ON THE EQUATIONS $z^m = F(x,y)$ AND $Ax^p + By^q = Cz^r$

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Abstract

We investigate integer solutions of the *superelliptic equation*

$$z^m = F(x,y), \tag{1}$$

where F is a homogeneous polynomial with integer coefficients, and of the *generalized Fermat equation*

$$Ax^p + By^q = Cz^r, \tag{2}$$

where A, B and C are non-zero integers. Call an integer solution (x,y,z) to such an equation *proper* if $\gcd(x,y,z) = 1$. Using Faltings' Theorem, we shall give criteria for these equations to have only finitely many proper solutions.

We examine (1) using a descent technique of Kummer, which allows us to obtain, from any infinite set of proper solutions to (1), infinitely many rational points on a curve of (usually) high genus, thus contradicting Faltings' Theorem (for example, this works if $F(t,1) = 0$ has three simple roots and $m \geqslant 4$).

We study (2) via a descent method which uses unramified coverings of $\mathbf{P}_1 \setminus \{0,1,\infty\}$ of *signature* (p,q,r), and show that (2) has only finitely many proper solutions if $1/p + 1/q + 1/r < 1$. In cases where these coverings arise from modular curves, our descent leads naturally to the approach of Hellegouarch and Frey to Fermat's Last Theorem. We explain how their idea may be exploited for other examples of (2).

We then collect together a variety of results for (2) when $1/p + 1/q + 1/r \geqslant 1$. In particular, we consider 'local–global' principles for proper solutions, and consider solutions in function fields.

Introduction

Faltings' extraordinary 1983 theorem **[15]** ('née *Mordell's Conjecture*' **[41]**) states that there are only finitely many rational points on any irreducible algebraic curve of genus > 1 in any number field. Two important immediate consequences are as follows.

THEOREM. *There are only finitely many pairs of rational numbers x, y for which $f(x,y) = 0$, if the curve so represented is smooth and has genus > 1.*

THEOREM. *If $p \geqslant 4$ and A, B and C are non-zero integers, then there are only finitely many triples of coprime integers x, y, z for which $Ax^p + By^p = Cz^p$.*

Here we shall see that, following various arithmetic descents, one can also apply Faltings' result to integral points on certain interesting surfaces.

(Vojta **[42]** and Bombieri **[2]** have now given quantitative versions of Faltings' Theorem. In principle, we can thus give an explicit upper bound to the number of solutions in each equation below, instead of just writing 'finitely many'.)

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The superelliptic equation. In 1929, Siegel [35] showed that a polynomial equation $f(x, y) = 0$ can have infinitely many *integral* solutions in some algebraic number field K only if a component of the curve represented has genus 0. In 1964, LeVeque [24] applied Siegel's ideas to prove that the equation

$$y^m = f(x) \tag{1}^*$$

has infinitely many integral solutions in some number field K if and only if $f(X)$ takes either the form $c(X-a)^e g(X)^m$ or the form $f(X) = c(X^2 - aX + b)^{m/2} g(X)^m$. In all other cases, one can obtain explicit upper bounds on solutions of (1)*, using Baker's method (see [34]).

By using a descent technique of Kummer, we can apply Faltings' Theorem to the *superelliptic equation* (1), much as LeVeque applied Siegel's Theorem to (1)*.

THEOREM 1. *Let $F(X, Y)$ be a homogeneous polynomial with algebraic coefficients, and suppose that there exists a number field K in which*

$$z^m = F(x, y) \tag{1}$$

has infinitely many K-integral solutions with the ideal $(x, y) = 1$, and the ratios x/y distinct. Then $F(X, Y) = cf(X, Y)^m$ times one of the following forms:

(i) $(X - \alpha Y)^a (X - \beta Y)^b$;
(ii) $g(X, Y)^{m/2}$, *where $g(X, Y)$ has at most 4 distinct roots;*
(iii) $g(X, Y)^{m/3}$, *where $g(X, Y)$ has at most 3 distinct roots;*
(iv) $(X - \alpha Y)^{m/2} g(X, Y)^{m/4}$, *where $g(X, Y)$ has at most 2 distinct roots;*
(v) $(X - \alpha Y)^a g(X, Y)^{m/2}$, *where $g(X, Y)$ has at most 2 distinct roots;*
(vi) $(X - \alpha Y)^{m/2} (X - \beta Y)^{am/3} (X - \gamma Y)^{bm/r}$, *where $r \leqslant 6$;*

where a, b and r are non-negative integers, c is a constant, $f(X, Y)$ and $g(X, Y)$ are homogeneous polynomials, and exponents $^{im/j}$ are always integers. Moreover, for each such F and m, there are number fields K in which (1) has infinitely many distinct, coprime K-integral solutions.

This result answers the last of the five questions posed by Mordell in his famous paper [28] (the others having been resolved by Siegel [35] and Faltings [15]). (Actually, Mordell conjectured finitely many *rational* solutions in his last three questions, where he surely meant *integral.*)

We deduce from Theorem 1 that there are only finitely many distinct, coprime K-integral solutions to (1) whenever $F(X, Y)$ has k ($\geqslant 3$) distinct simple roots (over $\mathbf{Q}$) and $m \geqslant \max\{2, 7-k\}$.

The generalized Fermat equation. One last result of Fermat has finally been re-proven [46]: that is, that there are no non-zero integer solutions to

$$x^p + y^p = z^p$$

when $p \geqslant 3$. (This corresponds to the case $p = q = r \geqslant 3$ and $A = B = C = 1$ of the *generalized Fermat equation*

$$Ax^p + By^q = Cz^r, \tag{2}$$

where A, B and C are non-zero integers.) Fortunately, Fermat never wrote down his proof, and many beautiful branches of number theory have grown out of attempts to re-discover it. In the last few years, there have been a number of spectacular advances in the theory of Fermat's equation, culminating in the work of Faltings [**15**], Ribet [**31**] and, ultimately, Wiles [**46**].

As we discussed above, Faltings' Theorem immediately implies that there are only finitely many triples of coprime integers x, y, z for which $x^p + y^p = z^p$. One might hope to also apply Faltings' Theorem directly to (2), since this is a curve in an appropriate *weighted* projective space. However, this curve often has genus 0 (for instance, if p, q and r are pairwise coprime), so that finiteness statements for proper solutions must be reached through a less direct approach.

It has often been conjectured that (2) has only finitely many proper solutions if $1/p + 1/q + 1/r < 1$, perhaps first by Brun [**6**] in 1914. This is easily proved to be true in function fields, and it follows for integers from the '*abc*'-*conjecture.* We shall use Faltings' Theorem to show the following.

THEOREM 2. *For any given integers p, q, r satisfying $1/p + 1/q + 1/r < 1$, the generalized Fermat equation*

$$Ax^p + By^q = Cz^r \tag{2}$$

has only finitely many proper integer solutions.

(Our proofs of Theorems 1 and 2 are extended easily to proper solutions in any fixed number field, and even those that are S-units.)

Catalan conjectured in 1844 that $3^2 - 2^3 = 1$ are the only powers of positive integers that differ by 1. Tijdeman proved this for sufficiently large powers ($>$ $\exp\exp\exp\exp(730)$: Langevin, 1976). One can unify and generalize the Fermat and Catalan Conjectures in the following.

THE FERMAT–CATALAN CONJECTURE. *There are only finitely many triples of coprime integer powers x^p, y^q, z^r for which*

$$x^p + y^q = z^r \quad \textit{with} \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1. \tag{2$'$}$$

This conjecture may be deduced from the *abc*-conjecture (see Subsection 5.2). There are five 'small' solutions (x, y, z) to the above equation:

$$1 + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4, \quad 7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2, \quad 3^5 + 11^4 = 122^2.$$

(Blair Kelly III, Reese Scott and Benne de Weger all found these examples independently.)

Beukers and Zagier have found five surprisingly large solutions:

$$17^7 + 76271^3 = 21063928^2, \quad 1414^3 + 2213459^2 = 65^7, \quad 9262^3 + 15312283^2 = 113^7,$$

$$43^8 + 96222^3 = 30042907^2, \quad 33^8 + 1549034^2 = 15613^3.$$

In Subsection 4.3, we shall use these solutions to write down examples of non-isogenous elliptic curves with isomorphic Galois representations on points of orders 7 and 8. We wonder whether there are any more solutions to (2): in particular, whether there are any with $p, q, r \geqslant 3$.

Given Theorem 2, it is natural to ask what happens in equation (2) when $1/p + 1/q + 1/r \geqslant 1$.

In the cases where $1/p + 1/q + 1/r = 1$, the proper solutions correspond to rational points on certain curves of genus one. It is easily demonstrated that for each such p, q, r, there exist values of A, B, C such that the equation has infinitely many proper solutions; some such examples are given in Section 6. There also exist values of A, B, C such that the equation has no proper solutions (which can be proved by showing that there are no proper solutions modulo some prime); though, for any A, B, C, there are number fields which contain infinitely many proper solutions (see Subsection 5.4).

In the cases where $1/p + 1/q + 1/r > 1$, the proper solutions give rise to rational points on certain curves of genus zero. However, even when the curve has infinitely many rational points, they may not correspond to proper solutions of the equation. Is there an easy way to determine whether equation (2) has infinitely many proper solutions?

In the case of conics ($p = q = r = 2$), Legendre proved the *local–global principle* in 1798; and using this we can determine easily whether (2) has any proper solutions. However, in Section 8 we shall see that there are no proper solutions for

$$x^2 + 29y^2 = 3z^3,$$

despite the fact that there are proper solutions everywhere locally, as well as a rational parametrization of solutions. We prove this using what we call a *class group obstruction*, which may be the only obstruction to a local–global principle in (2) when $1/p + 1/q + 1/r > 1$. We also study this obstruction for a family of equations of the form $x^2 + By^2 = Cz^r$.

It has long been known that there is no general local–global principle for (2) when $1/p + 1/q + 1/r = 1$. Indeed, Lind and Selmer gave the examples

$$u^4 - 17v^4 = 2w^2 \quad \text{and} \quad 3x^3 + 4y^3 = 5z^3,$$

respectively, of equations which are everywhere locally solvable but nonetheless have no non-trivial integer solutions. This obstruction is described by the appropriate *Tate–Šafarevič group*, which may be determined by an algorithm that is only known to work if the Birch–Swinnerton-Dyer Conjectures are true.

There are no local obstructions or class group obstructions to *any* equation

$$Ax^2 + By^3 = Cz^5 \tag{3}$$

if A, B and C are pairwise coprime. So are there are always infinitely many proper solutions? If so, is there a parametric solution to (3) with x, y and z coprime polynomials in A, B and C?

Application of modular curves. The driving principle behind the proof of Theorem 2 is a descent method based on coverings of *signature* (p, q, r) (see Section 3 for the definition). Sometimes, these coverings can be realized as coverings of modular curves. A lot more is known about the Diophantine properties of modular curves than about the properties of Fermat curves, thanks largely to the fundamental work of Mazur on the Eisenstein ideal [**26**]. Hence one can hope that descent using modular coverings yields new insights into such equations. The basic example for this is the covering $X(2p) \longrightarrow X(2)$ which is of signature (p, p, p), ramified over the three cusps of $X(2)$, and forms the basis for the Hellegouarch–Frey attack on Fermat's Last Theorem. Thanks to the deep work of Ribet, Taylor and Wiles, this approach has finally led to the proof of Fermat's Last Theorem; and there is a strong incentive for seeing whether other modular coverings of signature (p,q,r) will yield similar insights into the corresponding generalized Fermat equation (as also noted by Wiles in his Cambridge lectures). In Subsection 4.3 we shall give a classification of the coverings of signature (p,q,r) obtained from modular curves, and state some Diophantine applications.

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1. *Remarks and observations*

There are many remarks to be made about what has been written above. For instance, why the restrictions on pairs x, y in the statement of Theorem 1? What if A, B, C are not pairwise coprime in Theorem 2? We include remarks on these questions here, rather than weigh down the main body of the paper.

1.1 *Proper and improper solutions.* The study of integer solutions to homogeneous polynomials in three variables 'projectivizes' naturally to the study of rational points on curves, by simply de-homogenizing the equation. However, the study of integer solutions to non-homogeneous polynomials in three variables does not so naturally 'projectivize', because there are often parametric families of solutions with common factors that are of little interest from a number-theoretic viewpoint. As an example, look at the integer solutions to $x^3 + y^3 = z^4$. It is easy to find a solution for any fixed ratio x/y: if we want $x/y = a/b$, then simply take $z = a^3 + b^3$, $x = az$ and $y = bz$. This is not too interesting. However, if we do not allow x, y and z to have a large common factor, then we can rule out the above parametric family of solutions (and others), and show that there are only finitely many solutions.

In general, we shall define a *proper* solution to an equation (1) or (2), in some given number field K, to be a set of integer solutions (x, y, z) with the value of x/y fixed, and (x, y) dividing some given, fixed ideal of K (and we thus incorporate here the notion that the solutions may be S-units for a given finite set of primes S).

Notice that in this definition we consider a proper solution to be a *set* of integer solutions (x, y, z) with the value of x/y fixed. This is because one can obtain infinitely many solutions of (1) of the form $x\xi^m, y\xi^m, z\xi^d$ (where $d = \deg(F)$), and of (2) of the form $x\xi^{qr}, y\xi^{rp}, z\xi^{pq}$, as ξ runs over the units of K, given some initial solution (x, y, z). Thus a proper solution is really an equivalence class of solutions under a straightforward action of the unit group of the field. (Actually, if $F(x, y) = \xi$ is itself a unit of K, then $\xi^m = F(x\xi^{(m-1)/d}, y\xi^{(m-1)/d})$ is a proper solution to (1); and so, by Theorem 1, if $F(X, Y)$ has three distinct factors, then there are only finitely many such units. This well-known result also follows from Siegel's Theorem.)

Even when we work with a homogeneous equation like the Fermat equation, it is not always possible to 'divide out' a common factor (x, y) as we might when dealing with rational integer solutions: for instance, if the ideal (x, y) is irreducible and non-principal. (Even Kummer made this mistake, which Weil calls an 'unaccountable lapse' in Kummer's *Collected works.*) However, in this case let I and J be the ideals of smallest norm from the ideal class and inverse ideal class of $G = (x, y)$, respectively. Multiply each of x, y, z through by the generator of the principal ideal IJ, so that now $(x, y) = GIJ$. Since GJ is principal, we may divide through by the generator of that ideal, but then $(x, y) = I$, one of a finite set of ideals. Thus it makes sense to restrict solutions in (1) and (2) by insisting that (x, y) can divide only some fixed ideal of the field.

Let X be the affine surface defined by equation (2). From a geometric perspective, a proper solution (x, y, z) of (2) is the image of an integral point on the blowup of X at the origin, where, here, 'integral' is taken with respect to the special divisor (that is, the proper transform of $(0, 0, 0)$ in the blowup). In recent years, a beautiful theory of rational and integral points on surfaces has begun to emerge through the work of Vojta and Faltings. We make no use of it here, since our descent reduces the problem to results about curves. However, our approach is probably applicable for only a small class of surfaces, maybe just those that are equipped with a non-trivial action of the multiplicative group.

If the degree of F is coprime with m, then we can always construct a parametric improper solution of (1): since there exist positive integers r and s with $mr - \deg(F)s = 1$, we may take $x = aF(a,b)^s$, $y = bF(a,b)^s$, $z = F(a,b)^r$. More generally, if $g = \gcd(\deg(F), m) = mr - \deg(F)s$, then we can obtain a solution of (1) from a solution of $F(a,b) = c^g$ by taking $x = ac^s$, $y = bc^s$ and $z = c^r$.

Equation (2) may be approached similarly, and indeed its generalization to arbitrary diagonal equations (see [4]). The solutions to a diagonal equation $a_1X_1^{e_1} + \ldots + a_nX_n^{e_n} = 0$ may be obtained from the solutions of $a_1Y_1^{g_1} + \ldots + a_nY_n^{g_n} = 0$, where each $g_j = \gcd(e_j, L_j)$ and $L_j = \mathrm{lcm}[e_i,\ 1 \leqslant i \leqslant n,\ i \neq j]$. (If $g_j = e_js_j - L_jr_j$, then we may take $X_i = Y_i^{s_i}\prod_{j\neq i} Y_j^{r_jL_j/e_j}$.)

1.2 *What happens when A, B and C are not pairwise coprime?* Evidently, any common factor of all three of A, B and C in (2) may be divided out, so we may assume that $(A,B,C) = 1$. But what if A, B and C are not pairwise coprime?

If prime ℓ divides A and B but not C, then in any solution of (2), ℓ divides Cz^r and so z. Thus $Cz^r = C\ell^r z'^r$, and so we can rewrite $C\ell^r$ as C, and z' as z. But then ℓ divides each of A, B and C, and so we remove the common power of ℓ dividing them. If ℓ now divides only one of A, B and C, then there are no further such trivial manipulations, but if ℓ divides two of A, B and C, then we are forced to repeat this process. Sometimes this will go on *ad infinitum*, such as for the equation $x^3 + 2y^3 = 4z^3$. In general, it is easily decided whether this difficulty can be resolved.

PROPOSITION 1.1. *Suppose that α, β and γ are the exact powers of ℓ that divide A, B and C, respectively. If there is an integer solution to* (2), *then (p,q) divides $\alpha-\beta$, or (q,r) divides $\beta-\gamma$, or (r,p) divides $\gamma-\alpha$.*

Proof. Let a, b, c and d be the exact powers of ℓ dividing x, y, z and (Ax^p, By^q, Cz^r), respectively. Evidently, d must be equal to at least two of $\alpha + ap$, $\beta + bq$, $\gamma + cr$. From the Euclidean algorithm, we know that there exist integers a and b such that $ap - bq = \beta - \alpha$ if and only if (p,q) divides $\alpha - \beta$; the result follows from examining all three pairs in this way.

2. *Proper solutions of the superelliptic equation*

To prove Theorem 1, we first 'factor' the left-hand side of (1) into ideals in the field K (which may be enlarged to contain the splitting field extension for F), so that these ideals are mth powers of ideals, times ideals from some fixed, finite set. We then multiply these ideals through by ideals from some other fixed, finite set, to obtain principal ideals. Equating the generators of the ideals, modulo the unit group, we obtain a set of linear equations in X and Y. Taking linear combinations to eliminate X and Y, we have now 'descended' to a new variety to which we may be able to apply Faltings' Theorem. If not, we descend again and again, until we can.

The details of this proof are somewhat technical, and so we choose to illustrate them in the next subsection with a simple example.

2.1 *A generalization of Kummer's descent.* In 1975 Erdős and Selfridge **[14]** proved the beautiful result that the product of two or more consecutive integers can never be a perfect power. We conjecture that the product of three or more consecutive integers of an arithmetic progression $a \pmod q$ with $(a, q) = 1$ can never be a perfect power except in the two cases parametrized below. This is well beyond the reach of our methods here, though we now prove the following.

COROLLARY 2.1. *Fix integers $m \geqslant 2$ and $k \geqslant 3$ with $m + k \geqslant 6$. There are only finitely many k-term arithmetic progressions of coprime integers whose product is the mth power of an integer.*

If the product of a three-term arithmetic progression is a square (the case $k = 3$, $m = 2$), then we are led to the systems of equations $a = \lambda x^2$, $a+d = y^2$, $a+2d = \lambda z^2$, with $\lambda = 1$ or 2, so that $x^2 + z^2 = (2/\lambda)y^2$. This leads to the parametric solutions $(t^2 - 2tu - u^2)^2$, $(t^2 + u^2)^2$, $(t^2 + 2tu - u^2)^2$ and $2(t^2 - u^2)^2$, $(t^2 + u^2)^2$, $8t^2u^2$, where, in each case, $(t, u) = 1$ and $t + u$ is odd (for $\lambda = 1$ and 2, respectively).

Euler proved in 1780 that there are only trivial four-term arithmetic progressions whose product is a square, ruling out the case $k = 4$, $m = 2$. In 1782 he showed that there are only trivial integer solutions to $x^3 + y^3 = 2z^3$, which implies that there are no three-term arithmetic progressions whose product is a cube, ruling out the case $m = k = 3$.

Now fix integers $k \geqslant 3$ and $m \geqslant 2$ with $m + k \geqslant 7$, so that $2/k + 1/m < 1$. We shall assume that there exist infinitely many k-term arithmetic progressions of coprime integers whose products are all mth powers of integers; in other words, that there are infinitely many pairs of positive integers a and d for which

$$(a + d)(a + 2d) \dots (a + kd) = z^m \quad \text{with } (a, d) = 1. \tag{2.1}$$

For any $i \neq j$ we have that $(a + id, a + jd)$ divides

$$\big((a + id) - (a + jd), j(a + id) - i(a + jd)\big) = (i - j)(d, a) = (i - j).$$

Therefore, for each i, we have

$$a + id = \lambda_i z_i^m, \quad \text{for } i = 1, 2, \dots, k,$$

for some integers z_i, where each λ_i is a factor of $\left(\prod_{p \leqslant k-1} p\right)^{m-1}$. From elementary linear algebra, we know that we can eliminate a and d from any three such equations; explicitly taking $i = 1$, 2 and j above, we obtain

$$\lambda_j z_j^m = j\lambda_2 z_2^m - (j - 1)\lambda_1 z_1^m, \quad \text{for } j = 3, 4, \dots, k. \tag{2.2}$$

If $m \geqslant 4$, then any single such equation has only finitely many proper solutions, by Faltings' Theorem; and as there are only finitely many choices for the λ_i, this gives finitely many proper solutions to (2.1).

More generally, the collection of equations (2.2) defines a non-singular curve C as the complete intersection of hypersurfaces in $\mathbf{P}^{k-1}$. By considering the natural projection from C onto the Fermat curve in $\mathbf{P}^2$ defined by the single equation (2.2) with $j = 3$, we may use the Riemann–Hurwitz formula to deduce that C has genus

g given by

$$2g-2 = m^{k-3}\left(2\binom{m-1}{2}-2\right)+(k-3)m^2(m^{k-3}-m^{k-4})$$
$$= km^{k-1}\left(1-\frac{2}{k}-\frac{1}{m}\right) > 0,$$

since the degree of the covering map is m^{k-3}, and the only ramification points are where $z_j = 0$ for some $j \geqslant 4$ (and it is easy to show that $z_i = z_j = 0$ is impossible). Thus C has genus > 1, and so has only finitely many rational points, by Faltings' Theorem. Therefore (2.1) has only finitely many proper integer solutions.

Suppose that in equation (1),

$$F(X,Y) = a_0 Y^{r_0}\prod_{i=1}^{n}(X-\alpha_i Y)^{r_i},$$

where the α_i are distinct complex numbers, and the r_i are non-negative integers; we enlarge K, if necessary, to contain the α_i. Let S denote the multiset of integers $s > 1$, each counted as often as there are values of i for which $m/(m,r_i) = s$. Theorem 1 is implied by the following.

THEOREM 1′. *Suppose that there are infinitely many proper K-integral solutions to* (1), *in some number field K. Then one of the following holds:*
(i) $|S| \leqslant 2$;
(ii) $S \subseteq \{2,2,2,2\}$;
(iii) $S = \{3,3,3\}$;
(iv) $S = \{2,4,4\}$;
(v) $S = \{2,2,n\}$ *for some integer* n;
(vi) $S = \{2,3,n\}$ *for some integer* n, $3 \leqslant n \leqslant 6$.

Rewriting (1) as the ideal equation

$$(y)^{r_0}\prod_{i=1}^{n}(a_0x-\beta_i y)^{r_i} = (a_0)^{d-1-r_0}(z)^m,$$

with $\beta_i = a_0\alpha_i$, we proceed in the familiar way, analogous to the above. All ideals of the form $(y, a_0x-\beta_i y)$ and $(a_0x-\beta_i y, a_0x-\beta_j y)$ (with $i \neq j$) divide the ideals J and $(\beta_i-\beta_j)J$, respectively (where J is that fixed ideal which is divisible by (a_0x, y) for any proper solution of (1)). Therefore, by the unique factorization theorem for ideals, we have

$$(a_0x-\beta_i y)^{r_i} = \sigma_i\theta_i^m, \quad \text{for each } i,\ 1 \leqslant i \leqslant n,$$
$$(y)^{r_0} = \sigma_0\theta_0^m,$$

for some ideals θ_i of K and some set of ideal divisors σ_i of $(J')^{m-1}$, where

$$J' := J\left(\prod_{1\leqslant i<j\leqslant n}(\beta_i-\beta_j)\right).$$

We may factor both sides of each of the above equations in terms of their prime ideal divisors. If the exact power to which the prime ideal $\mathbf{p}$ divides $(a_0x-\beta_i y)$ or

(y) is e, and $\mathfrak{p}$ does not divide σ_i, then er_i must be divisible by m, and thus e is a multiple of $m/(m,r_i) = s_i$. Therefore, since all prime divisors $\mathfrak{p}$ of σ_i divide J', we can rewrite the above equations as

$$\begin{aligned}(a_0x - \beta_i y) &= \tau_i\theta_i^{s_i}, \quad \text{for each } i,\ 1 \leqslant i \leqslant n,\\ (y) &= \tau_0\theta_0^{s_0},\end{aligned} \tag{2.3}$$

where each τ_i divides $(J')^{s_i-1}$.

Let $\bar{\theta}_i$ and $\bar{\tau}_i$ be those ideals with smallest norm in the inverse ideal classes of θ_i and τ_i in K, respectively. Both $\bar{\theta}_i\theta_i = (z_i)$ and $\bar{\tau}_i\tau_i = (\omega_i)$ are principal ideals, by definition. Moreover, $\tau_i\theta_i^{s_i}$ is principal by (2.3), and thus so is $\bar{\tau}_i\bar{\theta}_i^{s_i} = (\lambda_i)$, say. Let λ be a fixed integer of the field, divisible by all of the λ_i. Multiplying (2.3) through by λ, we obtain

$$\begin{aligned}(a_0(\lambda x) - \beta_i(\lambda y)) &= ((\lambda/\lambda_i)\omega_i z_i^{s_i}), \quad \text{for each } i,\ 1 \leqslant i \leqslant n,\\ (\lambda y) &= ((\lambda/\lambda_0)\omega_0 z_0^{s_0}).\end{aligned}$$

In each of these ideal equations, the ideals involved are all principal, and so the integers generating the two sides must differ by a unit. Dirichlet's unit theorem tells us that the unit group U of K is finitely generated, and so U/U^{s_i} is finite; that is, for each i, the ratio of the generators of the two sides of the ith equation above, a unit, may be written as $u_i v_i^{s_i}$, where u_i is a unit from a fixed, finite set of representatives of U/U^{s_i}, and v_i is some other unit. Replacing $v_i z_i$ in the equations above by z_i, as well as λx by x and λy by y, we obtain

$$\begin{aligned}a_0x - \beta_i y &= u_i(\lambda/\lambda_i)\omega_i z_i^{s_i}, \quad \text{for each } i,\ 1 \leqslant i \leqslant n,\\ y &= u_0(\lambda/\lambda_0)\omega_0 z_0^{s_0}.\end{aligned}$$

Let $\rho_i = \lambda u_i\omega_i/\lambda_i$ for each i, and let L be the field K extended by $(\rho_i)^{1/s_i}$, $i = 0, 1, \ldots, n$, a finite extension.

Since J' has only finitely many prime ideal divisors, there are finitely many choices for the τ_i, and thus for the ω_i. Since the *class group* of K is finite, there can be only finitely many choices for the $\bar{\theta}_i$, and thus for the λ_i, and so for λ: let μ be an integer divisible by all of the possible λ. Therefore there are only finitely many possible choices for the ρ_i and so for the fields L: let M be the compositum of all possible such fields L. We now replace $(\rho_i)^{1/s_i}z_i$ by z_i in the equations above, to deduce the following.

There exists a number field M in which there are infinitely many proper M-integral solutions $x, y, z_0, z_1, \ldots, z_n$ to the system of equations

$$\begin{aligned}a_0x - \beta_i y &= z_i^{s_i}, \quad \textit{for each } i,\ 1 \leqslant i \leqslant n,\\ y &= z_0^{s_0}.\end{aligned} \tag{2.4}$$

Taking the appropriate linear combination of any three given equations in (2.4), we can eliminate x and y. Explicitly, if $1 \leqslant i < j < k \leqslant n$, then

$$(\beta_j - \beta_k)z_i^{s_i} + (\beta_k - \beta_i)z_j^{s_j} + (\beta_i - \beta_j)z_k^{s_k} = 0,$$

$$\text{and if } r_0 \geqslant 1, \text{ then } \quad z_i^{s_i} - z_j^{s_j} + (\beta_i - \beta_j)z_0^{s_0} = 0. \tag{2.5}$$

Note that we obtain a proper solution here, since the $(z_i^{s_i}, z_j^{s_j})$ all divide the fixed ideal $(\lambda)J$; and the $z_i^{s_i}/z_j^{s_j}$ are all distinct, for if $z_i^{s_i}/z_j^{s_j} = (z_i')^{s_i}/(z_j')^{s_j}$, then

$$\frac{a_0x - \beta_i y}{a_0x' - \beta_i y'} = \frac{a_0x - \beta_j y}{a_0x' - \beta_j y'},$$

and so $(\beta_i - \beta_j)(x/y - x'/y') = 0$, contradicting the hypothesis.

Notice that if F has n simple roots, then all of the corresponding $s_j = m$. Therefore, descending as we did above for (2.1), we see that (2.5) describes a curve of genus > 1 if $2/n + 1/m < 1$, and so we have proved the following.

PROPOSITION 2.1. *If $F(x,y)$ has n simple roots, where $2/n + 1/m < 1$, then there are only finitely many proper solutions to* (1) *in any given number field.*

2.2 *Iterating the descent, leading to the proof of Theorem* 1. The descent just described is entirely explicit; that is, we can compute precisely what variety we shall descend to. On the other hand, the descent described in Section 3 invokes the Riemann Existence Theorem at a crucial stage, and thus is not, *a priori*, so explicit. For this reason, we shall proceed as far as we can in the proof of Theorem 1′ using only the concrete methods of the previous subsection, which turn out to be sufficient unless the elements of the set S are pairwise coprime.

Indeed, if the elements of S are pairwise coprime, and are not case (i) or the third example in case (vi) of Theorem 1′, then there must be three elements $p, q, r \in S$ with $1/p + 1/q + 1/r < 1$. Therefore we can apply Theorem 2 to (2.5), and deduce that there are only finitely many proper solutions to (1).

Now suppose that there are infinitely many proper solutions to (1) in some number field. We need consider only those sets S in which some pair of elements have a common factor: say $pa, pb \in S$, where $p \geqslant 2$ and $a \geqslant b \geqslant 1$ are coprime. To avoid case (i), we may assume that S contains a third element $q \geqslant 2$. (For the rest of this section, 'case' refers to the case number of Theorem 1′.)

The equations (2.5) imply that there are infinitely many proper solutions of some equation of the form $Ax^p + By^p = Cz^q$ in an appropriate number field. So, applying Proposition 2.1 to this new equation, we deduce that $2/p + 1/q \geqslant 1$. Thus $p = 2$, 3 or 4, since $q \geqslant 2$.

Now suppose that S contains a fourth element, call it r, with $q \geqslant r \geqslant 2$. Applying the descent procedure of Subsection 2.1, we obtain infinitely many proper solutions to simultaneous equations of the form

$$c_1x^p + c_2y^p = c_3z^q \quad \text{and} \quad c_1'x^p + c_2'y^p = c_3'w^r.$$

Applying the descent procedure of Subsection 2.1 to the first equation here, we see from (2.4) that x^a and y^b can both be written as certain linear combinations of u^q and v^q, where u and v are integers of some fixed number field. Substituting these linear combinations into the second equation above, we see that Cw^r can be written as the value of a binary homogeneous form in u and v of degree pq. It is straightforward to check that this binary form can have only simple roots, and so, by Proposition 2.1, we have $2/pq + 1/r \geqslant 1$. This implies that $pq \leqslant 4$, since $r \geqslant 2$. On the other hand, $pq \geqslant 4$ since $p, q \geqslant 2$, and so we deduce that $p = q = 2$ and $r = 2$.

We have thus proved that if $\{pa, pb, q, r\}$ is a subset of S, then $p = q = r = 2$.

But then $\{2,2,2a,2b\}$ is a subset of S and, applying the same analysis to this new ordering of the set, we obtain that $2a = 2b = 2$. Therefore if S has four or more elements, then all of these elements must be equal to 2. If so, then we multiply together the linear equations (2.4) that arise from each $s_i = 2$, giving a form with $|S|$ simple roots whose value is a square. Proposition 2.1 implies that we must be in case (ii).

Henceforth we may assume that $S = \{pa, pb, q\}$, where $2/p + 1/q \geqslant 1$ and $p = 2$, 3 or 4, with $q \geqslant 2$, $a \geqslant b \geqslant 1$ and $(a,b) = 1$. If $a = 1$, then $b = 1$, and we must be in one of the cases (iii), (iv), (v), or the first example in (vi). So assume that $a \geqslant 2$.

From (2.5), we obtain a single equation of the form $Ax^{ap} + By^{bp} = Cz^q$. We could apply Theorem 2 to this equation, but instead prefer to continue with the explicit descents of Subsection 2.1. From (2.4), this equation now leads to p equations of the form

$$\alpha_i x^a + \beta_i y^b = z_i^q, \quad i = 1, 2, \ldots, p. \tag{2.6}$$

Eliminating the y^b term from the first two such equations, we obtain an equation of the form $x^a = \gamma_1 z_1^q + \gamma_2 z_2^q$; we deduce that $2/q + 1/a \geqslant 1$ by Proposition 2.1, and so $q \leqslant 4$.

If $(p,q) > 1$, then we may re-order S so that ap is the third element, and thus, by the same reasoning as above, $ap \leqslant 4$. However, since $a, p \geqslant 2$, this implies that $a = p = 2$, $b = 1$ and $q = 2$ or 4, and so we have case (iv) or (v). So we may assume now that $(p,q) = 1$ which, with all the above, leaves only the possibilities $p = 2$, $q = 3$ and $p = 3$, $q = 2$.

If $q = 3$, $p = 2$, then $a = 2$ or 3. This leads to the second and last examples in (vi), and $S = \{6,4,3\}$ which was already ruled out, taking 4 as the third element.

If $p = 3$, $q = 2$, then we can eliminate x^a and y^b from the three equations in (2.6) to obtain a conic in variables z_1, z_2, z_3. As is well known, the integral points on this may be parametrized by a homogeneous quadratic form in new variables u and v, say. Solving for x^a in (2.6), we now obtain that x^a is equal to the value of a homogeneous form in u and v, of degree 4. It is easy to check that the roots of this form must be simple, and so, by Proposition 2.1, $a \leqslant 2$, leading to the last example in (vi).

3. *Proper solutions of the generalized Fermat equation*

It has often been conjectured that

$$Ax^p + By^q = Cz^r \tag{2}$$

has only finitely many proper solutions if $1/p + 1/q + 1/r < 1$. One reason for this is that the whole Fermat–Catalan Conjecture follows from the '*abc*'-*conjecture* (see **[40]** and Subsection 5.2). Another reason is that the analogous result in function fields is easily proved (see Subsection 5.1). A simple heuristic argument is that there are presumably $N^{1/p+1/q+1/r+o(1)}$ integer triples (x,y,z) for which $-N \leqslant Ax^p + By^q - Cz^r \leqslant N$; and so if the values of $Ax^p + By^q - Cz^r$ are reasonably well-distributed on $(-N, N)$, then we should expect that 0 is so represented only finitely often if $1/p + 1/q + 1/r < 1$.

Let $S_{p,q,r}$ denote the surface in affine 3-space $\mathbf{A}^3$ defined by (2). When $p = q = r$, the proper solutions are in an obvious two-to-one correspondence with the rational points on a smooth projective curve in $\mathbf{P}^2$. The genus of this *Fermat curve* is $\binom{p-1}{2}$, which is > 1 when $p > 3$; and Faltings' Theorem then implies that such a projective curve has only finitely many rational points.

Define the *characteristic* of the generalized Fermat equation (2) to be

$$\chi(p,q,r) := \frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1.$$

Fix an embedding of $\bar{\mathbf{Q}} \subset \mathbf{C}$. Given a curve X, defined over $\bar{\mathbf{Q}}$, we shall consider absolutely irreducible algebraic covering maps $\pi : X \longrightarrow \mathbf{P}_1$, defined over $\bar{\mathbf{Q}}$. Such a covering map π is *Galois* if the group of fibre-preserving automorphisms of X has order exactly $d = \deg \pi$.

Moreover, if π is unramified over $\mathbf{P}_1 \setminus \{0,1,\infty\}$, and the ramification indices of the points over 0, 1 and ∞ are p, q and r, respectively, then we say that 'π has *signature* (p,q,r)'. One can show that such a map exists for all positive integers $p,q,r > 1$, by using the Riemann Existence Theorem. The (topological) fundamental group Π_1 of $\mathbf{P}_1 \setminus \{0,1,\infty\}$ is a group on three generators $\sigma_0, \sigma_1, \sigma_\infty$, satisfying the one relation $\sigma_0\sigma_1\sigma_\infty = 1$. (Here σ_i is represented by the appropriate loop winding once around the deleted point i.) Let $\Gamma_{p,q,r}$ be the group with three generators γ_0, γ_1 and γ_∞, satisfying the relations

$$\gamma_0^p = \gamma_1^q = \gamma_\infty^r = \gamma_0\gamma_1\gamma_\infty = 1.$$

The map sending σ_i to γ_i defines a surjective homomorphism from Π_1 to $\Gamma_{p,q,r}$. A standard result of group theory says that $\Gamma_{p,q,r}$ is infinite when $1/p + 1/q + 1/r \leqslant 1$, and has non-trivial finite quotients. Pick such a quotient, G. The homomorphism $\Pi_1 \longrightarrow \Gamma_{p,q,r} \longrightarrow G$ defines, in the usual way, a topological covering of $\mathbf{P}_1 \setminus \{0,1,\infty\}$ which is of signature (p,q,r) and has Galois group G. The Riemann Existence Theorem tells us that such a covering can be realized as an algebraic covering of algebraic curves over $\mathbf{C}$, and a standard specialization argument allows us to conclude that this covering map can be defined over some finite extension K of $\mathbf{Q}$. (For more details, see Theorem 6.3.1 on page 58, as well as the discussion in Sections 6.3 and 6.4, in **[32]**.)

From the Riemann–Hurwitz formula, we can compute the genus of X using the covering map obtained from the Riemann Existence Theorem:

$$2 - 2g = d(2 - 2\cdot 0) - \left(d - \frac{d}{p}\right) - \left(d - \frac{d}{q}\right) - \left(d - \frac{d}{r}\right) = d\chi(p,q,r).$$

Thus $g < 1$, $g = 1$, $g > 1$, according to whether $\chi(p,q,r) > 0$, $\chi(p,q,r) = 0$, $\chi(p,q,r) < 0$. Since g and d are non-negative integers, we have the following.

PROPOSITION 3.1. *For any positive integers $p,q,r > 1$, there exists a Galois covering $\pi : X \longrightarrow \mathbf{P}_1$ of signature (p,q,r). Let d be its degree, and let g be the genus of X.*

If $\chi(p,q,r) > 0$, then $g = 0$ and $d = 2/\chi(p,q,r)$.

If $\chi(p,q,r) = 0$, then $g = 1$.

If $\chi(p,q,r) < 0$, then $g > 1$.

Let $\pi : X \longrightarrow \mathbf{P}_1$ be such a covering map of signature (p,q,r). Since it is defined over $\bar{\mathbf{Q}}$, it can be defined in some finite extension K of $\mathbf{Q}$. By enlarging K if necessary, we can ensure that the automorphisms of $\mathrm{Gal}(X/\mathbf{P}_1)$ are also defined over K.

Given a point $t \in \mathbf{P}_1(K)\backslash\{0,1,\infty\}$, define $\pi^{-1}(t)$ to be the set of points $P \in X(\bar{\mathbf{Q}})$ for which $\pi(P) = t$; by definition, this is a set of cardinality d. Define L_t to be the field extension of K generated by the elements of $\pi^{-1}(t)$. Evidently, L_t is a Galois extension of K with degree at most d.

Define V to be the finite set of places in K for which the covering $\pi : X \longrightarrow \mathbf{P}_1$ has bad reduction.

For a given place v of K, let e_v be a fixed uniformizing element for v. Then, for any $t \in \mathbf{P}_1(K) \setminus \{0,1,\infty\} = K^* - 1$, we have $t = e_v^{\mathrm{ord}_v(t)}u$, where u is a v-unit and $\mathrm{ord}_v(t)$ is a fixed integer, independent of the choice of e_v. Define the arithmetic intersection numbers

$$\begin{aligned}(t \cdot 0)_v &:= \max(\mathrm{ord}_v(t), 0),\\ (t \cdot 1)_v &:= \max(\mathrm{ord}_v(t-1), 0),\\ (t \cdot \infty)_v &:= \max(\mathrm{ord}_v(1/t), 0).\end{aligned}$$

The following result of Beckmann [**1**] describes the ramification in L_t.

Proposition 3.2 (Beckmann). *Suppose that we are given a point $t \in \mathbf{P}_1(K) \setminus \{0,1,\infty\}$, and a place v of K which is not in the set V (defined above). If*

$$(t \cdot 0)_v \equiv 0 \pmod p), \quad (t \cdot 1)_v \equiv 0 \pmod q) \quad \textit{and} \quad (t \cdot \infty)_v \equiv 0 \pmod r), \tag{3.1}$$

then L_t is unramified at v.

Since this result is so fundamental to the proof of Theorem 2, we provide the following.

Proof of Proposition 3.2 *(Sketch).* It is shown in [**1**] that L_t is unramified when

$$(t \cdot 0)_v = (t \cdot 1)_v = (t \cdot \infty)_v = 0,$$

and v is not in V. Let $K(T) \subset K(X)$ denote the inclusion of function fields corresponding to the covering $X \longrightarrow \mathbf{P}_1$. Let $\bar{v}$ be a place of $\bar{K}$ above v. Completing at a place $\mathscr{P}$ above (v, X), one obtains an inclusion of Puiseux series fields

$$K_v((X)) \subset L_{\bar{v}}((X^{1/p})),$$

where $L_{\bar{v}}/K_v$ is unramified. If $(t \cdot 0)_v$ is not zero, then Puiseux series evaluated at $X = t$ converge, and we have

$$(L_t)_{\bar{v}} = L_{\bar{v}}((t^{1/p})).$$

The condition $(t \cdot 0)_v \equiv 0 \pmod p)$ implies that L_t is unramified above v. A similar argument holds if $(t \cdot 1)_v \neq 0$ or $(t \cdot \infty)_v \neq 0$ (by localizing at $(T-1)$ and $(1/T)$, respectively).

Proof of Theorem 2. Let (x, y, z) be a proper solution to the generalized Fermat equation

$$Ax^p + By^q = Cz^r, \tag{2}$$

and take $t = Ax^p/Cz^r$. The congruences in (3.1) are satisfied if v does not divide A, B or C, and so, by Proposition 3.2, L_t is unramified at any $v \notin V_{ABC}$ (the union of V and the places dividing ABC).

Minkowski's Theorem asserts that there are only finitely many fields with bounded degree and ramification; and we have seen that each L_t has degree $\leqslant d$, and all of its ramification is inside V_{ABC}. Thus there are only finitely many distinct fields L_t with $t = Ax^p/Cz^r$ arising from proper solutions x, y, z of (2); and therefore the compositum L of all such fields L_t is a finite extension of $\mathbf{Q}$.

Since the genus of X is > 1 and L is a number field, Faltings' Theorem implies that $X(L)$ is finite. Therefore there are only finitely many proper solutions x, y, z to (2), as $X(L)$ contains all d points of $\pi^{-1}(Ax^p/By^q)$ for each such solution.

This argument (with suitable modifications) also allows us to bound the number of proper solutions in arbitrary algebraic number fields.

Our proof here is similar to that of the weak Mordell–Weil Theorem: the role of the isogeny of an elliptic curve is played here by coverings of $\mathbf{P}_1 \setminus \{0, 1, \infty\}$ of signature (p, q, r), and Minkowski's Theorem is used in much the same way (see [**44**]).

Theorem 2 may be deduced directly from the *abc*-conjecture. In fact, unramified coverings of $\mathbf{P}_1 \setminus \{0, 1, \infty\}$ also play a key role in Elkies' result [**12**] that the *abc*-conjecture implies Mordell's Conjecture.

It is sometimes possible to be more explicit about the curve X and the covering map π, as we shall see in the next few sections.

4. *Explicit coverings when* $1/p + 1/q + 1/r < 1$

The curve X (of the proof in Section 3) can be realized as the quotient of the upper half plane by the action of a Fuchsian group Γ; that is, a discrete subgroup of $\mathrm{PSL}_2(\mathbf{R})$ with finite covolume. Actually, X is quite special among all curves of its genus, since it has many automorphisms. One can sometimes show that these automorphisms determine X uniquely over $\mathbf{C}$, and hence the curve X may be defined over $\mathbf{Q}$ using the descent criterion of Weil. Examples in which even the Galois action of Γ is defined over $\mathbf{Q}$ can be constructed using the rigidity method (see [**32**]).

Those finite groups G which occur as Galois groups of such coverings are said to be 'of signature (p, q, r)'. Evidently, such groups have generators α, β, γ for which

$$\alpha^p = \beta^q = \gamma^r = \alpha\beta\gamma = 1.$$

Because of the connection to the Fermat equation, it is natural to start with coverings of signature (p, p, p), where p is an odd prime. Although we are far from a satisfying classification of coverings of signature (p, p, p), we discuss the construction of a few examples in the next two subsections, which lead to the approaches of Kummer, and Hellegouarch and Frey [**17, 18**], for tackling Fermat's Last Theorem. In the third subsection, we extend the Hellegouarch–Frey method to some other cases of the generalized Fermat equation, by exploiting coverings coming from modular curves.

4.1 *Solvable coverings of signature* (p, p, p). Let $\pi : X \longrightarrow \mathbf{P}_1$ be a covering of signature (p, p, p) with solvable Galois group G. Let $G' = [G, G]$ be the derived group of G, and let $G^{\mathrm{ab}} := G/G'$ be the maximal abelian quotient of G. In fact, π is an unramified covering of a quotient of the pth Fermat curve.

PROPOSITION 4.1. *The group G^{ab} is isomorphic to either $\mathbf{Z}/p\mathbf{Z}$ or $\mathbf{Z}/p\mathbf{Z}\times\mathbf{Z}/p\mathbf{Z}$. The quotient curve $F = X/G'$ is isomorphic (over $\bar{\mathbf{Q}}$) to a quotient of the pth Fermat curve. The map $X \longrightarrow F$ is unramified.*

We may construct an example as follows. Let

$$L = \mathbf{Q}\left(T^{1/p}, \left(\frac{T^{1/p}-\zeta_p^i}{T^{1/p}-1}\right)^{1/p} \quad \text{for } 1 \leqslant i \leqslant p-1\right)$$

be an extension of $\mathbf{Q}(T)$, where ζ_p is a primitive pth root of unity. The inclusion $\mathbf{Q}(T) \subset L$ corresponds to a covering map $\pi : X \longrightarrow \mathbf{P}_1$ of signature (p,p,p) with Galois group

$$G = (\mathbf{Z}/p\mathbf{Z})^{p-1} \ltimes \mathbf{Z}/p\mathbf{Z},$$

where the action of $\mathbf{Z}/p\mathbf{Z}$ on $(\mathbf{Z}/p\mathbf{Z})^{p-1}$ in the semi-direct product is by the regular representation 'minus the trivial representation' (that is, the space of functions on $\mathbf{Z}/p\mathbf{Z}$ with values in $\mathbf{Z}/p\mathbf{Z}$ whose integral over the group is zero). Note that the action of G is defined over $\mathbf{Q}(\zeta_p)$. The group G^{ab} is isomorphic to $\mathbf{Z}/p\mathbf{Z}\times\mathbf{Z}/p\mathbf{Z}$, and X is isomorphic to an unramified covering of the pth Fermat curve with Galois group $(\mathbf{Z}/p\mathbf{Z})^{p-2}$. If $a^p+b^p=c^p$ is a non-trivial solution of the Fermat equation, then, setting $t=a^p/b^p$, one finds that L_t is the Galois closure of $\mathbf{Q}(\zeta_p,(a-\zeta_p c)^{1/p})$ over $\mathbf{Q}$. A crude analysis shows that $L_t/\mathbf{Q}(\zeta_p)$ is unramified outside the prime $(1-\zeta_p)$ above p. A clever manipulation (that may require replacing X by a covering which is isomorphic to it over $\bar{\mathbf{Q}}$), and a careful analysis of the ramification in L_t, lead to a contradiction by showing that such an extension cannot exist when p does not divide the class number of $\mathbf{Q}(\zeta_p)$. This gives a (vastly over-simplified) geometric perspective of Kummer's approach to Fermat's Last Theorem.

4.2 *Modular coverings of signature (p,p,p).* Let $X(N)$ be the modular curve classifying elliptic curves with full level N structure. The curve $X(2)$ of level 2 is isomorphic to $\mathbf{P}_1$, and has three cusps: let t be a function on $X(2)$ such that $t=0,1,\infty$ at these cusps. The natural projection

$$X(2p) \longrightarrow X(2)$$

is then a covering of signature (p,p,p) ramified over $t=0,1,\infty$. Its Galois group $\mathrm{PSL}_2(\mathbf{F}_p)$ is a non-abelian simple group. If $a^p+b^p=c^p$ is a non-trivial solution of the Fermat equation, then, setting $t=a^p/b^p$, one finds that t corresponds (via the moduli interpretation of $X(2)$) to the elliptic curve

$$Y^2 = X(X-a^p)(X+b^p)$$

(or its twist over $\mathbf{Q}(i)$). The field L_t is then the field generated by the points of order p of this curve; and so we recover the Hellegouarch–Frey strategy for tackling Fermat's Last Theorem (see also pages 193–197 of [**22**]).

4.3 *Modular coverings of signature (p,q,r).* Wiles' attack on Fermat's Last Theorem [**38**, **46**] uses the Hellegouarch–Frey approach via modular coverings, described above. Serre [**33**] has noted that this analysis can be extended to certain other equations of the form $x^p+y^p=cz^p$. In fact, what they do is to study Galois representations in $\mathrm{GL}_2(\mathbf{F}_p)$ arising out of the p-division points of suitable elliptic curves. It is thus natural to prove the following.

PROPOSITION 4.2. *Coverings $X \longrightarrow Y$ of signature (p,q,r) can arise only as pull-backs of the covering $X(p) \longrightarrow X(1)$ (up to $\bar{\mathbf{Q}}$-isomorphism), via an auxiliary covering $\phi : Y \longrightarrow X(1)$ where $Y \simeq \mathbf{P}_1$, for the following such coverings ϕ.*

$(2,3,p)$: *the identity covering $X(1) \longrightarrow X(1)$.*

$(3,3,p)$: *the degree two Kummer covering of the j-line, ramified over $j = 1728$ and $j = \infty$.*

$(2,p,p)$: *the covering $X_0(2) \longrightarrow X(1)$, where ϕ is the natural projection.*

$(3,p,p)$: *the degree two Kummer covering of the j-line, ramified over $j = 0$ and $j = 1728$.*

$(3,p,p)$: *the covering $X_0(3) \longrightarrow X(1)$, where ϕ is the natural projection.*

(p,p,p) : *the covering $X(2) \longrightarrow X(1)$, where ϕ is the natural projection.*

Analogously to Subsection 4.2, we let $t \in Y$ be the rational point arising from a solution to the appropriate generalized Fermat equation. The curve corresponding to t (that is, a curve with j-invariant $\phi(t)$) gives rise to a mod p Galois representation with very small conductor, and one can hope to derive a contradiction from this.

The equations $x^p + y^p = z^2$ and $x^p + y^p = z^3$. Given $a^p + b^p = c^2$, with (a,b,c) proper, we consider the curve

$$Y^2 = X^3 + 2cX^2 + a^pX$$

arising from the universal family over $X_0(2)$. The conductor of the associated mod p representation is a power of 2 (which can be made to divide 32, possibly after rearranging a and b).

Given $a^p - b^p = c^3$, with (a,b,c) proper, the classification result states that there are two 'Frey curves' that can be constructed, namely

$$Y^2 = X^3 + 3cX^2 + 4b^p$$

and

$$Y^2 = X^3 - 3(9a^p - b^p)cX + 2(27a^{2p} - 18a^pb^p - b^{2p}).$$

The second comes from a universal family on $X_0(3)$. Each of these curves gives rise to a mod p Galois representation whose conductor can be made to divide 54, by permuting a and b as necessary.

By analysing these representations (using a result of Kamienny on Eisenstein quotients over imaginary quadratic fields [**20**]), the first author proved, in [**7**], the following.

PROPOSITION 4.3. *Let $p > 13$ be prime. If the Shimura–Taniyama Conjecture is true, then we have the following.*

(i) *The equation $x^p + y^p = z^2$ has no non-trivial proper solutions when $p \equiv 1 \pmod 4$.*
(ii) *The equation $x^p + y^p = z^3$ has no non-trivial proper solutions when $p \equiv 1 \pmod 3$ and p is not a Mersenne prime. (A Mersenne prime is one of the form $2^q - 1$.)*

The equation $x^3+y^3=z^p$. Inspired by Gauss' proof that $x^3+y^3=z^3$ has no non-trivial solutions over $\mathbf{Q}(\zeta_3)$, where ζ_3 is a primitive cube root of unity (see **[30**, pp. 42–45]), we construct a 'Frey curve' corresponding to a proper solution (a,b,c) of

$$a^3+b^3=c^p,$$

where $p>3$ is prime. Since $\mathbf{Q}(\zeta_3)=\mathbf{Q}(\sqrt{-3})$ has class number one and a finite unit group whose order is not divisible by p, we may factor the right-hand side of the equation above so that all three factors

$$\alpha=a+b,\quad \beta=\zeta a+\zeta^2 b,\quad \bar{\beta}=\zeta^2 a+\zeta b$$

are pth powers in $\mathbf{Q}(\sqrt{-3})$, at least when 3 does not divide z. Furthermore, they satisfy

$$\alpha+\beta+\bar{\beta}=0,$$

and hence give rise to a solution of Fermat's equation of exponent p over $\mathbf{Q}(\sqrt{-3})$. Unfortunately, the Hellegouarch–Frey approach does not apply directly to Fermat's equation over number fields other than $\mathbf{Q}$ (in fact, $(\zeta,\zeta^2,-1)$ is a solution to $x^n+y^n=z^n$ in $\mathbf{Q}(\sqrt{-3})$ when $(6,n)=1$).

On the other hand, following Hellegouarch–Frey, we can consider the elliptic curve

$$E_f: y^2=x(x-\beta)(x+\bar{\beta})$$

defined over $\mathbf{Q}(\sqrt{-3})$. Expanding the right-hand side, the equation for E_f becomes

$$y^2=x^3-\sqrt{-3}(a-b)x^2+(a^2-ab+b^2)x.$$

Although this curve is not defined over $\mathbf{Q}$, a twist of E_f over $\mathbf{Q}((-3)^{1/4})$ is

$$E: y^2=x^3+3(a-b)x^2+3(a^2-ab+b^2)x.$$

The j-invariant and discriminant of E are

$$j=2^8 3^3\frac{a^3b^3}{c^{2p}},\quad \Delta=-2^4 3^3 c^{2p}.$$

The conductor of the mod p representation associated to E can be shown to divide $2^4 3^3$, and 54 if c is even. An analysis very similar to the one in [7] shows that this representation cannot exist when c is even, and hence we have the following.

PROPOSITION 4.4. *Let $p>13$ be prime. If the Shimura–Taniyama conjecture is true, then an even pth power cannot be expressed as a sum of two relatively prime cubes.*

The equation $x^2+y^3=z^p$. When $a^2+b^3=c^p$, the corresponding 'Frey curve' is

$$y^2=x^3+3bx+2a,$$

which has discriminant $1728c^p$; and the conductor of its associated Galois representation divides 1728. Because of the rather large conductor, the analysis along the lines of the previous section seems rather difficult. In fact, the equation $x^2+y^3=z^7$ does have a few proper solutions, including three rather large ones (mentioned in the Introduction in connection with the Fermat–Catalan Conjecture).

The proper solutions

$$(a,b,c) = (3,-2,1),\ (2213459,1414,65),\ (21063928,-76271,17)$$

lead to the following (possibly twisted) 'Frey curves', each with conductor $864c$:

$$y^2 = x^3 - 3x + 6, \quad y^2 = x^3 + 16968x + 35415344$$

and

$$y^2 = x^3 - 228813x - 42127856,$$

respectively. These have isomorphic Galois representations on the points of order 7. The 'Frey curve' corresponding to the proper solution $(15312283, 9262, 113)$ is

$$y^2 = x^3 + 27786x + 30624566.$$

The associated representation on the points of order 7 is isomorphic to that of the curve $y^2 = x^3 - 3x$, which has complex multiplication by $\mathbf{Z}[i]$. Since 7 is inert in $\mathbf{Z}[i]$, this mod 7 representation maps onto the normalizer of a non-split Cartan subgroup of $\mathrm{GL}_2(\mathbf{F}_7)$. These examples address a question posed by Mazur in the introduction of **[27]**. (Other examples of isomorphic mod 7 representations are given in **[21]**. We actually need to use the main theorem of **[21]** to prove what is asserted above. We are unable to check whether our isomorphisms are *symplectic*—that is, that they preserve the Weil pairing.) Recently, Noam Elkies has proved that there are infinitely many pairs of non-isogenous elliptic curves over $\mathbf{Q}$ giving rise to isomorphic Galois representations on the points of order 7.

The large solutions of $x^2 + y^3 = \pm z^8$ may be used similarly to construct non-isogenous elliptic curves with isomorphic Galois representations on the points of order 8 (which we leave to the reader).

5. *The generalized Fermat equation in function fields, and the abc-conjecture*

In most Diophantine questions it is much easier to prove good results in function fields (here we restrict ourselves to $\mathbf{C}[t]$). In Subsection 5.1 below we show that (2) has no proper $\mathbf{C}[t]$-solutions when $1/p + 1/q + 1/r \leqslant 1$. On the other hand, in Section 7, we shall exhibit proper $\mathbf{C}[t]$-solutions of (2) for each choice of p,q,r with $1/p + 1/q + 1/r > 1$. (All of this was first proved by Welmin **[45]** in 1904, and re-proved by an entirely different method by Silverman **[36]** in 1982.)

The proof of this result stems from an application of the *abc*-conjecture for $\mathbf{C}[t]$, which is easily proved. Its analogue for number fields is one of the most extraordinary conjectures of recent years, and implies many interesting things about the generalized Fermat equation (which we discuss in Subsection 5.2 and Section 9).

It is typical, in the theory of curves of genus 0 and 1, that if one finds a rational point, then it can be used to derive infinitely many other such points through some geometric process (except for 'torsion points'). However, it is not clear that new points derived on the curves corresponding to (2) will necessarily lead to new proper solutions of (2). In Subsection 5.3 we discuss a method of deriving new proper solutions by finding points on appropriate curves over $\mathbf{C}[t]$.

5.1 *Proper solutions in function fields.* Liouville (1879) was the first to realize that equations like (2), in $\mathbf{C}[t]$, can be attacked using elementary calculus. Relatively recently, Mason (**[25]**, but see also **[37]**) recognized that such methods can be applied to prove a very general type of result, the so-called '*abc*-conjecture'. A sharp version of Mason's result, which has appeared by now in many places, is as follows.

PROPOSITION 5.1. *Suppose that $a, b, c \in \mathbf{C}[t]$ satisfy the equation $a+b=c$, where a, b and c are not all constants and do not have any common roots. Then the degrees of a, b and c are less than the number of distinct roots of $a(t)b(t)c(t)=0$.*

Proof. Define $w(t)=\prod_{abc(\delta)=0}(t-\delta)$. Since $a+b=c$, we have $a'+b'=c'$ (where each y' means dy/dt), which implies that

$$\begin{aligned} aw(\log(a/c))' + bw(\log(b/c))' &= w(a(\log a)' + b(\log b)' - (a+b)(\log c)') \\ &= w(a'+b'-c') = 0. \end{aligned}$$

Therefore a divides $bw(\log(b/c))'$, and so a divides $w(\log(b/c))'$ since a and b have no common root. Evidently, $w(\log(b/c))' \neq 0$, else b and c would have the same roots, which by hypothesis is impossible unless b and c are both constants, but then a, b and c would all be constants, contradicting the hypothesis. Therefore the degree of a is at most the degree of $w(\log(b/c))'$. However, if $b/c=\prod_{bc(\delta)=0}(t-\delta)^{e_\delta}$, then $(\log(b/c))'=\sum_{bc(\delta)=0} e_\delta/(t-\delta)$, so that $w(\log(b/c))'$ is evidently an element of $\mathbf{C}[t]$ of degree lower than that of w. This gives the result for a, and the result for b and c is proved analogously.

Applying this to a solution of (2) proves a strong version of our 'Fermat–Catalan' Conjecture for $\mathbf{C}[t]$. Take $a=Ax^p$, $b=By^q$, $c=Cz^r$, to obtain $p\deg(x)$, $q\deg(y)$, $r\deg(z)<\deg(xyz)$ and so $1/p+1/q+1/r>1$.

The proposition above (and even the proof) may be generalized to n-term sums (see [**25**], [**5**] and [**43**]). From Theorem B of [**5**] we know that if $y_1, y_2, \ldots, y_n$ are non-constant polynomials, without (pairwise) common roots, whose sum vanishes, then $\dfrac{1}{n-2}\deg(y_j)$ is less than the number of distinct roots of $y_1y_2\ldots y_n$, for each j. Proceeding as above, we then deduce the following.

PROPOSITION 5.2. *If $p_1, p_2, \ldots, p_n$ are positive integers with*

$$1/p_1 + 1/p_2 + \ldots + 1/p_n \leqslant 1/(n-2),$$

then there do not exist non-constant polynomials $x_1, x_2, \ldots, x_n$, without (pairwise) common roots, such that $x_1^{p_1} + x_2^{p_2} + \ldots + x_n^{p_n} = 0$.

5.2 *The abc-conjecture for integers, and some consequences.* Proposition 5.1, and particularly its formulation, has influenced the statement of an analogous '*abc*-conjecture' for the rational integers (due to Oesterlé and Masser).

THE *abc*-CONJECTURE. *For any fixed $\varepsilon>0$, there exists a constant $\kappa_\varepsilon>0$ such that if $a+b=c$ in coprime positive integers, then*

$$c \leqslant \kappa_\varepsilon\, G(a,b,c)^{1+\varepsilon}, \quad \textit{where} \quad G(a,b,c) = \prod_{p|abc} p.$$

Fix $\varepsilon = 1/83$, and suppose that we are given a proper solution to (2)$'$ in which all terms are positive. Then

$$G(x^p, y^q, z^r) \leqslant xyz \leqslant |x^p|^{1/p}|y^q|^{1/q}|z^r|^{1/r} \leqslant |z^r|^{1/p+1/q+1/r} \leqslant |z^r|^{41/42},$$

since $1/p + 1/q + 1/r \leqslant 41/42$. Therefore, by the *abc*-conjecture, we have $z^r \leqslant \kappa_{1/83}^{83}$, and thus the solutions of $(2)'$ are all bounded. This implies the 'Fermat–Catalan' Conjecture; and indeed this argument may be extended to include all equations (2) where the prime divisors of ABC come from some fixed finite set (see **[40]**).

In **[12]** Elkies succeeded in applying the *abc*-conjecture (suitably formulated over arbitrary number fields) to any curve of genus > 1, and deduced that the *abc*-conjecture implies Faltings' Theorem. His proof inspired some of our work here, particularly Theorem 2.

The following generalization of the *abc*-conjecture has been proposed for equations with n summands, implying a result analogous to Proposition 5.2.

THE GENERALIZED *abc*-CONJECTURE. *For every integer $n \geqslant 3$, there is a constant $T(n)$ such that for every $T > T(n)$, there exists a constant $\kappa_T > 0$ such that if $x_1 + x_2 + \ldots + x_n = 0$ in coprime integers $x_1, x_2, \ldots, x_n$, and no subsum vanishes, then*

$$\max_j |x_j| \leqslant \kappa_T \left(\prod_{p | x_1 x_2 \ldots x_n} p \right)^T.$$

5.3 *Generating new proper integer solutions when $1/p + 1/q + 1/r \geqslant 1$.* Given integers p, q, r, we wish to find $f(t), g(t), h(t) \in \mathbf{Z}[t] \setminus \mathbf{Z}$, without common roots, for which

$$tf(t)^p + (1-t)g(t)^q = h(t)^r, \tag{5.1}$$

and the degrees of $f(t)^p$, $g(t)^q$ and $h(t)^r$ are equal (to d, say). Applying Proposition 5.1 to any such solution, we determine that $d + 1 < d/p + d/q + d/r + 2$, and so $1/p + 1/q + 1/r \geqslant 1$.

Now if we find a solution to (5.1), let

$$F(u,v) = v^{d/p} f(u/v), \quad G(u,v) = v^{d/q} g(u/v) \quad \text{and} \quad H(u,v) = v^{d/r} h(u/v).$$

Then, given any solution x, y, z to (2), we derive another:

$$X = xF(u,v), \quad Y = yG(u,v), \quad Z = zH(u,v), \tag{5.2}$$

where $u = Ax^p$ and $v = Cz^r$.

If x, y, z had been a proper solution to (2), so that $(u,v) = 1$, then

$$k = (AX^p, BY^q) = (uF(u,v)^p, vG(u,v)^q),$$

which divides

$$K = (u, G(0,1)^q)(F(1,0)^p, v)\,\text{Resultant}(f, g).$$

Thus k is easily determined from the congruence classes of u and v (mod K). We may thus divide out an appropriate integer from each of X, Y and Z to obtain a proper solution, provided that k is a $[p,q,r]$th power.

We measure the 'size' of a solution of (2) by the magnitude $|x^p y^q z^r|$. Thus our new proper solution is larger than our old proper solution unless

$$|X^p/k|\,|Y^q/k|\,|Z^r/k| \leqslant |x^p y^q z^r|,$$

that is, $|F^p/k|\,|G^q/k|\,|H^r/k| \leqslant 1$. Since each term here is an integer, this implies that either one of them is zero, or else they are all equal in absolute value. Thus either $f(u/v)g(u/v)h(u/v) = 0$, or $f(u/v)^p = g(u/v)^q = h(u/v)^r$ using (5.1) (here we do not allow $u = v$ or $u = 0$ since they would both imply $xyz = 0$).

5.4 *Number fields in which there are infinitely many solutions.* In Section 7 we shall give values of a, b, c for which $ax^p + by^q = cz^r$ has a parametric solution, for each choice of $p, q, r > 1$ with $1/p + 1/q + 1/r > 1$. Now ax^p is a pth power in $\mathbf{Q}(a^{1/p}, b^{1/q}, c^{1/r})$ (similarly by^q and cz^r), so we have a parametric solution, in this field, to $x^p + y^q = z^r$. Then, given any choice of coprime A, B, C, we can certainly choose the parameters in an appropriate number field so that A divides x^p, B divides y^q, and C divides z^r. This thus leads to a number field in which there are infinitely many solutions of (2).

In the last subsection we described a technique that allowed us, given one proper solution to (2), to generate infinitely many (except in a few easily found cases), provided one has an appropriate solution to (5.1). In Section 6 an appropriate solution will be found whenever $1/p + 1/q + 1/r = 1$. Thus given algebraic numbers x, y chosen so that C divides $Ax^p + By^q$, we can find z from (2), and then obtain infinitely many solutions to (2) by the method of (5.1). If our original choices of x, y lie in the torsion of the method of Subsection 5.3, then we may replace x by any number $\equiv x \pmod C$ (and similarly y by any number $\equiv y \pmod C$), and it is easily shown to work for some such choice.

For any $F(X, Y)$ and m satisfying the cases (i)–(vi) of Theorem 1, we claim that there are number fields K in which (1) has infinitely many proper K-integral solutions. To see this, start by taking K to be a field which contains $c^{1/m}$ as well as the roots of $F(t, 1) = 0$. Then we shall try to select X and Y so that each of the factors in cases (i)–(vi) is itself an mth power.

In (i) we can determine X and Y directly from the two linear equations $X - \alpha Y = u^m$, $X - \beta Y = v^m$, where u and v are selected to be coprime with each other and $\beta - \alpha$, but with $v - u$ divisible by $\beta - \alpha$.

In each of the cases (iii)–(vi) we obtain three linear equations in X and Y, which we can assume are each equal to a constant times an appropriate power of a new variable. Eliminating X and Y by taking the appropriate linear combination of the three linear equations, we reach an equation of the form (2), with $1/p + 1/q + 1/r \geqslant 1$. Above, we saw how to find number fields in which there are infinitely many proper solutions to such equations.

The only case not yet answered arises from case (ii) of Theorem 1, defining an equation (2) with $m = 2$ and F quartic. Select x and y to be large coprime integers, and $z = \sqrt{F(x, y)}$; by the appropriate modification of the Lutz–Nagell Theorem, we see that these can certainly be chosen to obtain a non-torsion point on the corresponding curve. Taking multiples of this point, we obtain an infinite sequence of solutions to $z^2 = F(x, y)$ in the same field. As in Subsection 1.1, we may replace x and y by appropriate multiples, to force (x, y) to belong to a certain finite set of ideals, and thus find proper solutions (we leave it to the reader to show that these must be distinct).

6. *The generalized Fermat equation when* $1/p + 1/q + 1/r = 1$

In each of these cases, the proper solutions to (2) correspond to rational points on certain curves of genus one. The coverings X are well known, and are to be found in the classical treatment of curves with complex multiplication; in fact, it has long been known that the equations $x^p + y^q = z^r$ with $xyz \neq 0$ and $1/p + 1/q + 1/r = 1$ have only one proper solution, namely $3^2 + 1 = 2^3$. Our discussion here is little more than a reformulation of the descent arguments of Euler and Fermat, from their studies of the Fermat equation for exponents 3 and 4.

In looking for appropriate solutions to (5.1), we note that we may look for suitable $\mathbf{Q}[t]$-points on the genus one curve $E_t : tf(t)^p + (1-t)g(t)^q = 1$ (taking $r = 3$, 6 and 4 below, respectively), which we shall be able to find by taking multiples of the point $(1,1)$. Thus, except in a few special cases, any one proper solution to (2) gives rise to infinitely many.

6.1 $Ax^3 + By^3 = Cz^3$: *the Fermat cubic.* The elliptic curve $E : v^3 = u^3 - 1$ has j-invariant 0 and complex multiplication by $\mathbf{Q}(\sqrt{-3})$. It has no non-trivial rational points, as was proved by Euler in 1753 (though an incomplete proof was proposed by Alkhodjandi as early as 972). In fact, the proper solutions to the equation

$$Ax^3 + By^3 + Cz^3 = 0$$

correspond to rational points on a certain curve of genus 1, which is a principal homogeneous space for E.

In 1886, Desboves [9] gave explicit expressions for deriving new proper solutions from old ones (essentially doubling the point on the associated curve). In fact, these identities correspond to doubling the point $(1,1)$ on E_t, obtaining

$$t(t-2)^3 + (1-t)(1+t)^3 = (1-2t)^3.$$

Thus if we begin with a solution (x,y,z) of $Ax^3 + By^3 = Cz^3$, then we have another solution to $AX^3 + BY^3 = CZ^3$ given by

$$X = x(u-2v), \quad Y = y(u+v), \quad Z = z(v-2u),$$

where $u = Ax^3$ and $v = Cz^3$ (and $k = (3, u+v)^3$). All cases where this fails to give a larger proper solution correspond to the point $(\pm 1, \pm 1, \pm 1)$ on $x^3 + y^3 = 2z^3$.

6.2 $Ax^2 + By^3 = Cz^6$: *another Fermat cubic.* The elliptic curve $E : v^2 = u^3 - 1$ also has j-invariant 0. The map $\pi : E \longmapsto \mathbf{P}_1$ defined by $\pi(u,v) = u^3 = t$ has degree 6 and signature $(3,2,6)$. The points $t = y^3/z^6$ in $\mathbf{P}_1(\mathbf{Q})$ derived from proper solutions of $x^2 = y^3 - z^6$ are in a natural 1–1 correspondence with the points $(u,v) = (y/z^2, x/z^3)$ in $E(\mathbf{Q})$. Euler showed that $E(\mathbf{Q})$ has rank 0, and hence $x^2 = y^3 - z^6$ has no non-trivial proper solutions. One can look similarly at rational points on twists of the curve E, when considering $Ax^2 = -By^3 + Cz^6$.

In fact, Bachet showed that, other than $3^2 - 2^3 = 1$, there are no non-trivial proper solutions to $x^2 - y^3 = z^6$.

Quintupling the point $(1,1)$ on E_t, we obtain

$$\begin{aligned} &t(t^{12} + 4680t^{11} - 936090t^{10} + 10983600t^9 - 151723125t^8 - 508608720t^7 \\ &\quad + 3545695620t^6 - 12131026560t^5 + 27834222375t^4 - 37307158200t^3 \\ &\quad + 27119434230t^2 - 10331213040t + 1937102445)^2 + (1-t)(t^8 - 2088t^7 \\ &\quad + 64908t^6 + 21384t^5 + 1917270t^4 - 5616216t^3 + 7007148t^2 \\ &\quad - 4251528t + 531441)^3 = (5t^4 + 360t^3 - 1350t^2 + 729)^6. \end{aligned}$$

A straightforward computation gives that k is always the sixth power of an integer dividing $2^8 3^6$. All cases where this fails to give a larger proper solution correspond to the points $(\pm 1, 1, \pm 1)$ on $4y^3 - 3x^2 = z^6$, and $(\pm 3, 2, \pm 1)$ on $x^2 - y^3 = z^6$.

6.3 $Ax^4 + By^4 = Cz^2$: *the curve with invariant* $j = 1728$. Fermat's only published account of his *method of descent* was his proof, in around 1636, that there are no non-trivial proper solutions to $x^4 + y^4 = z^2$, thus establishing his Last Theorem for exponent 4. In 1678 Leibniz showed that $x^4 - y^4 = z^2$ has no non-trivial proper solutions.

The elliptic curve $E : v^2 = u^3 - u$ has j-invariant 1728 and complex multiplication by $\mathbf{Q}(\sqrt{-1})$. The map $\pi : E \longmapsto \mathbf{P}_1$ defined by $\pi(u,v) = u^2 = t$ has degree 4 and signature $(4,2,4)$. The points $t = x^4/y^4$ in $\mathbf{P}_1(\mathbf{Q})$ derived from proper solutions of $x^4 - y^4 = z^2$ are in a natural 1–1 correspondence with the points $(u,v) = (x^2/y^2, xz/y^3)$ in $E(\mathbf{Q})$; and one can show easily that $E(\mathbf{Q})$ has rank 0.

Tripling the point $(1,1)$ on E_t, we obtain

$$t(t^2 + 6t - 3)^4 + (1 - t)(t^4 - 28t^3 + 6t^2 - 28t + 1)^2 = (3t^2 - 6t - 1)^4.$$

A straightforward computation gives that k is always the fourth power of an integer dividing 8. All cases where this fails to give a larger proper solution correspond to the point $(1,1,1)$ on $x^4 + y^4 = 2z^2$.

7. *The generalized Fermat equation when* $1/p + 1/q + 1/r > 1$

In each of these cases the proper solutions to (2) correspond to rational points on certain curves of genus zero. Sometimes, we can write down equations for Galois coverings of signature (p,q,r), which may allow us to exhibit infinitely many proper solutions to (2). To each such (p,q,r) we shall associate a certain (explicit) finite subgroup Γ of PGL_2, corresponding to the symmetries of a regular solid. The covering π is then given by the quotient map $\pi : \mathbf{P}_1 \longrightarrow \mathbf{P}_1/\Gamma$; and we may write down equations for π over $\mathbf{Q}$, even though the action of Γ may not be defined over $\mathbf{Q}$. Rational points on these coverings will then lead to infinitely many proper solutions to (2).

It is easy to show that there are infinitely many proper solutions of every equation $x^p + y^q = z^r$ with $1/p + 1/q + 1/r > 1$. If two of the exponents are 2, then the solutions are easy to parametrize; small examples in the other cases include

$$11^3 + 37^3 = 228^2, \quad 143^3 + 433^2 = 42^4, \quad 3^4 + 46^2 = 13^3 \quad \text{and} \quad 10^2 + 3^5 = 7^3.$$

7.1 $Ax^2 + By^2 = Cz^r$: *dihedral coverings.* The dihedral group $\Gamma = D_{2r} = \langle \sigma, \tau : \sigma^r = \tau^2 = (\sigma\tau)^2 = 1 \rangle$, of order $2r$, acts on $t \in X = \mathbf{P}_1$ by the actions $\sigma(t) = \zeta_r t$ and $\tau(t) = 1/t$, where ζ_r is a primitive rth root of unity. The function $(t^r + t^{-r})/4$ generates the field of invariants of Γ, and so

$$\pi_{2,2,r} : X \longrightarrow \mathbf{P}_1 \quad \text{defined by} \quad \pi_{2,2,r}(t) = (t^r + t^{-r})^2/4$$

is a covering map of signature $(2,2,r)$ with Galois group Γ. One can recover the parametric solution $(t^r + 1)^2 - (t^r - 1)^2 = 4t^r$ from π.

Parametric solutions to $x^2 + y^2 = z^r$ may be obtained by defining polynomials x and y from the formula $x(u,v) + iy(u,v) = (u + iv)^r$, with $z = u^2 + v^2$. Parametric solutions to $x^2+y^2=z^r$ may be obtained by taking $(u^r+2^{r-2}v^r)^2-(u^r-2^{r-2}v^r)^2 = (2uv)^r$. In each case, we obtain proper solutions whenever v is even and $(u,v)=1$.

To obtain a solution to (5.1), define polynomials f and h by $h-\sqrt{t}f = (1-\sqrt{t})(1-\sqrt{t}(1-t))^{2r}$, so that $tf^2+(1-t)(1-t(1-t)^2)^{2r} = h^2$. With some work we find that, in all cases, $k=1$ and our new proper solution is larger than our old one.

7.2 $Ax^3+By^3=Cz^2$: *tetrahedral coverings.* The group of rotations, Γ, which preserve a regular tetrahedron, is isomorphic to the alternating group on four letters. The covering map of degree 4,

$$\pi_1 : X' \longrightarrow \mathbf{P}_1 \quad \text{defined by} \quad \pi_1(t) = -(t-1)^3(t-9)/64t,$$

has signature $(3,2,3)$, since $1-\pi_1(t) = (t^2-6t-3)^2/64t$. Let X be the Galois closure of X' over $\mathbf{P}_1$. Since the covering map $\pi_2 : X \longrightarrow X'$ must be cyclic of degree 3, and ramified at both 0 and 9 in X', we may define it by $\pi_2(u) = 9/(1-u^3)$. The composition covering map $\pi_{2,3,3} = \pi_1 \circ \pi_2 : X \longrightarrow \mathbf{P}_1$ is then given by

$$\pi_{2,3,3}(u) = \frac{(u^3+8)^3u^3}{64(u^3-1)^3}, \quad \text{so that} \quad 1-\pi_{2,3,3}(u) = \frac{-(u^6-20u^3-8)^2}{64(u^3-1)^3}.$$

The general solution to $x^3+y^3=z^2$ splits into two parametrizations:

$$x = a(a^3-8b^3)/t^2, \quad y = 4b(a^3+b^3)/t^2, \quad z = (a^6+20a^3b^3-8b^6)/t^3,$$

where $(a,b)=1$, a is odd and $t=(3,a+b)$ (due to Euler, 1756); and

$$x = (a^4+6a^2b^2-3b^4)/t^2, \quad y = (3b^4+6a^2b^2-a^4)/t^2, \quad z = 6ab(a^4+3b^4)/t^3,$$

where $(a,b)=1$, 3 does not divide a, and $t=(2,a+1,b+1)$ (due to Hoppe, 1859).

One obtains infinitely many proper solutions of $x^3+y^3=Cz^2$ by taking $ab=Ct^2$ even, with $(a,b)=1$ and 3 not dividing a, in Euler's identity

$$(6ab+a^2-3b^2)^3+(6ab-a^2+3b^2)^3 = ab\{6(a^2+3b^2)\}^2.$$

Moreover, Gerardin (1911) gave a formula to obtain a new solution from a given one:

$$(a^3+4b^3)^3-(3a^2b)^3 = (a^3+b^3)(a^3-8b^3)^2.$$

A solution to (5.1) is given by

$$t(-7-42t+t^2)^3+(1-t)(1+109t-109t^2-t^3)^2 = (1-42t-7t^2)^3.$$

The prime divisors of k can be only 2 and 3, but k is not necessarily a sixth power, so proper solutions do not necessarily lead to new proper solutions of the same equation.

7.3 $Ax^2+By^3=Cz^4$: *octahedral coverings.* The group of rotations, Γ, which preserve a regular octahedron (or *cube*), is isomorphic to the permutation group on four letters. A map $\pi_{2,3,4} : \mathbf{P}_1 \longrightarrow \mathbf{P}_1$ of signature $(2,3,4)$ can be obtained by considering the projection $\mathbf{P}_1 \longrightarrow \mathbf{P}_1/\Gamma$, so that $\pi_{2,3,4}$ has degree $|\Gamma| = 24$. However, we may obtain an equation for $\pi_{2,3,4}$ without explicitly finding the Γ-invariants or even writing down the action of Γ, by observing that one can take $\pi_{2,3,4} = \phi \cdot \pi_{2,3,3}$, where $\phi : \mathbf{P}_1 \longrightarrow \mathbf{P}_1$ is a map of degree 2 for which

$$\phi(1) = \infty, \quad \phi(0) = \phi(\infty) = 0 \quad \text{and} \quad \phi \text{ is ramified over } 1.$$

The only function ϕ with these properties is $\phi(t) = -4t/(t-1)^2$, so that

$$\pi_{2,3,4}(u) = \frac{-2^8(u(u^3-1)(u^3+8))^3}{(u^6-20u^3-8)^4}$$

and

$$1-\pi_{2,3,4}(u) = \frac{((u^6+8)(u^6+88u^3-8))^2}{(u^6-20u^3-8)^4}.$$

We have a parametric solution to $x^2+y^3=z^4$ by taking $A=a^4$, $B=b^4$ and $C=4A-3B$ in

$$C^2(16A^2+408AB+9B^2)^2+\left(144AB-C^2\right)^3 = AB(24A+18B)^4.$$

This leads to a proper solution if b is odd, 3 does not divide a, and $(a,b)=1$.

We have a parametric solution to $x^2+y^4=z^3$ by taking $P=p^2$, $Q=q^2$ in

$$\begin{gathered}16PQ(P-3Q)^2(P^2+6PQ+81Q^2)^2(3P^2+2PQ+3Q^2)^2\\ +(3Q+P)^4(P^2-18PQ+9Q^2)^4 = (P^2-2PQ+9Q^2)^3(P^2+30PQ+9Q^2)^3.\end{gathered}$$

This leads to a proper solution if $p+q$ is odd, 3 does not divide p, and $(p,q)=1$.

There is an easy parametric solution to $108x^4+y^3=z^2$ obtained by taking $U=u^4$, $V=v^4$ in

$$108UV(U+V)^4+(U^2-14UV+V^2)^3 = (U^3+33U^2V-33UV^2-V^3)^2.$$

This leads to a proper solution if uv is even and $(u,v)=1$.

7.4 $Ax^2+By^3=Cz^5$: *Klein's icosahedron.* We follow [**19**, p. 657] in describing Klein's beautiful analysis of $x^2+y^3=z^5$. The group of rotations, Γ, which preserve a regular icosahedron, is isomorphic to the alternating group on five letters. A map $\pi_{2,3,5}:\mathbf{P}_1 \longrightarrow \mathbf{P}_1$ of signature $(2,3,5)$ can be obtained by considering the projection $\mathbf{P}_1 \longrightarrow \mathbf{P}_1/\Gamma$, with Γ thought of as a subgroup of PGL_2. The ramification points of orders 2, 3 and 5 occur, respectively, as the edge midpoints, face centres and vertex points of the icosahedron.

The zeros of $z(u,v) = uv(u^{10}+11u^5v^5-v^{10})$ in $\mathbf{P}_1(\mathbf{C})$ lie at $u/v=0$, ∞ and $\left(\frac{-1\pm\sqrt{5}}{2}\right)e^{2i\pi j/5}$, corresponding to the twelve vertices of the icosahedron under stereographic projection onto the Riemann sphere. The homogeneous polynomials

$$y(u,v) = \frac{1}{121}\det(\mathrm{Hessian}(z(u,v))) \quad\text{and}\quad x(u,v) = \frac{\partial(y,z)}{\partial(u,v)},$$

are invariant under the action of the icosahedral group. They satisfy the icosahedral relation $x(u,v)^2+y(u,v)^3 = 1728z(u,v)^5$ leading to Klein's identity,

$$\begin{gathered}(a^6+522a^5b-10005a^4b^2-10005a^2b^4-522ab^5+b^6)^2\\ -(a^4-228a^3b+494a^2b^2+228ab^3+b^4)^3 = 1728ab(a^2+11ab-b^2)^5.\end{gathered}$$

This gives proper solutions to $x^2+y^3=Cz^5$ if we take $ab=144Ct^5$, with $\gcd(a,b)=1$ and $a\not\equiv 2b \pmod 5$.

The factor $1728=12^3$ which appears above is familiar to amateurs of modular forms (it appears in connection with the modular function j). Klein observed that this is no accident, since our icosahedral covering can be realized as the covering of modular curves $X(5)\longrightarrow X(1)$, where $X(1)$ is the j-line (and, indeed, our tetrahedral and octahedral coverings above can be realized as the coverings $X(3)\longrightarrow X(1)$ and $X(4)\longrightarrow X(1)$, respectively).

8. *The 'class group' obstruction to a 'local–global' principle*

If 3 does not divide ab, then $z = (a^2 + 29b^2)/3$, $x = az$, $y = bz$ is a solution to

$$x^2 + 29y^2 = 3z^3. \tag{8.1}$$

Taking $a = b = 1$ gives $x = y = z = 10$; taking $a = 2$, $b = 1$ gives $x = 22$, $y = z = 11$. For every prime p, at least one of these two solutions has no more than one of x, y, z divisible by p; that is, there exist 'proper local solutions' to (8.1) for every prime p. So are there any proper solutions 'globally'?

Suppose that we are given a proper solution to (8.1). Factor (8.1) as an ideal equation:

$$(x + \sqrt{-29}y)(x - \sqrt{-29}y) = (3)(z)^3.$$

$G = (x + \sqrt{-29}y, x - \sqrt{-29}y)$ divides $(2x, 2\sqrt{-29}y, 3z^3) = (2, z)$, which equals 1, since if z were even, then x and y must both be odd, and so (8.1) would give $1 + 29 \equiv 0 \pmod 8$, which is false. Thus $G = 1$, and so (choosing the sign of y appropriately)

$$(x + \sqrt{-29}y) = (3, 1 + \sqrt{-29})\zeta_+^3 \quad \text{and} \quad (x - \sqrt{-29}y) = (3, 1 - \sqrt{-29})\zeta_-^3,$$

where $\zeta_+\zeta_- = (z)$. This implies that the ideal classes to which $(3, 1 \pm \sqrt{-29})$ belong must both be cubes inside the class group C of $\mathbf{Q}(\sqrt{-29})$. However, this is false, since they are both generators of C, which has order 6. Therefore (8.1) has no proper solutions, indicating that the 'local–global' principle fails.

It is not too hard to generalize this argument to obtain 'if and only if' conditions for the existence of proper solutions to (2), especially for carefully chosen values of A, B, C and r. We prove the following.

PROPOSITION 8.1. *Suppose that $r \geqslant 2$, and b and c are coprime positive integers with $b \equiv 1 \pmod 4$ and squarefree, and c odd.*

(i) *There are proper integer solutions to $x^2 + by^2 = cz^r$ if and only if there exist coprime ideals J_+, J_- in $\mathbf{Q}(\sqrt{-b})$ with $J_+J_- = (c)$, whose ideal classes are rth powers inside the class group of $\mathbf{Q}(\sqrt{-b})$.*

(ii) *There are proper local solutions to $x^2 + by^2 = cz^r$ at every prime p if and only if the Legendre symbol $(-b/p) = 1$ for every prime p dividing c; and when r is even, we have $(c/p) = 1$ for every prime p dividing b, as well as $c \equiv 1 \pmod 4$.*

Proof. Given proper integer solutions to $x^2 + by^2 = cz^r$, the proof of (i) is entirely analogous to the case worked out above. In the other direction, if the ideal class of J_+ is an rth power, we may select an integral ideal ζ_+ for which $J_+\zeta_+^r$ is principal, $(x + \sqrt{-b}y)$ say. Then $(x^2 + by^2) = (cz^r)$ where $(z) = \mathrm{Norm}(\zeta_+)$, and the result follows.

In (ii) it is evident that all of the conditions are necessary. We must show how to find a proper local solution at prime p given these conditions. It is well known that if prime p does not divide $2bc$, then there is a solution in p-adic units x, y to $x^2 + by^2 = c$, and so we can take $(x, y, 1)$. It is also well known that if prime p is odd and $(-b/p) = 1$, then there is a p-adic unit x such that $x^2 = -b$, and so we take $(x, 1, 0)$. Similarly, if $(c/p) = 1$, then there is a p-adic unit x such that $x^2 = c$, and so we take $(x, 0, 1)$. If r is odd and p does not divide c, then we may take

$(c^{(r+1)/2}, 0, c)$. Finally, if r is even and $c \equiv 5 \pmod 8$, then there is a p-adic unit x such that $x^2 = c - 4b$, so we take $(x, 2, 1)$.

The conditions for proper integer solutions, given above, depend on the value of (r, h) where h is the class number of $\mathbf{Q}(\sqrt{-b})$. On the other hand, the conditions for proper local solutions everywhere, given above, depend only on the parity of r. The local–global principle for conics tells us that these are the same for $r = 2$; it is thus evident that the conditions are not going to coincide if $(r, h) \geqslant 3$.

The techniques used here may be generalized to study when the value of an arbitrary binary quadratic form is equal to a given constant times the rth power of an integer. The techniques can also be modified to find obstructions to a local–global principle for equations $x^2 + by^4 = cz^3$, and probably to $x^3 + by^3 = cz^2$. On the other hand, there are never any local obstructions for equations $Ax^2 + By^3 = Cz^5$ which have A, B, C pairwise coprime: if p does not divide AB or AC or BC, then we can take $(AB^2, -AB, 0)$ or $(A^2C^3, 0, AC)$ or $(0, B^3C^2, B^2C)$, respectively. Could it be that such equations always have proper integer solutions?

9. *Conjectures on generalized Fermat equations*

9.1 *How many proper solutions can* (2) *have if* $1/p + 1/q + 1/r < 1$? It is evident that any equation of the form

$$(y_1^q z_2^r - y_2^q z_1^r)x^p + (z_1^r x_2^p - z_2^r x_1^p)y^q = (x_2^p y_1^q - x_1^p y_2^q)z^r$$

has the two solutions (x_i, y_i, z_i). If there are three solutions to an equation (2), then we may eliminate A, B and C using linear algebra to deduce that

$$x_1^p y_2^q z_3^r + x_2^p y_3^q z_1^r + x_3^p y_1^q z_2^r = x_1^p y_3^q z_2^r + x_2^p y_1^q z_3^r + x_3^p y_2^q z_1^r.$$

If $1/p + 1/q + 1/r$ is sufficiently small, then the generalization of the *abc*-conjecture (see Subsection 5.2) implies that this has only finitely many solutions. Thus there are only finitely many triples of coprime integers A, B, C for which (2) has more than two proper solutions. (Bombieri and Mueller [3] proved such a result unconditionally in $\mathbf{C}[t]$, since [5] and [43] provide the necessary generalization of the *abc*-conjecture.)

If $n = p = q = r$, then it is easy to determine A, B, C from the equation above. In fact, Desboves [8] proved that the set of coprime integers A, B, C, together with three given distinct solutions to $Ax^n + By^n = Cz^n$, is in 1–1 correspondence with the set of coprime integer solutions to

$$r^n + s^n + t^n = u^n + v^n + w^n \quad \text{with} \quad rst = uvw,$$

where $\{r^n, s^n, t^n\} \cap \{u^n, v^n, w^n\} = \emptyset$. Applying a suitable generalized *abc*-conjecture to this, we immediately deduce the following. There exists a number n_0 such that *if* $n \geqslant n_0$, *then there are at most two proper solutions to* $Ax^n + By^n = Cz^n$ *for any given non-zero integers* A, B, C. Moreover, there exist infinitely many triples A, B, C for which there do exist two proper solutions.

9.2 *Diagonal equations with four or more terms.* The generalized *abc*-conjecture implies that

$$a_1 x_1^{p_1} + a_2 x_2^{p_2} + \ldots + a_n x_n^{p_n} = 0$$

has only finitely many *proper* K-integral solutions, in every number field K, if $\sum_j 1/p_j$ is sufficiently small. Here are a few interesting examples of known solutions to such equations.

(i) Ryley proved that every integer can be written as the sum of three rational cubes. (This appeared in the *Ladies' Diary* (1825) 35.) For example, Mahler noted that $2 = (1+6t^3)^3 + (1-6t^3)^3 - (6t^2)^3$. Ramanujan gave a parametric solution for $x^3+y^3+z^3=t^3$:

$$(3a^2+5ab-5b^2)^3 + (4a^2-4ab+6b^2)^3 + (5a^2-5ab-3b^2)^3 = (6a^2-4ab+4b^2)^3.$$

Examples include $3^3+4^3+5^3=6^3$, and Hardy's taxi-cab number $1^3+12^3=9^3+10^3$.

(ii) Taking $u=(x_n-y_n)/2$, $v=y_n$, where $\left(\frac{x_n+y_n\sqrt{-3}}{2}\right) = \left(\frac{5+\sqrt{-3}}{2}\right)^n$, in Diophantos' identity

$$u^4+v^4+(u+v)^4 = 2(u^2+uv+v^2)^2, \tag{9.1}$$

gives proper solutions to $a^4+b^4+c^4=2d^n$; specifically,

$$\left(\frac{x_n+y_n}{2}\right)^4 + \left(\frac{x_n-y_n}{2}\right)^4 + y_n^4 = 2\times 7^{2n}. \tag{9.2}$$

(iii) Euler gave the first parametric solution to $x^4+y^4=a^4+b^4$, in polynomials of degree seven; an example is $59^4+158^4=133^4+134^4$. By a sophisticated analysis of Demjanenko's pencil of genus one curves on the surface $t^4+u^4+v^4=1$, Elkies **[11]** showed that there are infinitely many triples of coprime fourth powers of integers whose sum is a fourth power. (This radically contradicts Euler's Conjecture that for any $n \geqslant 3$, the sum of $n-1$ distinct nth powers of positive integers cannot be an nth power.) The smallest of these is

$$95800^4+217519^4+414560^4=422481^4.$$

(iv) In 1966, Lander and Parkin gave the first counterexample to Euler's Conjecture,

$$27^5+84^5+110^5+133^5=144^5.$$

In 1952, Swinnerton-Dyer had shown how to give a parametric solution to $a^5+b^5+c^5=x^5+y^5+z^5$; a small example is $49^5+75^5+107^5=39^5+92^5+100^5$.

(v) In 1976, Brudno gave a parametric solution to $a^6+b^6+c^6=x^6+y^6+z^6$ of degree 4; a small example is $3^6+19^6+22^6=10^6+15^6+23^6$.

We do know of various examples of

$$Ax^j+By^k+Cz^\ell=Dw^m \tag{4}$$

with infinitely many proper solutions and $1/j+1/k+1/\ell+1/m$ small, as follows.

(a) (9.2) is an example of an equation (4) having infinitely many proper solutions, with $1/j+1/k+1/\ell+1/m$ arbitrarily close to $3/4$. We can also obtain this by taking $u=x^p$ and $v=y^q$ in (9.1).

(b) In Section 6 we saw how to choose A,B,C for any given $1/p+1/q+1/r=1$ so that there are infinitely many proper solutions to (2). Substituting $u=Ax^p$ and $v=By^q$ of (2) into Diophantos' identity (9.1), we obtain infinitely many proper solutions of some equation (4) with exponents $(4p,4q,4r,2)$, so that $1/j+1/k+1/\ell+1/m=3/4$.

(c) By taking $t=2z^n$ in the identity $(t+1)^3-(t-1)^3=6t^2+2$, we obtain infinitely many proper solutions to $x^3 + y^3 = 24z^{2n} + 2w^m$; here $1/j + 1/k + 1/\ell + 1/m$ is arbitrarily close to $2/3$.

(d) Elkies [13] points out that by taking $t^2 + t - 1 = u^2$ and $t^2 - t - 1 = Av^2$, whenever this defines an elliptic curve of positive rank (for instance, when $A = 5$), in the identity $(t^2 + t - 1)^3 + (t^2 - t - 1)^3 = 2(t^6 - 1)$, we obtain infinitely many proper solutions to some equation (4) with $1/j + 1/k + 1/\ell + 1/m = 1/6 + 1/6 + 1/6 + 1/6 = 2/3$.

(e) Elkies [13] also points out that $\sum_{\alpha^4=1} \alpha((\alpha x)^2 + 2(\alpha x) - 2)^5 = 0$. Thus by taking $x^2 + 2x - 2 = ay^2$ and $x^2 - 2x - 2 = bz^2$ whenever this defines an elliptic curve of positive rank over $\mathbf{Q}(i)$, we obtain infinitely many proper solutions in $\mathbf{Z}[i]$ to some equation (4) with $1/j + 1/k + 1/\ell + 1/m = 1/10 + 1/10 + 1/5 + 1/5 = 3/5$.

(f) If we allow *improper* solutions, that is where pairs of the monomials in (4) have large common factors, then one can obtain $1/j + 1/k + 1/\ell + 1/m$ arbitrarily close to $1/2$ from the identity $x^{2n} + 2(xy)^n + y^{2n} = (x^n + y^n)^2$.

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