(\vec{e}_1(i), \xi_0^i)$ defines a Gross point of conductor cp. Thus, $(\vec{e}_n(i), \xi_0^i)$ defines a Gross point P_n^i of conductor cp^n, for all $n \geq 0$. For $g \in G_n$, let $e_n(i)^g$ denote the natural image of $(P_n^i)^g$ in $\mathcal{N}_{\mathrm{sp}}$. Let

$$\mathcal{L}_{p,n}\big(\mathcal{N}_{\mathrm{sp}}/H, P_0^i\big) := \sum_{g \in G_n} e_n(i)^g \cdot g^{-1} \in \mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[G_n].$$

The proof of Lemma 3.2 also shows that the elements $\mathcal{L}_{p,n}(\mathcal{N}_{\mathrm{sp}}/H, P_0^i)$ are compatible under the maps induced by the natural projections of group rings. Thus, we may define the partial p-adic L-function attached to $\mathcal{N}_{\mathrm{sp}}$ and P_0^i to be

$$\mathcal{L}_p\big(\mathcal{N}_{\mathrm{sp}}/H, P_0^i\big) := \lim_{\substack{\leftarrow \\ n}} \mathcal{L}_{p,n}\big(\mathcal{N}_{\mathrm{sp}}/H, P_0^i\big) \in \mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[[G_\infty]].$$

We observe that $\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/H, P_0^i)$ depends only on the Σ-orbit of P_0^i, up to multiplication by elements of G_∞.

Let I_H be the kernel of the augmentation map $\mathbb{Z}[[G_\infty]] \to \mathbb{Z}$. Like in the proof of Lemma 3.7, one checks that $\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/H, P_0^i)$ belongs to I_H. Write $\mathcal{L}'_p(\mathcal{N}_{\mathrm{sp}}/H, P_0^i)$ for the natural image of $\mathcal{L}_p(\mathcal{N}_{\mathrm{sp}}/H, P_0^i)$ in $\mathcal{N}_{\mathrm{sp}} \otimes I_H/I_H^2 = \mathcal{N}_{\mathrm{sp}} \otimes G_\infty$. Thus,

$$\mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/H, P_0^i\big) = \varprojlim_n \mathcal{L}'_{p,n}\big(\mathcal{N}_{\mathrm{sp}}/H, P_0^i\big),$$

where

$$\mathcal{L}'_{p,n}\big(\mathcal{N}_{\mathrm{sp}}/H, P_0^i\big) = \sum_{g \in G_n} e_n(i)^g \otimes g^{-1}. \tag{3}$$

We obtain the following lemma directly.

LEMMA 3.8. (i)

$$\mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/K\big) = \sum_{i=1}^{h} \mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/H, P_0^i\big).$$

(ii)

$$\mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/H\big) = \sum_{i=1}^{h} \mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/H, P_0^i\big) \cdot \delta_i^{-1}.$$

4. The theory of p-adic uniformization of Shimura curves. For more details on the results stated in this section, the reader is referred to [BC], [C], [Dr], [GvdP], and [BD2].

Let $\mathcal{B}$ be the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant N^-p, and let $\mathcal{R}$ be an Eichler order of $\mathcal{B}$ of level N^+. Denote by X the Shimura curve over $\mathbb{Q}$ associated with the order $\mathcal{R}$. We refer the reader to [BC] and [BD2, Sec. 4] for the definition of X via moduli. Here we content ourselves with recalling Cherednik's theorem, which describes a rigid-analytic uniformization of X. Write

$$\mathcal{H}_p := \mathbb{C}_p - \mathbb{Q}_p$$

for the p-adic upper half plane. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts (on the left) on $\mathcal{H}_p$ by linear fractional transformations. Thus, fixing an isomorphism

$$\psi : B_p \to M_2(\mathbb{Q}_p)$$

induces an action of Γ on $\mathcal{H}_p$. This action is discontinuous, and the rigid-analytic quotient $\Gamma\backslash\mathcal{H}_p$ defines the $\mathbb{C}_p$-points of a nonsingular curve $\mathcal{X}$ over $\mathbb{Q}_p$. The curves X and $\mathcal{X}$ are equipped with the action of Hecke algebras $\mathbb{T}_X$ and $\mathbb{T}_{\mathcal{X}}$, respectively (see [BC], [BD1]).

By Lemma 2.1, the action of Γ_+/Γ induces an involution W of $\mathcal{X}$. Let $\mathbb{Q}_{p^2}$ be the unique unramified quadratic extension of $\mathbb{Q}_p$ contained in $\mathbb{C}_p$, and let τ be the generator of $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$. Denote by $\natural \in H^1(\langle\tau\rangle, \mathrm{Aut}(\mathcal{X}))$ the class of the cocycle mapping τ to W, and write $\mathcal{X}^\natural$ for the curve over $\mathbb{Q}_p$ obtained by twisting $\mathcal{X}$ by $\natural$.

THEOREM 4.1 (Cherednik). *There is a Hecke-equivariant isomorphism $X \simeq \mathscr{X}^{\natural}$ of curves over $\mathbb{Q}_p$. In particular, X and $\mathscr{X}$ are isomorphic over $\mathbb{Q}_{p^2}$.*

Proof. See [C], [Dr], [BC].

Building on Theorem 4.1, the results in [GvdP] yield a rigid-analytic description of the jacobian of X. If $D = P_1 + \cdots + P_r - Q_1 - \cdots - Q_r \in \mathrm{Div}^0(\mathscr{H}_p)$ is a divisor of degree zero on $\mathscr{H}_p$, define the theta function

$$\vartheta(z; D) = \prod_{\epsilon \in \Gamma} \frac{(z - \epsilon P_1) \cdots (z - \epsilon P_r)}{(z - \epsilon Q_1) \cdots (z - \epsilon Q_r)} \ .$$

Write $\bar{\delta}$ for the natural image in $\bar{\Gamma}$ of an element δ of Γ. For all δ in Γ, the above theta function satisfies the functional equation

$$\vartheta(\delta z; D) = \phi_D\big(\bar{\delta}\big)\vartheta(z; D),$$

where ϕ_D is an element of $\mathrm{Hom}(\bar{\Gamma}, \mathbb{C}_p^\times) = \mathscr{N} \otimes \mathbb{C}_p^\times$ that does not depend on z. For $\gamma \in \Gamma$, the number $\phi_{(\gamma z)-(z)}(\bar{\delta})$ does not depend on the choice of $z \in \mathscr{H}_p$ and depends only on the image of γ in $\bar{\Gamma}$. This gives rise to a pairing

$$[\ ,\ ] : \bar{\Gamma} \times \bar{\Gamma} \to \mathbb{Q}_p^\times .$$

The pairing $[\ ,\ ]$ is bilinear and symmetric. The next proposition explains the relation between $[\ ,\ ]$ and the monodromy pairing $\langle\ ,\ \rangle : \bar{\Gamma} \times \bar{\Gamma} \to \mathbb{Z}$ defined in Section 2.

PROPOSITION 4.2. *The pairings $\langle\ ,\ \rangle$ and $\mathrm{ord}_p \circ [\ ,\ ]$ are equal.*

Proof. See [M, Th. 7.6].

It follows that $\mathrm{ord}_p \circ [\ ,\ ]$ is positive definite, so that the map

$$j : \bar{\Gamma} \to \mathscr{N} \otimes \mathbb{Q}_p^\times$$

induced by $[\ ,\ ]$ is injective and has discrete image. Set $\Lambda := j(\bar{\Gamma})$. Given a divisor D of degree-zero on $\mathscr{X}(\mathbb{C}_p) = \Gamma \backslash \mathscr{H}_p$, let $\tilde{D}$ denote an arbitrary lift to a degree-zero divisor on $\mathscr{H}_p$. The automorphy factor $\phi_{\tilde{D}}$ depends on the choice of the lift $\tilde{D}$, but its image in $(\mathscr{N} \otimes \mathbb{C}_p^\times)/\Lambda$ depends only on D. Thus, the assignment $D \mapsto \phi_{\tilde{D}}$ gives a well-defined map from $\mathrm{Div}^0(\mathscr{X}(\mathbb{C}_p))$ to $(\mathscr{N} \otimes \mathbb{C}_p^\times)/\Lambda$.

PROPOSITION 4.3. *The map $\mathrm{Div}^0(\mathscr{X}(\mathbb{C}_p)) \to (\mathscr{N} \otimes \mathbb{C}_p^\times)/\Lambda$ defined above is trivial on the group of principal divisors and induces a Hecke-equivariant isomorphism from the $\mathbb{C}_p$-points of the jacobian $\mathscr{J}$ of $\mathscr{X}$ to $(\mathscr{N} \otimes \mathbb{C}_p^\times)/\Lambda$.*

Proof. See [GvdP, VI.2 and VII.4] and also [BC, Ch. III].

Let

$$\Phi : \mathscr{N} \otimes \mathbb{C}_p^\times \to \mathscr{J}\big(\mathbb{C}_p\big)$$

stand for the map induced by (the inverse of) the isomorphism defined in Proposition 4.3.

Modular parametrizations, II. The map $\eta_f : \mathcal{N} \to \mathbb{Z}$ defined in Section 2 induces a map

$$\eta_f \otimes \mathrm{id} : \mathcal{N} \otimes \mathbb{C}_p^\times \to \mathbb{C}_p^\times.$$

The Jacquet-Langlands correspondence [JL] implies that the quotient abelian variety $\eta_f J$ is an elliptic curve $\mathbb{Q}$-isogenous to E. From now on, we assume that $E = \eta_f J$ is the *strong Weil curve* for the parametrization by the Shimura curve X. By an abuse of notation, we denote by η_f also the surjective map

$$J(\mathbb{C}_p) \to E(\mathbb{C}_p)$$

induced by η_f.

Let Λ^f be the submodule of Λ on which $\mathbb{T}$ acts via the character ϕ_f.

PROPOSITION 4.4. *The kernel $q^{\mathbb{Z}}$ of Φ_{Tate} is canonically equal to the module Λ^f, and the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Lambda & \longrightarrow & \mathcal{N} \otimes \mathbb{C}_p^\times & \xrightarrow{\Phi} & \mathcal{J}(\mathbb{C}_p) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_f} & & \downarrow{\scriptstyle \eta_f \otimes \mathrm{id}} & & \downarrow{\scriptstyle \eta_f} & & \\
0 & \longrightarrow & \Lambda^f & \longrightarrow & \mathbb{C}_p^\times & \xrightarrow{\Phi_{\mathrm{Tate}}} & E(\mathbb{C}_p) & \longrightarrow & 0
\end{array}$$

is Hecke-equivariant and commutes up to sign.

Proof. The rightmost square in the above diagram is a consequence of Proposition 4.3, combined with Theorem 4.1 and the fact that f is split-multiplicative at p. In order to obtain the leftmost square, it is enough to prove that the kernel of Φ_{Tate} is equal to Λ^f. Note that the target $\mathbb{C}_p^\times = \mathcal{N}^f \otimes \mathbb{C}_p^\times$ of the map $\eta_f \otimes \mathrm{id}$ is naturally a submodule of $\mathcal{N} \otimes \mathbb{C}_p^\times$, since the quotient of $\mathcal{N}$ by $\mathcal{N}^f$ is torsion free. By definition, $E(\mathbb{C}_p)$ may similarly be viewed as an abelian subvariety of $\mathcal{J}(\mathbb{C}_p)$. It follows that Φ_{Tate} can be described as the restriction of Φ to $\mathbb{C}_p^\times$. In particular, $\ker(\Phi_{\mathrm{Tate}})$ is equal to $\Lambda \cap \mathbb{C}_p^\times$. In turn, this last module is equal to Λ^f.

COROLLARY 4.5. *The integer $\hat{c}_p = |\langle e^f, e_f \rangle|$ (introduced in Lemma 3.3) is equal to c_p.*

Proof. Working through the definition of the maps in the diagram of Proposition 4.4 shows that $[e^f, e_f]$ is equal to $q^{\pm 1}$. The claim follows from Proposition 4.2.

5. p-adic Shintani cycles and special values of complex L-functions. Let $P_0 = (R_0, \xi_0) \pmod{B^\times}$ be a Gross point of conductor c. The point P_0 determines a p-adic cycle $\mathfrak{c}(P_0) \in \bar{\Gamma}$ in the following way. By strong approximation, we may assume that the representative (R_0, ξ_0) for P_0 is such that the oriented orders $R_0[1/p]$ and

$R[1/p]$ are equal. Thus, ξ_0 induces an embedding of $\mathbb{O}[1/p]$ into $R[1/p]$, which we still denote by ξ_0. The image by ξ_0 of a fundamental p-unit in $\mathbb{O}[1/p]$, having norm of even p-adic valuation, determines an element $\gamma = \gamma(P_0)$ of Γ. This element is well defined up to conjugation and up to inversion, and up to multiplication by the image of torsion elements of $\mathbb{O}^\times$.

More explicitly, write k for the order of $\sigma_{\mathfrak{p}}$ in $\mathrm{Pic}(\mathbb{O})$ (where $\sigma_{\mathfrak{p}}$ is as in Lemma 3.1), and set $\mathfrak{p}^k = (v)$ with $v \in \mathbb{O}$. Let ι be 1 (respectively, 2) if k is even (respectively, odd). Then γ is the image of $\xi_0(v)^\iota$ in Γ.

Definition. The p-adic Shintani cycle $\mathfrak{c} = \mathfrak{c}(P_0)$ attached to P_0 is the natural image of γ in $\bar{\Gamma}$.

This terminology is justified in Remark 5.4 below. Observe that $\mathfrak{c}$ is well defined up to sign.

Denote by $\mathbb{Z}[\mathscr{E}(\mathscr{G})]_{\mathrm{sp}}$ the maximal torsion-free quotient of $\mathbb{Z}[\mathscr{E}(\mathscr{G})]/(w+1)\mathbb{Z}[\mathscr{E}(\mathscr{G})]$, where w is any element of $\Gamma_+ - \Gamma$. Recall the element $\tilde{\eta}_f \in \tilde{\mathbb{T}}$ defined in the proof of Lemma 3.3, mapping to η_f by the natural projection $\tilde{\mathbb{T}} \to \mathbb{T}$. The next lemma relates the p-adic cycle $\mathfrak{c}$ to the image in $\mathscr{N}_{\mathrm{sp}}$ of the Gross point P_0.

LEMMA 5.1. *The natural images in $\mathbb{Z}[\mathscr{E}(\mathscr{G})]_{\mathrm{sp}}$ of $\mathfrak{c}$ and $\sum_{\sigma\in\Sigma} \iota P_0^\sigma$ are equal. In particular, $\eta_f \mathfrak{c}$ is equal to the image of $\sum_{\sigma\in\Sigma} \iota(\tilde{\eta}_f P_0^\sigma)$ in $\mathbb{Z}[\mathscr{E}(\mathscr{G})]$.*

Proof. (In order to visualize the geometric content of this proof, the reader may find it helpful to refer to Figure 1 in Section 3.) Set $P_i := \sigma_{\mathfrak{p}}^i P_0$, for $i = 0, \ldots, k-1$. By Lemma 3.1(1) and the definition of the action of U_p on the Bruhat-Tits tree, we can fix representatives $(\vec{e}_i, \xi_0)$ for the Gross points P_i so that the $\vec{e}_i$ are consecutive oriented edges of $\mathscr{T}$. With notation as at the beginning of this section, let $\gamma_+ \in \Gamma_+$ be the image of $\xi_0(v)$. Thus, $\gamma = \gamma_+^\iota$. Call v_0 the origin of $\vec{e}_0$. If $\iota = 1$, the even vertex of the edge $\vec{e}_{k-1}$ is equal to γv_0. If $\iota = 2$, that is, γ_+ belongs to $\Gamma_+ - \Gamma$, then

$$\vec{e}_0, \ldots, \vec{e}_{k-1}, \gamma_+\vec{e}_0, \ldots, \gamma_+\vec{e}_{k-1}$$

is a sequence of consecutive oriented edges, and the even vertex of $\gamma_+\vec{v}_{k-1}$ is equal to γv_0. Note that $\sum_{\sigma\in\Sigma} \iota P_0^\sigma$ is equal in $\mathbb{Z}[\vec{\mathscr{E}}(\mathscr{G}_+)]$ to $\vec{e}_0 + \vec{e}_1 + \cdots + \vec{e}_{k-1}$ if $\iota = 1$ and equal to

$$\vec{e}_0 + \vec{e}_1 + \cdots + \vec{e}_{k-1} + \gamma_+\vec{e}_0 + \gamma_+\vec{e}_1 + \cdots + \gamma_+\vec{e}_{k-1}$$

if $\iota = 2$. Denote by e_i the unoriented edge of $\mathscr{T}$ corresponding to $\vec{e}_i$, and let w be any element of $\Gamma_+ - \Gamma$. By the definition of the bijection κ of Lemma 2.2, the following equalities hold in $\mathbb{Z}[\mathscr{E}(\mathscr{G})]$:

$$\kappa^{-1}\big(\vec{e}_0 + \cdots + \vec{e}_{k-1}\big) = e_0 + we_1 + \cdots + e_{k-2} + we_{k-1} \quad \text{if } \iota = 1,$$

$$\kappa^{-1}\big(\vec{e}_0 + \cdots + \vec{e}_{k-1} + \gamma_+\vec{e}_0 + \gamma_+\vec{e}_1 + \cdots + \gamma_+\vec{e}_{k-1}\big) = e_0 + we_1 + \cdots + e_{k-1} + w(\gamma_+e_0) + (\gamma_+e_1) + \cdots + w(\gamma_+e_{k-1}) \quad \text{if } \iota = 2.$$

Projecting the right-hand sides of the above equalities to $\mathbb{Z}[\mathscr{E}(\mathscr{G})]_{\mathrm{sp}}$, and taking into account the fact that w acts as -1 on this module, gives in both cases $\mathrm{path}(v_0, \gamma v_0)$.

The next proposition elucidates the relation between the p-adic Shintani cycle defined above and the special values of the complex L-function of E/K. Following the notation of Section 3, fix Gross points $P_0 = P_0^1, \dots, P_0^h$ that are representatives for the Σ-orbits of the Gross points of conductor c, and list the elements of Δ so that $[\delta_i P_0^1] = [P_0^i]$, where $[P_0^i]$ denotes the Σ-orbit of P_0^i. As above, the Gross point P_0^i determines a p-adic Shintani cycle $\mathfrak{c}_i \in \bar{\Gamma}$, with $\mathfrak{c}_1 = \mathfrak{c}$. Given a complex character $\chi : \Delta \to \mathbb{C}^\times$ of Δ, set

$$\mathfrak{c}_H := \sum_{i=1}^{h} \mathfrak{c}_i \otimes \delta_i^{-1} \in \bar{\Gamma} \otimes \mathbb{Z}[\Delta],$$

$$\mathfrak{c}_{K,\chi} := \chi(\mathfrak{c}_H) = \sum_{i=1}^{h} \mathfrak{c}_i \otimes \chi(\delta_i)^{-1} \in \bar{\Gamma} \otimes \mathbb{Z}[\chi].$$

If χ is the trivial character, we also write $\mathfrak{c}_K$ as a shorthand term for $\mathfrak{c}_{K,\chi}$. Extend the pairing $\langle\ ,\ \rangle$ on $\bar{\Gamma}$ to a hermitian pairing on $\bar{\Gamma} \otimes \mathbb{Z}[\chi]$.

PROPOSITION 5.2. *Suppose that χ is primitive. The following equality holds:*

$$\left\langle \eta_f \mathfrak{c}_{K,\chi}, \mathfrak{c}_{K,\chi} \right\rangle = \frac{L(E/K, \chi, 1)}{\Omega_f} \sqrt{d} \cdot (\iota u)^2 \cdot n_f.$$

Proof. In view of Lemma 5.1, this is simply a restatement of the results of [Gr] and [Dag].

Recall the maps $j : \bar{\Gamma} \to \mathcal{N} \otimes \mathbb{Q}_p^\times$ and $\eta_f \otimes \mathrm{id} : \mathcal{N} \otimes \mathbb{C}_p^\times \to \mathbb{C}_p^\times$ defined in Section 4. By abuse of notation, we denote the maps obtained by extending scalars to $\mathbb{Z}[\chi]$ in the same way.

COROLLARY 5.3. *The equality*

$$(\eta_f \otimes \mathrm{id})\big(j(\mathfrak{c}_{K,\chi})\big) = q \otimes \rho$$

holds in $\mathbb{Q}_p^\times \otimes \mathbb{Z}[\chi]$, where $\rho \in \mathbb{Z}[\chi]$ satisfies

$$|\rho|^2 = \frac{L(E/K, \chi, 1)}{\Omega_f} \sqrt{d} \cdot (\iota u)^2 \cdot n_f.$$

Proof. By Proposition 4.4 combined with the definition of η_f given in Section 2, ρ is equal to $\langle \mathfrak{c}_{K,\chi}, e^f \rangle \in \mathbb{Z}[\chi]$. Hence

$$\begin{aligned} |\rho|^2 &= \left\langle \mathfrak{c}_{K,\chi}, e^f \right\rangle \left\langle e^f, \mathfrak{c}_{K,\chi} \right\rangle \\ &= \left\langle \eta_f \mathfrak{c}_{K,\chi}, \mathfrak{c}_{K,\chi} \right\rangle. \end{aligned}$$

The claim follows from Proposition 5.2.

Remark 5.4. Let F be a real quadratic field and let $\psi : F \to M_2(\mathbb{Q})$ be an embedding. Assume that ψ maps the ring of integers $\mathcal{O}_F$ to the Eichler order $M_0(N)$ of integral matrices with lower-left entry divisible by N. Since the homology group $H_1(X_0(N), \mathbb{Z})$ can be identified with the maximal torsion-free abelian quotient of $\Gamma_0(N)$, the image by ψ of a fundamental unit in $\mathcal{O}_F$ of norm 1 determines an integral homology cycle $\mathfrak{s} \in H_1(X_0(N), \mathbb{Z})$. Shintani [Sh] proved that the cycle $\mathfrak{s}$ encodes the special values of the classical L-series over F attached to newforms on $X_0(N)$. In light of Proposition 5.2, the element $\mathfrak{c}$ can be viewed as a p-adic analogue of the cycle $\mathfrak{s}$.

6. p-adic Shintani cycles and derivatives of p-adic L-functions. Let P_0 be a Gross point of conductor c. In Section 5, we attached to P_0 a p-adic cycle $\mathfrak{c} \in \bar{\Gamma}$, and proved in Proposition 5.2 that $\mathfrak{c}$ is related to the special values of the complex L-function of E/K. Our main result (Theorem 6.1 below) shows that $\mathfrak{c}$ is also related to the first derivative of the p-adic L-function defined in Section 3. By combining these results, we obtain Theorem 1.1.

Write $\underline{j}$ for the composite map

$$\bar{\Gamma} \xrightarrow{j} \mathcal{N} \otimes \mathbb{Q}_p^\times \to \mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Q}_p^\times \to \mathcal{N}_{\mathrm{sp}} \otimes G_\infty,$$

where the second map is induced by the natural projection of $\mathcal{N}$ onto $\mathcal{N}_{\mathrm{sp}}$, and the third map is induced by $\mathrm{rec}_p : \mathbb{Q}_p^\times \to G_\infty$. Our main result is the following.

THEOREM 6.1. *The following equality holds up to sign in $\mathcal{N}_{\mathrm{sp}} \otimes G_\infty$:*

$$\mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/H, P_0\big)^\iota = \underline{j}(\mathfrak{c}).$$

Recall the definition of the elements $\mathfrak{c}_H$ and $\mathfrak{c}_K$ given in Section 5. By Lemma 3.8, we obtain the following corollary directly.

COROLLARY 6.2. (i) *The following equality holds up to sign in $\mathcal{N}_{\mathrm{sp}} \otimes \mathbb{Z}[\Delta] \otimes G_\infty$:*

$$\mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/H\big)^\iota = \underline{j}\big(\mathfrak{c}_H\big).$$

(ii) *The following equality holds up to sign in $\mathcal{N}_{\mathrm{sp}} \otimes G_\infty$:*

$$\mathcal{L}'_p\big(\mathcal{N}_{\mathrm{sp}}/K\big)^\iota = \underline{j}\big(\mathfrak{c}_K\big).$$

By applying the operator η_f to both sides of the equalities of Corollary 6.2, and using Corollary 4.5 and the definitions of the p-adic L-functions attached to $\mathcal{N}_{\mathrm{sp}}$ and E, we find the following.

COROLLARY 6.3. (i) *The following equality holds up to sign in $\mathbb{Z}[\Delta] \otimes G_\infty$:*

$$c_p \mathcal{L}'_p(E/H)^\iota = \underline{j}\big(\eta_f \mathfrak{c}_H\big).$$

(ii) *The following equality holds up to sign in G_∞:*

$$c_p \mathscr{L}'_p(E/K)^\iota = \underline{j}\big(\eta_f \mathfrak{c}_K\big).$$

Proof of Theorem 1.1. Combine Corollary 6.3 with Corollary 5.3.

By combining Corollary 6.3 with Corollary 5.3, we also obtain the following generalization of Theorem 1.1. Let $\mathscr{L}'_p(E/K,\chi)$ stand for the element $\chi(\mathscr{L}'_p(E/H))$ of $G_\infty \otimes \mathbb{Z}[\chi]$.

THEOREM 6.4. *Suppose that χ is primitive. The following equalities hold up to sign:*

$$c_p \mathscr{L}'_p(E/K,\chi) = \mathrm{rec}_p(q) \otimes \rho \quad \text{in } G_\infty \otimes \mathbb{Z}\left[\frac{1}{2}\right][\chi]$$

and

$$\mathscr{L}'_p(E/K,\chi) = \frac{\mathrm{rec}_p(q)}{\mathrm{ord}_p(q)} \otimes \rho \quad \text{in } G_\infty \otimes \mathbb{Q}[\chi],$$

where

$$|\rho|^2 = \frac{L(E/K,\chi,1)}{\Omega_f} \cdot d^{1/2} u^2 n_f.$$

COROLLARY 6.5. *The derivative $\mathscr{L}'_p(E/K,\chi)$ is nonzero in $G_\infty \otimes \mathbb{Q}[\chi]$ if and only if the classical special value $L(E/K,\chi,1)$ is nonzero.*

Proof. By Theorem 6.4, one is reduced to showing that $\mathrm{rec}_p(q)$ is a nontorsion element of G_∞; that is, q^{p-1} does not belong to the kernel of the reciprocity map. But elements in this kernel are algebraic over $\mathbb{Q}$, and q is known to be transcendental by a result of Barré-Sirieix, Diaz, Gramain, and Philibert [B-SDGP].

Remark 6.6. Theorem 1.1 was conjectured in [BD1, Sec. 5.1] in a slightly different form. We conclude this section by studying the compatibility of Theorem 1.1 (and its generalization Theorem 6.4) with the conjectures of [BD1]. For simplicity, assume throughout this remark that the elliptic curve E is semistable so that N is squarefree, and that E is isolated in its isogeny class so that the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the ℓ-torsion points of E is irreducible for all primes ℓ.

Let $p_1 \cdots p_n q_1 \cdots q_n$ be a prime factorization of the squarefree integer pN^-, with $p_1 = p$. Denote by X_1 the Shimura curve X, and by X_{n+1} the classical modular curve $X_0(N)$. For $i = 2, \ldots, n$, denote by X_i the Shimura curve associated with an Eichler order of level $N^+ p_1 \cdots p_{i-1} q_1 \cdots q_{i-1}$ in the indefinite quaternion algebra of discriminant $p_i \cdots p_n q_i \cdots q_n$. Since E is modular, the Jacquet-Langlands correspondence [JL] implies that E is parametrized by the jacobian J_i of the curve X_i, $i = 1, \ldots, n+1$. Let

$$\phi_i : J_i \to E$$

be the *strong* Weil parametrization of E by J_i. Thus, the morphism ϕ_i has connected kernel, and its dual $\phi_i^\vee : E \to J_i$ is injective. The endomorphism $\phi_i \circ \phi_i^\vee$ of E is multiplication by an integer d_{X_i}, called the *degree* of the modular parametrization of E by the Shimura curve X_i.

If $\ell \mid N$, denote by c_ℓ the order of the group of connected components of E at ℓ.

THEOREM 6.7 (Ribet-Takahashi). *Under our assumptions*

(i)

$$\frac{d_{X_0(N)}}{d_X} = c_{p_1} \cdots c_{p_n} c_{q_1} \cdots c_{q_n};$$

(ii)

$$\left\langle e^f, e^f \right\rangle = d_X c_p.$$

Proof. Part (i) follows from Theorem 1 of [RT]. Part (ii) follows from Section 2 of [RT]. The results of [RT] exclude the case where N^+ is prime, but a forthcoming paper of Takahashi will deal with this case as well.

By combining Theorem 6.7 with the relation $\Omega_f = d_{X_0(N)} \cdot \Omega_E$, where Ω_E is the complex period of E, we find that the formula of Theorem 1.1 (and likewise for Theorem 6.4) becomes

$$\mathcal{L}'_p(E/K) = \frac{\mathrm{rec}_p(q)}{\mathrm{ord}_p(q)} \sqrt{L(E/K,1)\Omega_E^{-1} \cdot d^{1/2} u^2 \prod_{\ell \mid N^-} c_\ell^{-1}},$$

which is the same as Conjecture 5.3 of [BD1].

7. Proof of Theorem 6.1. First, we give an explicit description of certain group actions on the p-adic upper half plane and on the Bruhat-Tits tree depending on our choice of a Gross point P_0 of conductor c. Then, we compute the value $\underline{j}(\mathfrak{c})$, for $\mathfrak{c}$ as in Sections 5 and 6.

I. Group actions on $\mathcal{H}_p$ and $\mathcal{T}$. Let $K_p := K \otimes \mathbb{Q}_p$. Our choice of a prime $\mathfrak{p}$ above p determines an identification of $K_p = K_{\mathfrak{p}} \times K_{\bar{\mathfrak{p}}}$ with $\mathbb{Q}_p \times \mathbb{Q}_p$.

As in Section 5, choose a representative (R_0, ξ_0) for the Gross point P_0 such that $R_0[1/p]$ and $R[1/p]$ are equal. Let $(\vec{e}_0, \xi_0)$ be a pair corresponding to P_0, and denote by v_0 the origin of $\vec{e}_0$. Set $R_{0,p} := R_0 \otimes \mathbb{Z}_p$, and let $\underline{R}_{0,p}$ be the maximal order of B_p corresponding to v_0. Recall the isomorphism

$$\psi : B_p \to M_2(\mathbb{Q}_p)$$

fixed in Section 4. We may, and do from now on, choose ψ so that

(i) ψ maps $\underline{R}_{0,p}$ onto $M_2(\mathbb{Z}_p)$;

(ii) $\psi \circ \xi_0$ maps $(x, y) \in K_p = \mathbb{Q}_p \times \mathbb{Q}_p$ to the diagonal matrix $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$.

Condition (i) allows us to identify $\mathcal{T} = \mathbb{Q}_p^\times \underline{R}_{0,p}^\times \backslash B_p^\times$ with $\mathrm{PGL}_2(\mathbb{Z}_p) \backslash \mathrm{PGL}_2(\mathbb{Q}_p)$. Viewing $K_p^\times$ as a subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ thanks to the embedding $\psi \circ \xi_0$ yields actions of $K_p^\times$ on $\mathcal{H}_p$ and on $\mathcal{T} = \mathrm{PGL}_2(\mathbb{Z}_p) \backslash \mathrm{PGL}_2(\mathbb{Q}_p)$, factoring through $K_p^\times / \mathbb{Q}_p^\times$.

Identify this last group with $\mathbb{Q}_p^\times$ by mapping a pair (x, y) modulo $\mathbb{Q}_p^\times$ to xy^{-1}. Under this identification, an element x of $\mathbb{Q}_p^\times$ acts on $\mathcal{H}_p$ as multiplication by x, and on $\mathcal{T}$ as conjugation by the matrix $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Recall the element $v \in \mathbb{O} \subset K_p^\times$ defined in Section 5 by $\mathfrak{p}^k = (v)$. Identify, as above, v with an element $\underline{w}$ of $\mathbb{Q}_p^\times$. Note that $\underline{w}$ is equal to p^k times a p-adic unit. Set $\tilde{G}_\infty := \mathbb{Q}_p^\times = p^{\mathbb{Z}} \times \mathbb{Z}_p^\times$. Define the quotients of $\tilde{G}_\infty$,

$$\tilde{\Sigma} := \mathbb{Q}_p^\times / \mathbb{Z}_p^\times = p^{\mathbb{Z}}, \qquad \tilde{G}_n := p^{\mathbb{Z}} \times \left(\mathbb{Z}_p / p^n \mathbb{Z}_p\right)^\times, \; n \geq 1.$$

To simplify slightly the computation, assume from now on that $\mathbb{O}^\times = \{\pm 1\}$. (If $\mathbb{O}^\times \neq \{\pm 1\}$, then K has discriminant -3 or -4, and the exact sequences below have to be modified to account for the nontrivial units of $\mathbb{O}$. The computations in this case follow closely those presented in the paper.) Class field theory yields the exact sequence

$$0 \to \langle \underline{w} \rangle \to \tilde{G}_\infty \xrightarrow{\mathrm{rec}_p} G_\infty \to 0$$

and the induced sequences

$$0 \to \langle \underline{w} \rangle \to \tilde{\Sigma} \to \Sigma \to 0, \qquad 0 \to \langle \underline{w} \rangle \to \tilde{G}_n \to G_n \to 0.$$

For $n \geq 0$, denote by $\mathbb{Z}_p^{(n)} \subset \tilde{G}_\infty$ the subgroup of elements of $\mathbb{Z}_p^\times$ that are congruent to one modulo p^n.

Definition. We say that a vertex v of $\mathcal{T}$ has *level* n, and write $\ell(v) = n$, if the stabilizer of v for the action of $\tilde{G}_\infty$ is equal to $\mathbb{Z}_p^{(n)}$. Likewise, we say that an edge e of $\mathcal{T}$ has *level* n, and write $\ell(e) = n$, if the stabilizer of e for the action of $\tilde{G}_\infty$ is $\mathbb{Z}_p^{(n)}$.

Note that the group $\tilde{G}_n$ ($\tilde{\Sigma}$ if $n = 0$) acts simply transitively on the vertices and edges of level n. By definition of the action of $\tilde{G}_\infty$ on $\mathcal{T}$, v_0 is a vertex of level 0. Thus, the set of vertices of level 0 is equal to the $\tilde{\Sigma}$-orbit of v_0. More generally, the set of vertices of level n can be described as the $\tilde{G}_n$-orbit of a vertex v_n whose distance from v_0 is n and whose distance from all the other vertices in the orbit $\tilde{\Delta} v_0$ is $> n$.

By using the standard coordinate, identify $\mathbb{P}^1(\mathbb{C}_p)$ with $\mathbb{C}_p \cup \{\infty\}$ and $\mathcal{H}_p$ with $\mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$. In particular, view 0 and ∞ as elements of $\mathbb{P}^1(\mathbb{Q}_p)$. Recall the element $\gamma = \gamma(P_0)$ of Γ defined in Section 5. Since the reduced norm of γ has positive valuation, our choice of the isomorphism ψ yields

$$\lim_{n \to +\infty} \gamma^n z = 0, \qquad \lim_{n \to -\infty} \gamma^n z = \infty \tag{4}$$

for all $z \in \mathcal{H}_p$. Note also that 0 and ∞ are the fixed points for the action of $\tilde{G}_\infty$ on $\mathbb{P}^1(\mathbb{C}_p)$.

Let $\mathcal{H}_p(\mathbb{Q}_{p^2}) = \mathbb{Q}_{p^2} - \mathbb{Q}_p$ be the $\mathbb{Q}_{p^2}$-points of the p-adic upper half plane. Define the *reduction map*

$$r : \mathcal{H}_p\left(\mathbb{Q}_{p^2}\right) \to \mathcal{V}(\mathcal{T})$$

as follows. Given $z \in \mathcal{H}_p(\mathbb{Q}_{p^2})$, let $\mathcal{Q}_z$ denote the stabilizer of z in $\mathrm{GL}_2(\mathbb{Q}_p)$, together with the zero matrix. Then $\mathcal{Q}_z$ is a field isomorphic to $\mathbb{Q}_{p^2}$, and this gives rise to an embedding of $\mathbb{Q}_{p^2}$ in $M_2(\mathbb{Q}_p)$ (well defined up to an isomorphism of $\mathbb{Q}_{p^2}$). Write $\mathbb{Z}_{p^2}$ for the ring of integers of $\mathbb{Q}_{p^2}$, and let S be the unique maximal order of $M_2(\mathbb{Q}_p)$ containing the image of $\mathbb{Z}_{p^2}$ by the above embedding. We have $r(z) = S$. (See also [BD2, Sec. 1].)

LEMMA 7.1. (1) *The reduction map r is $\mathrm{GL}_2(\mathbb{Q}_p)$-equivariant. In particular, r is equivariant for the group actions defined above.*

(2) *Write $\mathbb{Z}_{p^2} = \mathbb{Z}_p\alpha + \mathbb{Z}_p$. We have $r^{-1}(v_0) = \mathbb{Z}_p^\times\alpha + \mathbb{Z}_p$.*

(3) *If z_1 and z_2 are mapped by r to adjacent vertices of respective levels n and $n+1$, then $z_1 z_2^{-1} \equiv 1 \pmod{p^n}$.*

Proof. (1) Let z be an element of $\mathcal{H}_p(\mathbb{Q}_{p^2})$, and let B be a matrix in $\mathrm{GL}_2(\mathbb{Q}_p)$. If $f : \mathbb{Q}_{p^2} \to M_2(\mathbb{Q}_p)$ is an embedding fixing z, then BfB^{-1} is an embedding fixing Bz. Suppose that S is the maximal ideal containing $f(\mathbb{Z}_{p^2})$. Then $BSB^{-1} = S * B^{-1}$ is the maximal ideal containing the image of $\mathbb{Z}_{p^2}$ by BfB^{-1}. Thus, $r(Bz) = S * B^{-1}$, as was to be shown.

(2) Assume that p is greater than 2. Then, we may assume that $\alpha = \sqrt{\nu}$, where the integer ν is not a square modulo p. (The case where $p = 2$ can be dealt with in a similar way—for instance, by taking $\alpha = (1+\sqrt{-3})/2$.) A direct computation shows that

$$\mathcal{Q}_{\sqrt{\nu}} = \left\{ \begin{pmatrix} b & a\nu \\ a & b \end{pmatrix} : a, b \in \mathbb{Q}_p \right\}.$$

Mapping the above matrix to $a\sqrt{\nu}+b$ yields an isomorphism of $\mathcal{Q}_{\sqrt{\nu}}$ onto $\mathbb{Q}_{p^2}$. Thus, $r(\sqrt{\nu})$ is equal to $v_0 = M_2(\mathbb{Z}_p)$. Given $z = a\sqrt{\nu}+b \in \mathcal{H}_p(\mathbb{Q}_{p^2})$, we have $z = B\sqrt{\nu}$, where B is the matrix $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$. By (1),

$$r(z) = BM_2(\mathbb{Z}_p)B^{-1}.$$

But $BM_2(\mathbb{Z}_p)B^{-1} = M_2(\mathbb{Z}_p)$ if and only if B belongs to $\mathrm{GL}_2(\mathbb{Z}_p)$, that is, a belongs to $\mathbb{Z}_p^\times$.

(3) Set $r(z_1) = v_1$ and $r(z_2) = v_2$. The edge joining v_1 to v_2 has level $n+1$. Since $\tilde{G}_\infty = \mathbb{Q}_p^\times$ acts transitively on the edges of level $n+1$, there is $g \in \mathbb{Q}_p^\times$ such that gv_1 and gv_2 have distance from v_0 equal to n and $n+1$, respectively. With notation as in the proof of (2) of this proposition, write $gz_i = a_i\sqrt{\nu}+b_i$, $i = 1, 2$, where $a_i, b_i \in \mathbb{Z}_p$, $gcd(a_i, b_i) = 1$, and $p^n \parallel a_1$, $p^{n+1} \parallel a_2$. Thus, the vertex gv_i is represented by the matrix

$$A_i = \begin{pmatrix} a_i & b_i \\ 0 & 1 \end{pmatrix}.$$

Our assumption on gv_1 and gv_2 implies that the column $\left(\begin{smallmatrix} b_2 \\ 1 \end{smallmatrix}\right)$ of A_2 is a $\mathbb{Z}_p$-linear combination of the columns of A_1. It follows that $b_1 \equiv b_2 \pmod{p^n}$, and hence

$$z_1 z_2^{-1} = g z_1 \big(g z_2\big)^{-1} \equiv 1 \big(\mathrm{mod}\, p^n\big).$$

II. The calculation. Given $\delta \in \Gamma$, write as usual $\bar{\delta}$ for the natural image of δ in $\bar{\Gamma}$. We now compute explicitly the value of $j(\mathfrak{c})(\bar{\delta}) = [\mathfrak{c}, \bar{\delta}]$, for $\delta \in \Gamma$. We begin with the following lemma.

LEMMA 7.2. *Given $\delta \in \Gamma$, we have*

$$j(\mathfrak{c})(\bar{\delta}) = \prod_{\epsilon \in \mathcal{S}} \frac{\epsilon \delta z_0}{\epsilon z_0} ,$$

where z_0 is any element in $\mathcal{H}_p$, and $\mathcal{S}$ is any set of representatives for $\langle \gamma \rangle \backslash \Gamma$.

Proof (Cf. [M, Th. 2.8]). Let $\mathcal{S}'$ be any set of representatives for $\Gamma / \langle \gamma \rangle$. In view of the formulae (4), for any z_0 and a in $\mathcal{H}_p$ we have the chain of equalities

$$\begin{aligned}
j(\mathfrak{c})(\bar{\delta}) &= \prod_{\epsilon \in \Gamma} \frac{z_0 - \epsilon a}{z_0 - \epsilon \gamma a} \cdot \frac{\delta z_0 - \epsilon \gamma a}{\delta z_0 - \epsilon a} \\
&= \prod_{\epsilon \in \mathcal{S}'} \prod_{n=-\infty}^{+\infty} \frac{z_0 - \epsilon \gamma^n a}{z_0 - \epsilon \gamma^{n+1} a} \cdot \frac{\delta z_0 - \epsilon \gamma^{n+1} a}{\delta z_0 - \epsilon \gamma^n a} \\
&= \prod_{\epsilon \in \mathcal{S}'} \lim_{N \to +\infty} \frac{z_0 - \epsilon \gamma^{-N} a}{z_0 - \epsilon \gamma^{N+1} a} \cdot \frac{\delta z_0 - \epsilon \gamma^{N+1} a}{\delta z_0 - \epsilon \gamma^{-N} a} \\
&= \prod_{\epsilon \in \mathcal{S}'} \frac{z_0 - \epsilon \infty}{z_0 - \epsilon 0} \cdot \frac{\delta z_0 - \epsilon 0}{\delta z_0 - \epsilon \infty} \\
&= \prod_{\epsilon \in \mathcal{S}'} \frac{\epsilon^{-1} \delta z_0}{\epsilon^{-1} z_0} .
\end{aligned}$$

Note that $(\mathcal{S}')^{-1}$ is a set of representatives for $\langle \gamma \rangle \backslash \Gamma$, and any set of representatives for $\langle \gamma \rangle \backslash \Gamma$ can be obtained in this way. The claim follows.

LEMMA 7.3. *Let d be an edge of $\mathcal{T}$, let n be a positive integer, and let $\mathcal{S}$ be a set of representatives for $\langle \gamma \rangle \backslash \Gamma$. Then the set $\{\epsilon \in \mathcal{S} : \ell(\epsilon d) \leq n\}$ is finite.*

Proof. If $\{\epsilon_i\}$ is a sequence of distinct elements of $\mathcal{S}$ such that $\ell(\epsilon_i d) \leq n$, we can find integers k_i such that $\gamma^{k_i} \epsilon_i d$ describes only finitely many edges. This contradicts the discreteness of Γ.

We say that two elements of $\bar{\Gamma}$ are *linearly independent* if they generate a rank-two free abelian subgroup of $\bar{\Gamma}$.

PROPOSITION 7.4. (1) *Suppose that $\mathfrak{c}$ and $\bar{\delta}$ are linearly independent in $\bar{\Gamma}$. There exists a set $\mathcal{S}$ of representatives for $\langle \gamma \rangle \backslash \Gamma$ such that if ϵ belongs to $\mathcal{S}$, then all the elements of the coset $\epsilon \langle \delta \rangle$ belong to $\mathcal{S}$.*

(2) *There exists a set* $\mathcal{S} = \mathcal{S}_0 \coprod \mathcal{S}_1$ *of representatives for* $\langle\gamma\rangle\backslash\Gamma$ *such that*

(i) *the set* $\mathcal{S}_0$ *contains a finite number of elements that are mapped by the isomorphism* ψ *to diagonal matrices of* $\mathrm{PGL}_2(\mathbb{Q}_p)$*;*

(ii) *if* ϵ *belongs to* $\mathcal{S}_1$*, then all the elements of the coset* $\epsilon\langle\gamma\rangle$ *belong to* $\mathcal{S}_1$.

Proof (Cf. [M, Lemma 2.7]). (1) Consider a decomposition of Γ as a disjoint union of double cosets

$$\Gamma = \coprod_{\bar{\epsilon}\in\bar{\mathcal{S}}} \langle\gamma\rangle\bar{\epsilon}\langle\delta\rangle.$$

We claim that we may take $\mathcal{S}$ to be $\{\bar{\epsilon}\delta^m : \bar{\epsilon}\in\bar{\mathcal{S}},\ m\in\mathbb{Z}\}$. For, if $\bar{\epsilon}\delta^m = \gamma^r\bar{\epsilon}\delta^n$, we find $\delta^{m-n} = \bar{\epsilon}^{-1}\gamma^r\bar{\epsilon}$. Projecting this relation to $\bar{\Gamma}$ gives $m = n$.

(2) Consider a decomposition of Γ as a disjoint union of double cosets

$$\Gamma = \coprod_{\bar{\epsilon}\in\bar{\mathcal{S}}} \langle\gamma\rangle\bar{\epsilon}\langle\gamma\rangle.$$

Define $\mathcal{S}_1$ to be the set of elements of Γ of the form $\bar{\epsilon}\gamma^m$, $m\in\mathbb{Z}$, where $\bar{\epsilon}\in\bar{\mathcal{S}}$ is such that $\langle\gamma\rangle\bar{\epsilon}\gamma^n \neq \langle\gamma\rangle\bar{\epsilon}\gamma^m$ whenever $m\neq n$. As for $\mathcal{S}_0$, we claim that it can be taken to be the set of elements $\bar{\epsilon}\in\bar{\mathcal{S}}$ that do not satisfy the above condition. In such a case, there is a relation $\gamma^r\bar{\epsilon}\gamma^n = \bar{\epsilon}\gamma^m$ for integers r and $m\neq n$. Then, $\gamma^r = \bar{\epsilon}\gamma^{m-n}\bar{\epsilon}^{-1}$. By projecting this equality to $\bar{\Gamma}$, we see that $m-n = r$, and hence $\bar{\epsilon}$ and γ^r commute. Since γ^r is mapped by ψ to the diagonal matrix $\begin{pmatrix} \underline{w}^{tr} & 0 \\ 0 & 1\end{pmatrix}$, where $\mathrm{ord}_p(\underline{w}) = k > 0$, a direct computation shows that $\bar{\epsilon}$ is also diagonal (and thus commutes with γ). Now consider the group of all the diagonal matrices in $\psi(\Gamma)$. Since Γ is discrete, this group is the product of a finite group by a cyclic group containing the group generated by γ. In conclusion, the set $\mathcal{S}_0$ is finite, and

$$\coprod_{\bar{\epsilon}\in\mathcal{S}_0} \langle\gamma\rangle\bar{\epsilon}\langle\gamma\rangle = \coprod_{\bar{\epsilon}\in\mathcal{S}_0} \langle\gamma\rangle\bar{\epsilon}.$$

The claim follows.

In the computation of $\underline{j}(\mathfrak{c})(\bar{\delta})$, we can assume that either

(I) $\mathfrak{c}$ and $\bar{\delta}$ are linearly independent; or

(II) $\bar{\delta} = \mathfrak{c}$.

(In fact, if the rank of $\bar{\Gamma}$ is greater than 1, it is enough to consider elements as in the first case, since the linear map $j(\mathfrak{c})$ is completely determined by the values $j(\mathfrak{c})(\bar{\delta})$, for $\mathfrak{c}$ and $\bar{\delta}$ linearly independent.) In case (I), we use the notation $\mathcal{S}_1 := \mathcal{S}$, and the symbol $\mathcal{S}_1$ always refers to a choice of representatives for $\langle\gamma\rangle\backslash\Gamma$ as in Proposition 7.4(1). In case (II), the symbol $\mathcal{S} = \mathcal{S}_0 \coprod \mathcal{S}_1$ stands for a choice of representatives as in Proposition 7.4(2).

LEMMA 7.5. *Let* $\delta\in\Gamma$ *be as in case (I) or (II) above. Then, the images in* G_∞ *by the reciprocity map of* $j(\mathfrak{c})(\bar{\delta})$ *and* $\prod_{\epsilon\in\mathcal{S}_1} \epsilon\delta z_0/\epsilon z_0$ *are equal.*

Proof. In case (I), there is nothing to prove. In case (II), Proposition 7.4 combined with a direct computation shows that

$$\prod_{\epsilon\in\mathcal{S}_0}\frac{\epsilon\gamma z_0}{\epsilon z_0}=\underline{w}^{\iota\#(\mathcal{S}_0)}.$$

Since $\underline{w}$ is in the kernel of the reciprocity map, the claim follows.

By Lemma 7.5, we are now reduced to computing the product $\prod_{\epsilon\in\mathcal{S}_1}\epsilon\delta z_0/\epsilon z_0$, with δ as in case (I) or (II).

We begin with some preliminary remarks. Fix an edge e of level equal to an odd integer n, having v as its vertex of level n. Moreover, assume that the distance of v from v_0 is also equal to n. Note that the image in $\mathcal{M}$ of e is equal to the image in $\mathcal{M}$ of a Gross point of conductor cp^n.

Given $\tilde{\sigma}\in\tilde{G}_n$, define $\mu_{\tilde{\sigma}}$ to be equal to 1 (respectively, -1) if $\tilde{\sigma}v$ has odd (respectively, even) distance from v_0. If $\iota=1$, observe that $\mu_{\tilde{\sigma}}$ depends only on the image $\bar{\sigma}$ of $\tilde{\sigma}$ in Σ under the projection induced by the reciprocity map; in this case, we write $\mu_{\bar{\sigma}}$ instead of $\mu_{\tilde{\sigma}}$. If $\iota=2$, then $\mu_{\tilde{\sigma}}$ is constant on the elements $\tilde{\sigma}$ that have the same image in Σ and p-adic valuation of the same parity; moreover, the values of $\mu_{\tilde{\sigma}}$ corresponding to different parities are opposite. In this case, if $\tilde{\sigma}$ projects in Σ to $\bar{\sigma}$ and $\mathrm{ord}_p(\tilde{\sigma})$ is even, we let $\mu_{\bar{\sigma}}$ stand for $\mu_{\tilde{\sigma}}$.

Given an edge d of $\mathcal{T}$, and $\tilde{\sigma}\in\tilde{G}_n$, write $\tilde{\sigma}e\equiv d$ if the edge $\tilde{\sigma}e$ is $\mathcal{S}_1$-equivalent to d, and $\sigma e\approx d$ if the element σe of $\mathcal{M}$ is Γ-equivalent to d. If $\iota=1$, the relation $\tilde{\sigma}e\equiv d$ implies that $\sigma e\approx d$. If $\iota=2$, $\tilde{\sigma}e\equiv d$ yields $\sigma e\approx d$ when $\mathrm{ord}_p(\tilde{\sigma})$ is even, and $\sigma e\approx wd$, with $w\in\Gamma_+-\Gamma$, when $\mathrm{ord}_p(\tilde{\sigma})$ is odd.

Recall that ω_d denotes the order of the stabilizer in Γ of d.

LEMMA 7.6. (1) *Suppose that $\iota=1$. If the odd integer n is sufficiently large, the projection $\tilde{G}_n\to G_n$ induces a ω_d-to-1 map*

$$\left\{\tilde{\sigma}\in\tilde{G}_n:\tilde{\sigma}e\equiv d\right\}\to\{\sigma\in G_n:\sigma e\approx d\}.$$

(2) *Suppose that $\iota=2$. If the odd integer n is sufficiently large, the projection $\tilde{G}_n\to G_n$ induces ω_d-to-1 maps*

$$\left\{\tilde{\sigma}\in\tilde{G}_n:\tilde{\sigma}e\equiv d,\ \mathrm{ord}_p(\tilde{\sigma})\ \textit{even}\right\}\to\{\sigma\in G_n:\sigma e\approx d\}$$

and

$$\{\tilde{\sigma}\in\tilde{G}_n:\tilde{\sigma}e\equiv d,\ \mathrm{ord}_p(\tilde{\sigma})\ \textit{odd}\}\to\{\sigma\in G_n:\sigma e\approx wd\}.$$

Proof. (1) Suppose that $\tilde{\sigma}_1e\equiv d$ and $\tilde{\sigma}_2e\equiv d$; that is, $\tilde{\sigma}_1e=\epsilon_1d$ and $\tilde{\sigma}_2e=\epsilon_2d$, for ϵ_1 and ϵ_2 in $\mathcal{S}_1$. If $\tilde{\sigma}_1$ and $\tilde{\sigma}_2$ have the same image in G_n, then $\tilde{\sigma}_1=\underline{w}^r\tilde{\sigma}_2$ for $r\in\mathbb{Z}$, and hence $\gamma^r\epsilon_2d=\epsilon_1d$. If $r\neq0$, that is, $\tilde{\sigma}_1\neq\tilde{\sigma}_2$ and $\epsilon_1\neq\epsilon_2$, then $\gamma^r\epsilon_2\epsilon_1^{-1}$ is a nontrivial element of the stabilizer in Γ of ϵ_1d, which is a group of cardinality ω_d. Conversely, if $\tilde{\sigma}_1e=\epsilon_1d$ for $\epsilon_1\in\mathcal{S}_1$ and if β is a nontrivial element of the stabilizer of $\epsilon_1 d$, we have $\tilde{\sigma}_1 e = \beta\epsilon_1 d$. Write $\beta\epsilon_1 = \gamma^r\epsilon_2$, $r \in \mathbb{Z}$, $\epsilon_2 \in \mathcal{S}$. Then $\epsilon_1 \neq \epsilon_2$. Note that if n is large, then ϵ_2 belongs to $\mathcal{S}_1$. We obtain $\underline{w}^{-r}\tilde{\sigma}_1 e = \epsilon_2 d$. This concludes the proof of part (1).

(2) The proof is exactly the same as that of part (1). Let

$$\operatorname{path}(v_0, \delta v_0) = d_1 - d_2 + \cdots + d_{s-1} - d_s \in \mathbb{Z}[\mathcal{E}(\mathcal{T})].$$

(Note that s is even, since δ belongs to Γ.) Write $d_j = \{v_j^e, v_j^o\}$, where v_j^e is the even vertex of d_j, and v_j^o is the odd vertex of d_j. Note that we have

$$\begin{aligned} v_j^o &= v_{j+1}^o \quad \text{for } j = 1, 3, \ldots, s-1,\\ v_j^e &= v_{j+1}^e \quad \text{for } j = 2, 4, \ldots, s-2,\\ v_s^e &= \delta v_1^e. \end{aligned}$$

Fix $z_0 \in \mathcal{H}_p(\mathbb{Q}_{p^2})$ such that $r(z_0) = v_0$. We may choose elements z_j^o and z_j^e in $\mathcal{H}_p(\mathbb{Q}_{p^2})$ such that $r(z_j^o) = v_j^o$, $r(z_j^e) = v_j^e$, and

$$\begin{aligned} z_j^o &= z_{j+1}^o \quad \text{for } j = 1, 3, \ldots, s-1,\\ z_j^e &= z_{j+1}^e \quad \text{for } j = 2, 4, \ldots, s-2,\\ z_1^e &= z_0, \qquad z_s^e = \delta z_0. \end{aligned}$$

Hence

$$\left(\epsilon z_1^o\right)\left(\epsilon z_2^o\right)^{-1}\cdots\left(\epsilon z_{s-1}^o\right)\left(\epsilon z_s^o\right)^{-1} = 1, \qquad \left(\epsilon z_2^e\right)\left(\epsilon z_3^e\right)^{-1}\cdots\left(\epsilon z_{s-2}^e\right)\left(\epsilon z_{s-1}^e\right)^{-1} = 1,$$

so that

$$\prod_{\epsilon\in\mathcal{S}_1}\frac{\epsilon\delta z_0}{\epsilon z_0} = \prod_{\epsilon\in\mathcal{S}_1}\left(\frac{\epsilon z_1^o}{\epsilon z_1^e}\right)\left(\frac{\epsilon z_2^o}{\epsilon z_2^e}\right)^{-1}\cdots\left(\frac{\epsilon z_s^o}{\epsilon z_s^e}\right)^{-1}.$$

Fix a large *odd* integer n. For each $1 \le j \le s$, let $\mathcal{S}(j)$ be the set of elements ϵ in $\mathcal{S}_1$ such that ϵd_j has level less than or equal to n. Lemma 7.3 shows that the sets $\mathcal{S}(j)$ are finite. By Lemma 7.1, we have the congruence

$$\prod_{\epsilon\in\mathcal{S}_1}\frac{\epsilon\delta z_0}{\epsilon z_0} \equiv \prod_{\epsilon\in\mathcal{S}(1)}\left(\frac{\epsilon z_1^o}{\epsilon z_1^e}\right)\prod_{\epsilon\in\mathcal{S}(2)}\left(\frac{\epsilon z_2^o}{\epsilon z_2^e}\right)^{-1}\cdots\prod_{\epsilon\in\mathcal{S}(s)}\left(\frac{\epsilon z_s^o}{\epsilon z_s^e}\right)^{-1} \pmod{p^n}. \tag{5}$$

Each of the factors in the right-hand side of equation (5) can be broken up into three contributions:

$$\prod_{\mathcal{S}(j)}\frac{\epsilon z_j^o}{\epsilon z_j^e} = \prod_{\ell(\epsilon v_j^o)<n}\epsilon z_j^o \cdot \prod_{\ell(\epsilon v_j^e)<n}(\epsilon z_j^e)^{-1}\cdot\prod_{\ell(\epsilon d_j)=n}(\epsilon z_j^{\pi_j})^{\mu_j},$$

where $\pi_j = o$ (respectively, $\pi_j = e$) if the distance of the furthest vertex of ϵd_j from v_0 is odd (respectively, even) and where we set $\mu_j = 1$ in the first case and $\mu_j = -1$

in the second case. By our choice of the set $\mathcal{S}_1$ as in Proposition 7.4, the first two factors in this last expression cancel out in formula (5). Hence we obtain

$$\prod_{\epsilon\in\mathcal{S}_1}\frac{\epsilon\delta z_0}{\epsilon z_0}\equiv\prod_{\ell(\epsilon d_1)=n}\left(\epsilon z_1^{\pi_1}\right)^{\mu_1}\cdot\prod_{\ell(\epsilon d_2)=n}\left(\epsilon z_2^{\pi_2}\right)^{-\mu_2}\cdots\prod_{\ell(\epsilon d_s)=n}\left(\epsilon z_s^{\pi_s}\right)^{-\mu_s}(\mathrm{mod}\, p^n).$$

As in the remarks before Lemma 7.6, let e be an edge of level n such that its vertex v of level n has distance from v_0 also equal to n. Choose any $z\in\mathcal{H}_p(\mathbb{Q}_{p^2})$ with $r(z)=v$. Since $\tilde{G}_n$ acts simply transitively on the set of edges of level n, Lemma 7.1 gives

$$\prod_{\epsilon\in\mathcal{S}_1}\frac{\epsilon\delta z_0}{\epsilon z_0}\equiv\prod_{\tilde{\sigma}e\equiv d_1}\left(\tilde{\sigma}z\right)^{\mu_{\tilde{\sigma}}}\cdot\prod_{\tilde{\sigma}e\equiv d_2}\left(\tilde{\sigma}z\right)^{-\mu_{\tilde{\sigma}}}\cdots\prod_{\tilde{\sigma}e\equiv d_s}\left(\tilde{\sigma}z\right)^{-\mu_{\tilde{\sigma}}}(\mathrm{mod}\, p^n).$$

By Lemma 7.6, we obtain

$$\prod_{\epsilon\in\mathcal{S}_1}\frac{\epsilon\delta z_0}{\epsilon z_0}\equiv\prod_{\tilde{\sigma}e\equiv d_1}\tilde{\sigma}^{\mu_{\tilde{\sigma}}}\cdot\prod_{\tilde{\sigma}e\equiv d_2}\tilde{\sigma}^{-\mu_{\tilde{\sigma}}}\cdots\prod_{\tilde{\sigma}e\equiv d_s}\tilde{\sigma}^{-\mu_{\tilde{\sigma}}}\cdot\left(z^M\right)(\mathrm{mod}\, p^n),$$

where

$$M=\begin{cases}\langle\mathrm{path}(v_0,\delta v_0),\sum_{\sigma\in G_n}\mu_{\bar{\sigma}}\sigma e\rangle & \text{if } \iota=1,\\ \langle\mathrm{path}(v_0,\delta v_0),\sum_{\sigma\in G_n}(\mu_{\bar{\sigma}}-\mu_{\bar{\sigma}}w)\sigma e\rangle & \text{if } \iota=2.\end{cases}$$

By Lemma 2.3, the duality $\langle\ ,\ \rangle$ induces a pairing on $H_1(\mathcal{G},\mathbb{Z})\times\mathcal{M}$. In the case $\iota=1$, one sees directly that $\sum_{\sigma\in G_n}\mu_{\bar{\sigma}}\sigma e$ has trivial image in $\mathcal{M}$, so that M is zero. Consider now the case $\iota=2$. Since we are interested in computing $\underline{j}(\mathfrak{c})(\bar{\delta})$, we need only consider the image of the homomorphism $j(\mathfrak{c})$ in $\mathcal{N}_{\mathrm{sp}}\otimes\mathbb{Q}_p^\times$. Thus, we may view the above pairing as being defined on $H_1(\mathcal{G},\mathbb{Z})^-\times\mathcal{M}_{\mathrm{sp}}$, where $H_1(\mathcal{G},\mathbb{Z})^-$ indicates the "minus" eigenspace for the action of w on $H_1(\mathcal{G},\mathbb{Z})$, and we may assume from now on that $\mathrm{path}(v_0,\delta v_0)$ belongs to $H_1(\mathcal{G},\mathbb{Z})^-$. One checks that the image $\iota\sum_{\sigma\in G_n}\mu_{\bar{\sigma}}\sigma e$ in $\mathcal{M}_{\mathrm{sp}}$ of the element $\sum_{\sigma\in G_n}(\mu_{\bar{\sigma}}-w\mu_{\bar{\sigma}})\sigma e$ is trivial, so that also in this case, M is zero. Hence, in all cases,

$$\prod_{\epsilon\in\mathcal{S}_1}\frac{\epsilon\delta z_0}{\epsilon z_0}\equiv\prod_{\tilde{\sigma}e\equiv d_1}\tilde{\sigma}^{\mu_{\tilde{\sigma}}}\cdot\prod_{\tilde{\sigma}e\equiv d_2}\tilde{\sigma}^{-\mu_{\tilde{\sigma}}}\cdots\prod_{\tilde{\sigma}e\equiv d_s}\tilde{\sigma}^{-\mu_{\tilde{\sigma}}}(\mathrm{mod}\, p^n).$$

Let $\mathrm{rec}_{p,n}:\tilde{G}_\infty\to G_n$ be the composite of the reciprocity map with the natural projection of G_∞ onto G_n. Suppose that $\iota=1$. By Lemma 7.6, the above relation yields the equality in G_n:

$$\mathrm{rec}_{p,n}\left(\prod_{\epsilon\in\mathcal{S}_1}\frac{\epsilon\delta z_0}{\epsilon z_0}\right)=\prod_{\sigma e\approx d_1}\sigma^{\omega_{d_1}\mu_{\bar{\sigma}}}\cdot\prod_{\sigma e\approx d_2}\sigma^{-\omega_{d_2}\mu_{\bar{\sigma}}}\cdots\prod_{\sigma e\approx d_s}\sigma^{-\omega_{d_s}\mu_{\bar{\sigma}}}.$$

Recall the derivative $\mathscr{L}'_{p,n}(\mathscr{N}_{\mathrm{sp}}/H, P_0) \in \mathscr{N}_{\mathrm{sp}} \otimes G_n$ defined in formula (3) at the end of Section 3. By the definition of the bijection κ of Lemma 2.2, the right-hand side of the above equality can be written as

$$\mathscr{L}'_{p,n}\big(\mathscr{N}_{\mathrm{sp}}/H, P_0\big)\big(\bar{\delta}\big) = \langle \mathrm{path}(v_0, \delta v_0), \sum_{g \in G_n} e_n(i)^g \otimes g^{-1} \rangle,$$

where, by an abuse of notation, $\sum_{g\in G_n} e_n(i)^g \otimes g^{-1}$ is viewed as an element of $\mathscr{M}_{\mathrm{sp}} \otimes G_n$. When $\iota = 2$, a similar computation shows that

$$\iota \mathscr{L}'_{p,n}\big(\mathscr{N}_{\mathrm{sp}}/H, P_0\big)\big(\bar{\delta}\big) = \mathrm{rec}_{p,n}\left(\prod_{\epsilon \in \mathscr{S}_1} \frac{\epsilon \delta z_0}{\epsilon z_0}\right).$$

By passing to the limit, one obtains in all cases

$$\iota \mathscr{L}'_{p}\big(\mathscr{N}_{\mathrm{sp}}/H, P_0\big)\big(\bar{\delta}\big) = \mathrm{rec}_{p}\left(\prod_{\epsilon \in \mathscr{S}_1} \frac{\epsilon \delta z_0}{\epsilon z_0}\right).$$

In other words, by definition of the map $\underline{j}$,

$$\mathscr{L}'_{p}\big(\mathscr{N}_{\mathrm{sp}}/H, P_0\big)^{\iota} = \underline{j}(\mathfrak{c}),$$

as was to be shown.

References

[B-SDGP] K. Barré-Sirieix, G. Diaz, F. Gramain, and G. Philibert, *Une preuve de la conjecture de Mahler-Manin*, Invent. Math. **124** (1996), 1–9.

[BD1] M. Bertolini and H. Darmon, *Heegner points on Mumford-Tate curves*, Invent. Math. **126** (1996), 413–456.

[BD2] ———, *Heegner points, p-adic L-functions, and the Cerednik-Drinfeld uniformization*, Invent. Math. **131** (1998), 453–491.

[Boi] X. Boichut, *On the Mazur-Tate-Teitelbaum p-adic conjecture for elliptic curves with bad reduction at p*, in preparation.

[BC] J-F. Boutot and H. Carayol, "Uniformisation p-adique des courbes de Shimura: les théorèmes de Cerednik et de Drinfeld" in *Courbes modulaires et courbes de Shimura (Orsay, 1987/1988)*, Astérisque **196–197**, Soc. Math. France, Marseille, 1992, 45–158.

[C] I. V. Cherednik, *Uniformization of algebraic curves by discrete arithmetic subgroups of* $\mathrm{PGL}_2(k_w)$ *with compact quotient spaces* (in Russian), Math. Sb. (N.S.) **100** (1976), 59–88; English trans. in Math. USSR-Sb. **29** (1976), 55–78.

[Dag] H. Daghigh, *Modular forms, quaternion algebras, and special values of L-functions*, McGill University, Ph.D. thesis, 1997.

[Dr] V. G. Drinfeld, *Coverings of p-adic symmetric regions*, Funct. Anal. Appl. **10** (1976), 29–40.

[GvdP] L. Gerritzen and M. van der Put, *Schottky Groups and Mumford Curves*, Lecture Notes in Math. **817**, Springer-Verlag, Berlin, 1980.

[GS] R. Greenberg and G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math. **111** (1993), 407–447.

[Gr] B. H. Gross, "Heights and the special values of L-series" in *Number Theory (Montreal, 1985)*, H. Kisilevsky and J. Labute, eds., CMS Conf. Proc. **7**, Amer. Math. Soc., Providence, 1987, 115–187.

[GZ] B. H. Gross and D. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225–320.

[JL] H. Jacquet and R. P. Langlands, *Automorphic Forms on* GL_2, Lecture Notes in Math. **114**, Springer-Verlag, Berlin, 1970.

[KKT] K. Kato, M. Kurihara, and Tsuji, forthcoming work.

[Kl] C. Klingenberg, "On p-adic L-functions on Mumford curves" in *p-adic Monodromy and the Birch and Swinnerton-Dyer Conjecture (Boston, 1991)*, B. Mazur and G. Stevens, eds., Contemp. Math. **165**, Amer. Math. Soc., Providence, 1991, 277–315.

[M] Y. I. Manin, *p-adic automorphic functions*, J. Soviet Math. **5** (1976), 279–333.

[MTT] B. Mazur, J. Tate, and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton-Dyer*, Invent. Math. **84** (1986), 1–48.

[RT] K. Ribet and S. Takahashi, *Parametrizations of elliptic curves by Shimura curves and by classical modular curves*, Proc. Nat. Acad. Sci. U.S.A. **94** (1997), 11110–11114.

[Sch] P. Schneider, "Rigid-analytic L-transforms" in *Number Theory (Noordwijkerhout, 1983)*, Lecture Notes in Math. **1068**, Springer-Verlag, Berlin, 1984.

[Se] J-P. Serre, *Arbres, amalgames, SL_2*, Astérisque **46**, Soc. Math. France, Marseille, 1977.

[Sh] T. Shintani, *On the construction of holomorphic cusp forms of half integral weight*, Nagoya Math. J. **58** (1975), 83–126.

[T] J. Teitelbaum, *Values of p-adic L-functions and a p-adic Poisson kernel*, Invent. Math. **101** (1990), 395–410.

[Vi] M-F.Vigneras, *Arithmétique des algèbres des quaternions*, Lecture Notes in Math. **800**, Springer-Verlag, Berlin, 1980.

Bertolini: Dipartimento di Matematica, Università di Pavia, Strada Ferrata 1, 27100 Pavia, Italy

Darmon: Department of Mathematics, McGill University, 805 Sherbrooke Street West, Montreal, Quebec H3A-2K6, Canada