NOTES

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A Continued Fraction and Permutations With Fixed Points

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The set Σ_n of all permutations on $\Omega_n = \{1, 2, 3, \dots, n\}$ is partitioned into FPF_n permutations acting without fixed points and FP_n acting with at least one fixed point. These numbers are related by

$$|\Sigma_n| = n! = FPF_n + FP_n \quad \text{for} \quad n \geq 0,$$

with the convention that $FPF_0 = 1$ and $FP_0 = 0$. We shall index all sequences from 0.

Let V denote the two-dimensional real vector space of sequences $\{t_n\}$ satisfying the recurrence

$$t_{n+1} = n(t_n + t_{n-1}).$$

To convince the reader that V is worthy of study, consider the following:

Theorem. *The sequences $\{n!\}$, $\{FPF_n\}$, and $\{FP_n\}$ belong to V.*

Proof. The proof that $\{n!\} \in V$ is trivial. To show that $\{FPF_n\} \in V$ we observe that any fixed-point-free permutation in Σ_{n+1} can be obtained from the identity permutation in exactly one of two ways:

(1) by applying a permutation with a unique fixed point $i \in \Sigma_n$ to the first n elements, and then transposing $n+1$ and i, or:
(2) by applying a fixed-point-free permutation to the first n elements, followed by a transposition of $n+1$ and some $i \in \Omega_n$.

There are n possible choices for i, FPF_{n-1} possible permutations on $1, \dots, n$ which leave only i fixed, and FPF_n fixed-point-free permutations on the first n elements. So $FPF_{n+1} = n(FPF_n + FPF_{n-1})$, and hence $\{FPF_n\} \in V$. The last statement of the theorem follows from the identity $FP_n + FPF_n = n!$. ■

The vector space V also contains a sequence which converges to 0, which means that the null sequences form a proper (one-dimensional) subspace of V. This is the content of the next theorem:

Main Theorem. *The sequence $\{I_n\}$ given by*

$$I_n = \int_0^\infty (e^{-x} - 1)^n e^{-e^{-x} - x}\, dx = \int_0^1 (-t)^n e^{t-1}\, dt$$

belongs to V. Moreover, $I_0 = 1 - 1/e$, $I_1 = -1/e$, $\lim_{n\to\infty} I_n = 0$, and $\forall n$, $|I_n| < 1$.

Proof. That $\{I_n\}$ satisfies the appropriate recurrence follows from integration by parts. It is a trivial exercise to evaluate I_0 and I_1. The limit of $\{I_n\}$, as well as the bound, follow from dominance of I_n by $\int_0^\infty e^{-x}\,dx$. ■

Let us denote by $S(a,b)$ the sequence $\{t_n\} \in V$ satisfying the initial conditions $t_0 = a$, $t_1 = b$. The previous results can be summarized as follows:

$$\begin{aligned} S(0,1) &= \{FP_n\}, \\ S(1,0) &= \{FPF_n\}, \\ S(1,1) &= \{n!\} = \{FP_n + FPF_n\}, \\ S\left(1-\frac{1}{e}, -\frac{1}{e}\right) &= \{I_n\}. \end{aligned}$$

Choosing an appropriate basis for V, one has

$$S(0,1) = \left(1-\frac{1}{e}\right)S(1,1) - S\left(1-\frac{1}{e}, -\frac{1}{e}\right),$$

which leads to the following result:

$$FP_n = \left(1-\frac{1}{e}\right)n! - \int_0^1 t^n e^{t-1}\,dt. \qquad (*)$$

The approximation $FP_n \approx (1-1/e)n!$ is well known in elementary combinatorial probability, where the result is usually stated as:

THEOREM. *After a random shuffle of a deck of cards, the chances that at least one card will remain fixed is very close to* $1 - 1/e$.

Formula $(*)$ provides a closed form expression for the error term.

Our investigation concludes with an evaluation of a continued fraction which is closely related to the previous formula:

THEOREM.

$$\frac{1}{e-1} = \cfrac{1}{1+\cfrac{2}{2+\cfrac{3}{3+\cdots}}}.$$

Proof. The right-hand side converges and is equal to $\lim_{n\to\infty} p_n/q_n$, where $\{p_n\}$, $\{q_n\} \in V$, and $p_0 = q_1 = 1$, $p_1 = q_0 = 0$. So

$$\cfrac{1}{1+\cfrac{2}{2+\cdots}} = \lim_{n\to\infty}\frac{S(1,0)}{S(0,1)} = \frac{1/e}{1-1/e} = \frac{1}{e-1}. \quad ■$$

The theory of the recurrence relations determining the p_n, q_n appears in [**1**, p. 492], [**2**, p. 15]. An evaluation of the continued fraction is given as an exercise in [**1**, Ch. XXXIV, Ex. 24, p. 576], but it is arrived at quite differently, using a classical formula of Gauss which gives continued fraction expansions of ratios of hypergeometric series.

REFERENCES

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Smoothness From Finite Points

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1. Introduction. Like other branches of mathematics, infinitesimal analysis allows some informality; but careless use of terms may lead to wrong answers.

Unfortunately, there is a serious error in the definition of tangent plane given in [2]. It is closely related to the well-known "Schwarz Accordion," which is treated in [4, p. 124] (with incorrect algebra). There also seems to be a tendency to blur the distinction between "standard" and "finite." Nevertheless, the main result of [2, p. 434], can be reclaimed with correct definitions. That theorem is actually proved, if one replaces the definition of tangent plane as in section 3, below. Moreover, the corrected proof is certainly simpler then the classical one we first learned from [1], for it goes more directly to the point.

Ours is not the only possible correction to Henle's definition of tangent plane. It suffices to require that the three close points that define the nearly tangent planes, form a triangle with no infinitesimal angles.

We use the notation set up in [3] chapters 1 and 1*. $\mathbb{R}^*$ denotes the nonstandard extension of the set of real numbers $\mathbb{R}$.

We consider $\mathbb{R}$ a subset of $\mathbb{R}^*$. $\mathbb{R}^{n*} = (\mathbb{R}^*)^n$ $(n = 2, 3)$.

An element r of $\mathbb{R}^*$ is standard, finite, or infinitesimal if, respectively, $r \in \mathbb{R}$, $|r| < s$ for some s in $\mathbb{R}^+$, or $|r| < s$ for all $s \in \mathbb{R}^+$. An element of $\mathbb{R}^{n*}$, $(n = 1, 2, 3)$, is *standard*, *finite*, or *infinitesimal* if all its coordinates are, respectively, standard, finite, or infinitesimal. We say x is infinitely close to I in $\mathbb{R}^{n*}$, $x \approx y$, if $x - y$ is infinitesimal.

Also, for $n = 1, 2, 3$, if $a \in \mathbb{R}^{n*}$ is finite and b is the (unique) standard element of $\mathbb{R}^{n*}$ which is infinitely close to a, we say b is the *standard part* of a and write $b = \mathrm{st}\, a$.

Finally, functions and their nonstandard extensions will be denoted with the same symbol.

2. A critique of [2]. Defining closeness of planes with the difference in normal unit vectors is dangerous: a plane is not close to itself if one takes opposite normals. But let us assume that a choice of unit normals was implicitly assumed.

According to Henle's definition [2], a tangent plane to a surface S, at a point u of S, is infinitely close to any plane determined by a three-tuple (u, v, w) of infinitely close, noncollinear points on the surface. This is simply false. This