Rigid Meromorphic Cocycles

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(joint work with Alice Pozzi, Jan Vonk)

A *rigid meromorphic cocycle* is a class in $H^1(\Gamma, \mathcal{M}^\times)$, where $\Gamma := \mathbf{SL}_2(\mathbb{Z}[1/p])$ is Ihara's group and $\mathcal{M}^\times$ is the multiplicative group of non-zero rigid meromorphic functions on the Drinfeld p-adic upper half-plane $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$, endowed with its natural action of Γ by Möbius transformations.

It is a prototypical example of more general objects arising in a tentative "p-adic Borcherds theory" which was briefly alluded to at the end of the lecture.

The main motivation (so far) for studying rigid meromorphic cocycles lies in their eventual connection with explicit class field theory for real quadratic fields. More precisely, a point $\tau \in \mathcal{H}_p$ is called a *real multiplication* (RM) point if it satisfies the following equivalent properties:

(1) The field $\mathbb{Q}(\tau)$ is a real quadratic field;
(2) The stabiliser of τ in Γ is infinite.

When τ is an RM point, its stabiliser Γ_τ has rank one, and is generated up to torsion by an *automorph* $\gamma_\tau \in \Gamma$, which can be chosen consistently by fixing appropriate orientations. The *value* of the rigid meromorphic cocycle J at τ is defined to be

$$J[\tau] := J(\gamma_\tau)(\tau) \in \mathbb{C}_p^\times \cup \{0, \infty\}.$$

Although the rigid meromorphic function $J(\gamma_\tau)$ depends on the choice of a representative one-cocycle, the value of this function at τ depends only on the class of J in cohomology. The value $J[\tau]$ also depends only on the Γ-orbit of τ, i.e., $J[\gamma\tau] = J[\tau]$ for all $\gamma \in \Gamma$.

The stabiliser of the RM point τ in the matrix ring $M_2(\mathbb{Z}[1/p])$ is isomorphic to a $\mathbb{Z}[1/p]$-order, denoted $\mathcal{O}$, in the real quadratic field $F = \mathbb{Q}(\tau)$. Global class field theory gives a canonical identification

$$\mathrm{Pic}^+(\mathcal{O}) = \mathrm{Gal}(H_\tau/F),$$

where $\mathrm{Pic}^+(\mathcal{O})$ denotes the class group in the narrow sense of $\mathcal{O}$ — i.e., the Picard group of projective $\mathcal{O}$-modules equipped with an orientation at ∞. The abelian extension H_τ of F is called the *narrow ring class field* attached to τ. Together with cyclotomic fields, the narrow ring class fields generate almost the full maximal abelian extension of F.

The following conjecture was proposed in [DV1]:

Conjecture 1. *If J is a rigid meromorphic cocycle, then there is a finite extension H_J of $\mathbb{Q}$ — the "field of definition" of J — for which $J[\tau]$ belongs to the compositum H_J and H_τ, for all RM points τ of $\mathcal{H}_p$.*

Until recently, almost all of the evidence for Conjecture 1 has been numerical and experimental, but recent progress based on the theory of p-adic deformations of modular forms and their associated Galois representations has led to strong theoretical evidence as well, so that Conjecture 1 now appears to lie within the scope of available techniques.

The lecture focussed first on a few simple examples of rigid meromorphic cocycles and related objects:

(1) the *Dedekind-Rademacher cocycle*, an avatar of Siegel units;
(2) *elliptic modular cocycles* attached to elliptic curves of conductor p;
(3) genuine rigid meromorphic cocycles, which play the role of meromorphic modular functions like the j-function.

It concluded by attempting to place the theory of rigid meromorphic cocycles in the more general framework of automorphic forms on orthogonal groups.

Let $\mathcal{A}^\times \subset \mathcal{M}^\times$ be the multiplicative group of rigid analytic functions on $\mathcal{H}_p$. It turns out that there are no interesting genuine rigid analytic cocycles: the group $H^1(\Gamma, \mathcal{A}^\times)$ is generated by the class J_{triv} given by

$$J_{\mathrm{triv}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) := (cz + d),$$

whose value at an RM point is the fundamental unit of the associated order. A richer class of examples is obtained by relaxing the definition and considering one-cochains that satisfy the cocycle relation up to certain multiplicative periods. The simplest example of such a "cocycle modulo periods" is the *Dedekind-Rademacher cocycle*.

To define this cocycle, we begin by noting the canonical identification

$$H^2(\Gamma, \mathbb{Q}) = H^1(\Gamma_0(p), \mathbb{Q})$$

arising from the fact that Γ is an amalgamated product of two copies of $\mathbf{SL}_2(\mathbb{Z})$ intersecting in a subgroup that is conjugate to $\Gamma_0(p)$. (This in turn follows from the transitive action of Γ on the edges of the Bruhat-Tits tree, in which the vertex and edge stabilisers are conjugate to $\mathbf{SL}_2(\mathbb{Z})$ and $\Gamma_0(p)$ respectively.) The *Dedekind-Rademacher two-cocycle* is the class $\alpha_{\mathrm{DR}} \in H^2(\Gamma, \mathbb{Z})$ that corresponds, under this identification, to the *Dedekind-Rademacher homomorphism* $\varphi_{\mathrm{DR}} \in H^1(\Gamma_0(p), \mathbb{Z})$ given by

$$\varphi_{\mathrm{DR}}(\gamma) := \frac{1}{2\pi i} \int_{z_0}^{\gamma z_0} E_2^{(p)}(z), \qquad E_2^{(p)} := \mathrm{dlog}\left(\frac{\Delta(pz)}{\Delta(z)}\right).$$

The one-cocycle φ_{DR} and the two-cocycle α_{DR} thus encode the periods of the Eisenstein series $E_2^{(p)}$ arising from the logarithmic derivative of the Siegel unit $\Delta(pz)/\Delta(z)$ on the open modular curve $Y_0(p)$.

The key fact underlying the construction of the Dedekind-Rademacher cocycle is that $p^{\alpha_{\mathrm{DR}}} \in H^2(\Gamma, \mathbb{C}_p^\times)$ *is trivialised* in the group $H^2(\Gamma, \mathcal{A}^\times)$. The following theorem from [DPV2] refines an earlier construction from [DD]:

Theorem 2. *There is a one-cochain $J_{\mathrm{DR}} \in C^1(\Gamma, \mathcal{A}^\times)$ satisfying*

$$\gamma_1 J_{\mathrm{DR}}(\gamma_2) \div J_{\mathrm{DR}}(\gamma_1\gamma_2) \times J_{\mathrm{DR}}(\gamma_1) = p^{\alpha_{\mathrm{DR}}(\gamma_1,\gamma_2)}, \qquad \textit{for all } \gamma_1, \gamma_2 \in \Gamma.$$

The essential triviality of $H^1(\Gamma, \mathcal{A}^\times)$ ensures that the one co-chain J_{DR} of Theorem 2 is well defined, up to powers of J_{triv} and one-coboundaries. Furthermore, it satisfies the one-cocycle relation up to powers of p. Its image in $H^1(\Gamma, \mathcal{A}^\times/p^{\mathbb{Z}})$ is therefore well-defined up to powers of J_{triv}. This image is called the *Dedekind-Rademacher cocycle*, and is also denoted by J_{DR}, by a slight abuse of notation.

Theorem 3. *Let τ be an RM point in $\mathcal{H}_p$. Up to torsion in $\mathbb{Q}_p(\tau)^\times$, the value $J_{\mathrm{DR}}[\tau]$ belongs to $\mathcal{O}_{H_\tau}[1/p]^\times$.*

This proof of this theorem emerged from a gradual series of developments over almost two decades:

1. *Relation with the Gross-Stark conjecture.* The relation

$$\log_p(\mathrm{Norm}^{\mathbb{Q}_p^2}_{\mathbb{Q}_p}(J_{\mathrm{DR}}[\tau])) = L_p'(\mathcal{C}_\tau, 0),$$

where $L_p(\mathcal{C}_\tau, s)$ is the partial p-adic L-function attached to the narrow ideal class $\mathcal{C}_\tau := [1, \tau]$ was obtained in [DD]. This result shows the algebraicity properties of $J_{\mathrm{DR}}[\tau]$ asserted in Theorem 3 are satisfied by its *norm* from $(F \otimes \mathbb{Q}_p)^\times$ to $\mathbb{Q}_p^\times$, assuming Gross's p-adic analogue of the Stark conjecture on leading terms of abelian L-series at $s = 0$ [Gr1].

2. *Proof of the Gross-Stark conjecture in rank one.* It was then shown in [DDP] that the Gross-Stark conjecture is true for the first derivatives of these abelian L-series, hence that $J_{\mathrm{DR}}[\tau]$ satisfies the predicted algebraicity properties, *up to torsion and elements of $(F \otimes \mathbb{Q}_p)^\times$ of norm one.* What makes Gross's p-adic analogue of the Stark conjecture more approachable than the original archimedean conjectures, which are still completely open, is the availability of the theory of p-adic deformations of modular forms and their associated p-adic Galois representations, along with the reciprocity law of global class field theory. In a sense, these ingredients are used to parlay class field theory for abelian extensions of F into explicit class field theory for F.

3. *The work of Dasgupta and Kakde.* The ambiguity by elements of norm one in $\mathbb{Q}_{p^2}^\times$ is a serious limitation of the results following from [DDP]. It is addressed in a remarkable series of recent works by Samit Dasgupta and Mahesh Kakde, who show that the full Theorem 3 (up to torsion) would follow from Gross's "tame refinement" of the p-adic Gross-Stark conjecture [Gr2] and the Brumer–Stark conjecture. They are then able to *prove* these conjectures, by significantly refining and extending the techniques of [DDP] to the tame setting.

4. *Modular generating series.* An ongoing project with Alice Pozzi and Jan Vonk [DPV2] aims to give an alternate, more "genuinely p-adic" proof of Theorem 3 by realising the RM values of the Dedekind-Rademacher cocycle as the fourier coefficients of a modular generating series. This approach is suggested by [DPV1]

which studies the fourier coefficients of the ordinary projection of the first derivative, with respect to the weight, of the diagonal restriction of a p-adic family of Hilbert modular Eisenstein series attached to totally odd characters of F.

The methods used to understand the RM values of $J_{\rm DR}$ are somewhat roundabout, relying crucially on p-adic deformations of modular forms and their associated Galois representations. These techniques do not dispel the mystery surrounding the deeper arithmetic meaning of rigid meromorphic cocycles. To underscore this point, the lecture then turned to a discussion of *elliptic cocycles*.

An elliptic rigid analytic cocycle can be attached to an elliptic curve E of conductor p, or rather to its normalised newform f of weight two, whose real and imaginary periods are encoded by two one-cocycles φ_f^+ and φ_f^- in $H^1(\Gamma_0(p),\mathbb{Z})$. Let α_f^+ and α_f^- be the corresponding two-cocycles in $H^2(\Gamma,\mathbb{Z})$. The following trivialisation result, in which $\alpha_{\rm DR}$ is replaced by $\alpha_f^{\pm}$, and the prime p by the Tate period $q_E \in \mathbb{Q}_p^\times$ attached to E over $\mathbb{Q}_p$, was shown in [Da] to follow from the "exceptional zero conjecture" of Mazur, Tate and Teitelbaum [MTT] proved by Greenberg and Stevens [GS].

Theorem 4. *There are one-cochains $J_{\rm f}^+, J_f^- \in C^1(\Gamma,\mathcal{A}^\times)$ satisfying*

$$\gamma_1 J_f^{\pm}(\gamma_2) \div J_f^{\pm}(\gamma_1\gamma_2) \times J_f^{\pm}(\gamma_1) = q_E^{\alpha_f^{\pm}(\gamma_1,\gamma_2)}, \qquad \textit{for all } \gamma_1,\gamma_2\in\Gamma,$$

up to torsion in $(F\otimes\mathbb{Q}_p)^\times$.

In particular, the one-cochains J_f^+ and J_f^- satisfy the cocycle relation up to powers of q_E and up to torsion. After raising them to a suitable integer power to remove the torsion ambiguity, their images in $H^1(\Gamma,\mathcal{A}^\times/q_E^{\mathbb{Z}})$ are called the *even and odd elliptic modular cocycles* attached to f, and denoted $J_f^{\pm}$ by abuse of notation.

The RM values $J_f^{\pm}[\tau]$ are then canonical elements of $\mathbb{C}_p^\times/q_E^{\mathbb{Z}} = E(\mathbb{C}_p)$. The following conjecture, an elliptic analogue of Theorem 3, was proposed in [Da].

Conjecture 5. *The RM values $J_f^+[\tau]$ and $J_f^-[\tau]$ belong to $E(H_\tau)$.*

The evidence for this conjecture so far is largely experimental [DP], [GMS], and the speaker can discern no proof on the horizon: the tools used to handle the Dedekind-Rademacher cocycle are not available in this setting, and would perhaps need to be supplemented with a more geometric perspective on the theory of rigid cocycles.

The lecture concluded by returning to rigid meromorphic cocycles. In contrast with the fact that $H^1(\Gamma,\mathcal{A}^\times)$ is essentially trivial, the following theorem from [DV1] shows that rigid meromorphic cocycles exist in abundance:

Theorem 6. *Assume that $p-1$ divides 12, i.e, $p = 2, 3, 5, 7,$ or 13. For all RM points τ, there is a unique $J_\tau \in H^1(\Gamma, \mathcal{M}^\times)$ for which the rigid meromorphic functions $J_\tau(\gamma)$ have divisor supported in the Γ-orbit of τ.*

In [DV1], it is conjectured that the quantity

$$J_p(\tau_1, \tau_2) := J_{\tau_1}[\tau_2],$$

for pairs (τ_1, τ_2) of RM points (with co-prime discriminants, say) behave "in essentially all respects" like the differences $j(\tau_1) - j(\tau_2)$ of singular moduli studied in the work of Gross and Zagier (with τ_1 and τ_2 CM points in the Poincaré upper half-plane). The p-adic logarithm of $J_p(\tau_1, \tau_2)$ can thus be envisaged as a kind of p-adic Green's function evaluated on the pair of RM cycles attached to τ_1 and τ_2.

For example, let $\tau_1 = \varphi := (1+\sqrt{5})/2$ be the golden ratio, and $\tau_2 = 2\sqrt{2}$, which are roots of primitive binary quadratic forms of discriminants $D_1 = 5$ and $D_2 = 32$ respectively. The associated ring class fields are $H_1 = \mathbb{Q}(\sqrt{5})$ and $H_2 = \mathbb{Q}(\sqrt{2}, i)$. The pair (τ_1, τ_2) belongs to $\mathcal{H}_p \times \mathcal{H}_p$ when $p = 3$ or $p = 13$, and a computer calculation reveals that

$$\begin{aligned} J_3(\varphi, 2\sqrt{2}) &\equiv (33 + 56i)/(5 \cdot 13) \pmod{3^{600}}, \\ J_{13}(\varphi, 2\sqrt{2}) &\equiv (1 + 2\sqrt{-2})/3 \pmod{13^{100}}. \end{aligned}$$

The table below lists the prime factorisations of the quantities $(D_1D_2 - t^2)/4$ when t ranges over the even integers between 0 and 12:

t	$(160 - t^2)/4$	$a_3(t)$	$a_{13}(t)$
0	$2^3 \cdot 5$	1	1
2	$3 \cdot 13$	13	3
4	$2^2 3^2$	1	1
6	31	1	1
8	$2^3 \cdot 3$	1	1
10	$3 \cdot 5$	5	1
12	2^2	1	1

The two rightmost columns are obtained by picking out the terms in the second column that are divisible by 3 and 13 respectively, and taking the remaining factors. These are exactly the primes that appear in the experimentally observed factorisations of $J_3(\tau_1, \tau_2)$ and $J_{13}(\tau_1, \tau_2)$. Such patterns are reminiscent of the recipes for the factorisation of singular moduli in the theorem of Gross and Zagier [GZ].

The restriction on p in Theorem 6 is made to ensure that the modular curve $X_0(p)$ has genus zero, i.e., that there are no weight two cusp forms of level p. For general primes p, there is an *obstruction* to producing a rigid meromorphic cocycle with a prescribed rational RM divisor, which lies in the space of weight two cusp forms on $\Gamma_0(p)$ – or equivalently, by the Shimura correspondence, in (the Kohnen subspace of) the space $M_{3/2}(4p)$ of modular forms of weight 3/2 and level $4p$. Let $M^{!!}_{1/2}(4p)$ be the space of weakly holomorphic modular forms of weight 1/2 and level $4p$ which are regular at all the cusps except ∞ and have integer fourier coefficients at that cusp. The following theorem, which produces a systematic supply of rigid meromorphic cocycles for all primes p, is part of a work in progress [DV2]:

Theorem 7. *There is an injective homomorphism*

$$\mathrm{BL}^\times : M^{!!}_{1/2}(4p) \to H^1(\Gamma, \mathcal{M}^\times)$$

satisfying all the formal properties of Borcherds' multiplicative "singular theta-lift". (Notably, the RM divisor of $\mathrm{BL}^\times(g)$ *is encoded in the principal part of* g*.)*

This result suggests the following immediate generalisation, placing the theory of rigid meromorphic cocycles in the broader setting of p-arithmetic subgroups of orthogonal groups. Let V be a quadratic space over $\mathbb{Q}$ of real signature (r,s), and let $d := r+s$ be its dimension. Let $L \subset V$ be a $\mathbb{Z}[1/p]$-lattice in V, and assume for simplicity that it is equal to its dual. The orthogonal group $\Gamma := O(L)$ of this lattice is a p-arithmetic group which acts on the real symmetric space

$$X_\infty := O(V)/(O(r)\times O(s))$$

of dimension rs, as well as on the p-adic symmetric space

$$X_p := \tilde{X}_p/\mathbb{C}_p^\times, \qquad \tilde{X}_p := \{x \in V\otimes\mathbb{C}_p \text{ with } \langle x,x\rangle = 0\} - \bigcup_{\langle v,v\rangle=1} (\mathbb{Q}_p v)^\perp,$$

the union being taken over all vectors $v \in V\otimes\mathbb{Q}_p$ of norm 1. The domain X_p can be identified with the $\mathbb{C}_p$-points of a rigid analytic space over $\mathbb{Q}_p$. Let $\mathcal{M}^\times$ be the multiplicative group of non zero rigid meromorphic functions on X_p. Theorem 7 is generalised in [DV3] to give an injective homomorphism

$$\mathrm{BL}^\times : M^{!!}_{2-d/2}(4p) \to H^s(\Gamma, \mathcal{M}^\times)$$

with the requisite properties, most importantly, that the divisor of $\mathrm{BL}^\times(g)$ is related to a collection of "rational quadratic divisors" on X_p which can be read off from the principal part of g. The construction of $\mathrm{BL}^\times$ rests crucially on the Kudla-Millson theta kernels [KM] with coefficients in the homology of the real manifold $X_\infty/O(V,\mathbb{Z})$, and on the variation of these theta-kernels in p-adic families.

In the case of signature $(3,0)$, taking V to be the space of trace zero elements on a definite quaternion algebra B over $\mathbb{Q}$ equipped with the norm form, the image of the p-adic Borcherds lift consists of Γ-invariant meromorphic functions on $\mathcal{H}_p$ with divisor supported on CM points. The rigid analytic quotient $\Gamma\backslash\mathcal{H}_p$ is identified with the $\mathbb{C}_p$-points of a Shimura curve attached to B, thanks to the uniformisation theory of Cerednik Drinfeld. The p-adic Borcherds lift in this case leads to a new, "purely p-adic" proof of the theorem of Gross-Kohnen-Zagier asserting that a generating series made from Heegner divisors of varying discriminants is a modular form of weight 3/2.

In the case of signature $(2,1)$, taking V to be the space of trace zero elements in the matrix ring $M_2(\mathbb{Q})$ and $L := M_2(\mathbb{Z}[1/p])\cap V$, the image of the p-adic Borcherds lift is the "original" space of rigid meromorphic cocycles for $\Gamma := \mathbf{SL}_2(\mathbb{Z}[1/p])$ that provided the starting point for this lecture.

The case of signature $(3,1)$ is noteworthy in light of the "accidental" isomorphisms relating the orthogonal group of this signature and the Bianchi group $\mathbf{SL}_2(\mathcal{O}_K)$ where K is an imaginary quadratic field. That the resulting rigid meromorphic cocycles should have arithmetic significance is suggested by the work of Mak Trifkovic [Tr] and of Daniel Barrera and Chris Williams [BW] on analytic cocycles in the Bianchi setting. Peter Scholze has proposed [Sch] that (additive, weight one) rigid analytic cocycles on the Bianchi modular group might also be relevant for understanding the conjectural correspondence between Maass forms with Laplace eigenvalue $1/4$ and even two-dimensional Artin representations.

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