189-235A: Basic Algebra I
Assignment 7

Due: Monday, November 4.

Section 6.1: 34, 40, 43, 44.

Section 6.2: 29, 31, 33.

8. Let $\mathbf{Q}(\sqrt{-5}) = \{a+b\sqrt{-5}, a, b \in \mathbf{Q}\}$, and $\mathbf{Z}[\sqrt{-5}] = \{a+b\sqrt{-5}, a, b \in \mathbf{Z}\}$.
a) Show that $\mathbf{Q}(\sqrt{-5})$ is a field, and that $\mathbf{Z}[\sqrt{-5}]$ is a subring. It is called the *ring of integers* of $\mathbf{Q}(\sqrt{-5})$ and plays the role of the usual integers in the arithmetic of $\mathbf{Q}(\sqrt{-5})$.
b) Show that the invertible elements in $\mathbf{Z}[\sqrt{-5}]$ are exactly 1 and -1.
c) Show that the elements 2, 3, $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are irreducible. (I.e., they cannot be written in the form ab where $a, b \neq \pm 1$.)
d) Using c), show that the ring $\mathbf{Z}[\sqrt{-5}]$ is not a unique factorization ring. (I.e., the "integers" in $\mathbf{Z}[\sqrt{-5}]$ cannot be written uniquely as a product of irreducible elements.)
e) Show that the ideals $(2, 1+\sqrt{-5})$, $(3, 1+\sqrt{-5})$, and $(3, 1-\sqrt{-5})$ are not principal, and that they are *irreducible*, i.e.,they cannot be factored further into products of non-trivial ideals.
f) If I and J are ideals, define the product IJ to be the ideal generated by the elements of the form ij with $i \in I$ and $j \in J$. Show that $(2, 1+\sqrt{5})^2 = (2)$, $(3, 1+\sqrt{-5})(3, 1-\sqrt{-5}) = (3)$, and conclude that the ideal (6) factorizes as a product of 4 (non-principal) ideals: $(6) = (2, 1+\sqrt{-5})^2(3, 1+\sqrt{-5})(3, 1-\sqrt{-5})$.
Remark: It can be shown that this factorization of the principal ideal (6) into a product of irreducible ideals is *unique*, up to the order of the factors. This is a general phenomenon: although the ring $\mathbf{Z}[\sqrt{-5}]$ fails to satisfy unique factorization, its *ideals* can be expressed uniquely as products of irreducible ideals. The introduction of ideals in the late 19-*th* century by Dedekind was an attempt to salvage unique factorization in such rings, by showing it was true on the level of ideals which were viewed as a kind of "ideal number". This is where the terminology comes from...