189-235A: Basic Algebra I
Assignment 5
Due: Wednesday, October 9

Section 4.1: 1, 16, 18.

Section 4.2: 15, 16.

Section 4.3: 2, 10, 14.

9. *Integer quaternions.*
a) Recall the ring H of real quaternions introduced in exercise 35, sec. 3.1. A quaternion is said to be *even* if it is of the form $a1 + bi + cj + dk$, where a, b, c, d are integers. It is said to be *odd* if a, b, c, and d are *each* halves of odd integers. Let R be the set of quaternions which are either even or odd. Show that R is a subring of H, and that it is a non-commutative ring with identity. Show also that the product of two even or odd quaternions is even, and that the product of an even and an odd quaternion is odd.
b) Define a "complex conjugation" on H by the rule

$$\overline{a1 + bi + cj + dk} := a1 - bi - cj - dk.$$

Show that if α and β are two quaternions, then

$$\overline{\alpha\beta} = \overline{\beta} \cdot \overline{\alpha}.$$

(Note the change in the order of multiplication!)
c) If α belongs to H, define $||\alpha|| := \alpha\overline{\alpha}$. Show that if $\alpha = a1 + bi + cj + dk$, then

$$||\alpha|| = a^2 + b^2 + c^2 + d^2.$$

Conclude that if $\alpha \in R$ then $||\alpha|| \in \mathbf{Z}$.
d) Show that $||\alpha\beta|| = ||\alpha|| \cdot ||\beta||$. (Remember in your proof that multiplication in H is not commutative!)
e) Using f), show that, if m and n are integers which can be expressed as a sum of 4 (integer) squares, then their product mn can also be expressed as a

sum of four integer squares. Use your proof to express $161 = 7 \cdot 23$ as a sum of four squares.

Part e) illustrates the usefulness of the ring R for number theory, and in particular for the study of representations of integers as sums of four squares. A deeper study of the ring theoretic structure of R leads to the following beautiful theorem of Lagrange: *Every positive integer can be expressed as a sum of four squares.* You should try to test this theorem empirically to get a feeling for what it says; and try to find a proof! In a later assignment, we will develop a Euclidean algorithm in R and use it to prove Lagrange's theorem.