Symmetric spaces and Orthogonal Groups

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Given a group G acting transitively on a set S, a common strategy for understanding both objects is to fix an element $x \in S$ and examine the quotient ${}^{G}\!/_{\mathrm{Stab}(x)}$. Under this identification, S is isomorphic, as a G-set, to the quotient ${}^{G}\!/_{\mathrm{Stab}(x)}$, effectively equipping S with a homogeneous structure. Moreover, if S carries additional structure—such as that of a topological space or a differentiable manifold—we can often transfer this structure to the quotient ${}^{G}\!/_{\mathrm{Stab}(x)}$. Beginning with a rational quadratic space V, the groups $G = \mathbf{O}(p,q)$ or $\mathbf{SO}(p,q)$ of the extension of scalars $V(\mathbf{R})$ already come with a topology induced from the one on $\mathrm{M}_{p+q}(\mathbf{R})$. If K is a maximal compact subgroup, then the resulting quotient, $\mathfrak{D} = {}^{G}\!/_{K}$ has the structure of a *symmetric space*- a specific class of differential manifold-. The specific case where $(p,q) = (2,n)$ has been extensively studied as the associated symmetric spaces admit complex structures and may therefore be considered as "algebraic" objects when compactified; for example, the hyperbolic upper half-plane $\mathcal{H}$ corresponds to the symmetric space associated to the group $\mathbf{SO}(2,1)$.
In our case we will consider, the group $\mathbf{SO}(3,1)$ from the previous example whose associated symmetric space can be realized as the 3-dimensional hyperbolic upper-half space. We begin by presenting $\mathcal{H}_3$ and then the isomorphism between it and $\mathbf{SO}(3,1)$.

1 The Upper-half Space

We denote by $\mathcal{H}_3$ the set $\{z + vj \mid z \in \mathbf{C}, v \in \mathbf{R}^{>0}\}$ where j is a formal symbol. Viewing $\mathcal{H}_3$ as the subset of $\mathbf{R}^3$ consisting of all triples of (i,j,k) with positive k coordinate naturally makes it into a differentiable manifold. We denote by $\partial\mathcal{H}_3$ the boundary of the upper half-space given by the plane $k = 0$ in $\mathbf{R}^3$.Letting, $z = x + iy$, we equip $\mathcal{H}_3$ with the hyperbolic metric induced from the line element

$$ds^2 = \frac{dx^2 + dy^2 + dv^2}{v^2}. \tag{1}$$

Under this assignment, we can define the geodesic between two points $P,Q \in \mathcal{H}_3$ as the unique path $\gamma(t) : [0,1] \to \mathcal{H}_3$ such that the integral

$$\int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2 + v'(t)^2}}{v'(t)^2} dt$$

is minimized. If P and Q have the same z coordinate, such a geodesic is given by a *Euclidean* straight line passing vertically through both points and going to infinity. Otherwise, it is given by a *Euclidean* semi-circle intersecting $\partial\mathcal{H}_3$ orthogonally and passing through both points. Similarly, hyperplanes in $\mathcal{H}_3$ are either given by Eucledian 2-dimensional planes perpendicular to the boundary or Euclidean hemispheres also intersecting the boundary orthogonally.

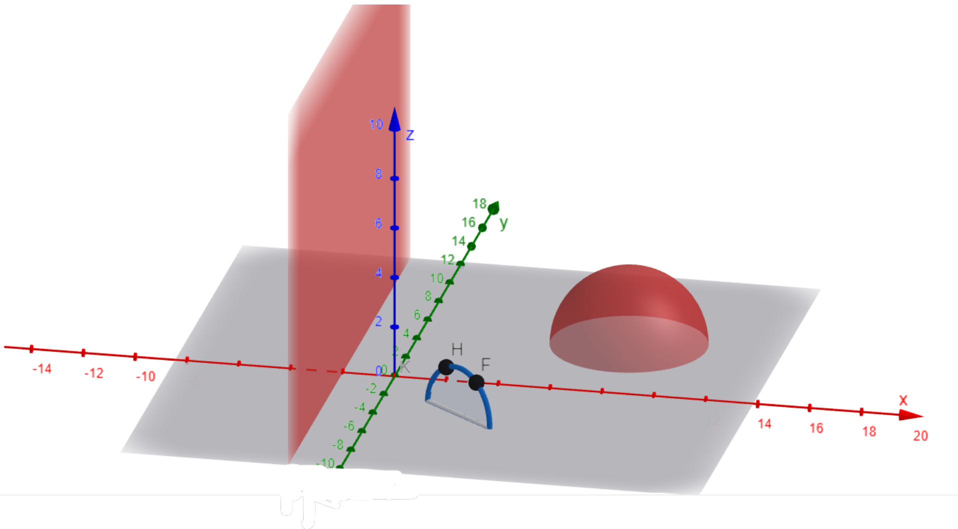

Figure 1: Visualization of the upper half-space $\mathscr{H}_3$ with boundary $\partial\mathscr{H}_3$ depicted in grey. The two possible types of hyperplanes are depicted in red. The blue curve shows the geodesic path between the points H and F.

The group $\mathbf{SL}_2(\mathbf{C})$ = of 2×2 complex matrices with determinant one acts isometrically on $\mathscr{H}_3$. For $x = z + j \in \mathscr{H}_3$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(\mathbf{C})$ we have

$$g * x = \frac{(az+b)(\bar{c}z+\bar{d}) + a\bar{c}v^2}{|cz+d|^2 + |c|^2v^2} + \frac{v}{|cz+d|^2+|c|^2v^2} j. \qquad \text{[EGM98]} \qquad (2)$$

Another way to write the action of $\mathbf{SL}_2(\mathbf{C})$ on the upper-half space is provided by the natural inclusion of $\mathscr{H}_3$ into the quaternion algebra $\mathbf{H}(-1,-1)_{/\mathbf{R}}$ by the assignment

$$z + jv = x + iy + jv \longmapsto x + iy + jv + 0k.$$

We note that this map is not an embedding, as $\mathscr{H}_3$ has no internal group structure; moreover the image of $\mathscr{H}_3$ in $\mathbf{H}(-1,-1)$ is not closed under multiplication. However, for the image P of x in $\mathbf{H}(-1,-1)_{/\mathbf{R}}$, we may write the action of $g \in \mathbf{SL}_2(\mathbf{C})$ as

$$g * P = (aP+b)(cP+D)^{-1}.$$

inside the quaternion algebra.[1] Using this formula, it one checks that the identity matrix I and $-I$ perform the same transformation on $\mathscr{H}_3$; we thus let $G := \mathbf{PSL}_2(\mathbf{C}) \curvearrowright \mathscr{H}_3$.

Proposition 1.1. *The group G acts transitively on $\mathscr{H}_3$.*

Proof. Let $x = z + jv \in \mathscr{H}_3$. We consider the matrix

$$g_x = \begin{pmatrix} \sqrt{v} & \dfrac{z}{\sqrt{v}} \\ 0 & \dfrac{1}{\sqrt{v}} \end{pmatrix} \in G.$$

Applying g_x to the point $j = (0,0,1) \in \mathscr{H}_3$ we get that

$$g * j = (aj+b)(cj+d)^{-1} = (\sqrt{v}j + \frac{z}{\sqrt{v}})(\frac{1}{\sqrt{v}})^{-1} = z + jv.$$

Therefore, for another point $y \in \mathscr{H}_3$, the transformation $g_x^{-1}g_y \in G$ maps x to y making the action of G on $\mathscr{H}_3$ transitive. □

[1] We write $(aP+b)(cP+D)^{-1}$ instead of $\frac{aP+B}{cP+D}$ to emphasize that the multiplication in $\mathbf{H}(-1,-1)$ is non-commutative.

Proposition 1.2. *The stabilizer of the point $j \in \mathcal{H}_3$ in G is the projective special unitary group* $\mathbf{PSU}(2,\mathbf{C})$.

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Stab}(j)$, we compute the product of g with its hermitian adjoint $g^\dagger$ which gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} a\bar{a}+b\bar{b} & a\bar{c}+b\bar{d} \\ c\bar{a}+d\bar{b} & c\bar{c}+d\bar{d} \end{pmatrix} = \begin{pmatrix} a\bar{a}+b\bar{b} & a\bar{c}+b\bar{d} \\ \bar{a\bar{c}+b\bar{d}} & c\bar{c}+d\bar{d} \end{pmatrix}$$

As g stabilizes j, (2) implies that $a\bar{c}+b\bar{d} = 0$ and $|c|^2+|d|^2 = 1$ so that the above matrix has the form $\begin{pmatrix} a\bar{a}+b\bar{b} & 0 \\ 0 & 1 \end{pmatrix}$. Since $g \in G$, $\det(g) = 1$ which implies that $a\bar{a}+b\bar{b} = 1$ and thus $gg^\dagger = I$ so that g is unitary. On the other hand, an arbitrary matrix in $\mathbf{SU}_2(\mathbf{C})$ may be given in the form $M = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ where $|a|^2+|b|^2 = 1$ so that the action on j is given by

$$\frac{a\bar{b}-\bar{b}a}{|a|^2+|b|^2} + \frac{1}{|a|^2+|b|^2} j = \frac{1}{\det(M)} j = j.$$

Thus $M \in \mathrm{Stab}(j)$ which proves the claim. □

Thus we identify $\mathcal{H}_3$ with the quotient $\mathbf{PSL}_2(\mathbf{C}) / \mathbf{PSU}_2(\mathbf{C})$ by sending a point $x = z+vj$ to the class of the matrix g_x sending j to x.

2 The Negative Grassmanian

In this section, we establish the isomorphism between the symmetric space attached to the group $\mathbf{SO}(3,1)$ and the upper-half space $\mathcal{H}_3$ presented in the previous section. We first recall the example which gave rise to this situation.

We let $D < 0$ be a square-free integer and considered the four dimensional rational vector space

$$V = \mathbf{Q} \oplus \mathbf{Q} \oplus F$$

where $F = \mathbf{Q}(\sqrt{D})$ is the quadratic imaginary extension. For $x = (a,b,\omega) \in V$, we considered the quadratic form $Q(x) = \omega\bar{\omega} - ab$. Under this assignment, the extension of scalars $V(\mathbf{R}) = V \otimes_{\mathbf{Q}} \mathbf{R}$ was a real quadratic space of type $(3,1)$. Using the Clifford algebra associated to V we determined the Spin groups of our space as$\mathrm{Spin}_V = \mathbf{SL}_2(F)$ and $\mathrm{Spin}_V(\mathbf{R}) = \mathbf{SL}_2(\mathbf{C})$; the latter satisfying the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{SL}_2(\mathbf{C}) \xrightarrow{\rho} \mathbf{SO}(3,1) \xrightarrow{\vartheta} \{\pm 1\} \tag{3}$$

We identified V and $V(\mathbf{R})$ with the isometric vector spaces

$$V \cong \left\{ \begin{pmatrix} a & w \\ \bar{\omega} & b \end{pmatrix} \mid a,b \in \mathbf{Q},\ \omega \in F \right\}$$

$$V(\mathbf{R}) \cong \left\{ \begin{pmatrix} a & w \\ \bar{\omega} & b \end{pmatrix} \mid a,b \in \mathbf{R},\ \omega \in \mathbf{C} \right\}$$

where the quadratic form of an element X was given by $-\det(X)$; the induced bilinear form was given by $(X,Y) = -\mathrm{Tr}(X \cdot Y^\sigma)$ where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\sigma = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

This identification allowed us to express the action of $P \in \mathbf{SL}_2(\mathbf{C})$ on $X \in V$ as $X \mapsto PXP^\dagger$ where $P^\dagger$ is P's Hermitian adjoint.

To realize the symmetric space attached to V, we consider the negative *Grassmanian* which consists of all maximum negative definite subspaces of $V(\mathbf{R})$. Since our space is of type (3,1) this is the set:

$$\mathrm{Gr}^-(V) = \left\{ l \subset V(\mathbf{R}) \mid \dim l = 1,\ Q_l < 0 \right\}.$$

As elements of $\mathbf{SO}_V$ preserves the form on individual vectors and map subspaces to subspaces, the group naturally acts on $\mathrm{Gr}^-(V)$; moreover this action is transitive. For all lines $l \in \mathrm{Gr}^-(V)$ We can pick a basis element z_l allowing us to fix an orientation on Gr⁻(V). We say that an element $g \in \mathbf{SO}(3,1)$ is *orientation preserving* if for all $l \in \mathrm{Gr}^-(V)$, $g * z_l = z_p$ for some $p \in$ Gr⁻(V) and denote by $\mathbf{SO}(3,1)^+$ the subgroup of orientation preserving elements. This group also acts transitively on $\mathrm{Gr}^-(V)$ and is in fact isomorphic to the identity component $\ker \vartheta$ of $\mathbf{SO}(3,1)$ from (3), hence the notation. We will not elaborate much on this but it is a well-known result that for an n-dimensional real vector space W, all Grassmanians of the form

$$\mathrm{Gr_k}(W) = \{ w \subset W \mid \dim w = k \}$$

are smooth projective varieties of dimension $(n-k)k$ in $\mathbb{P}^n(\mathbf{R})$(see [Sha13, p. 42]); more importantly they are real differentiable manifolds of the same dimension. In our case $\mathrm{Gr}_1^-(V)$ is an open subset of $\mathrm{Gr}_1(V)$ which makes the negative Grassmanian into differentiable manifold.
The stabilizer in $\mathbf{SO}(3,1)^+$ of a fixed line l_0 is isomorphic to the group $\mathbf{SO}(3) \times \mathbf{SO}(1)^+ \cong \mathbf{SO}(3)$. Indeed, since l_0 is negative definite, V can be decomposed as $V = l_0 \oplus l_0^\perp$ such that $l_0^\perp$ is quadratic space of type $(3,0)$. An isometry of V fixing l_0 is then given by an isometry of l_0 and an isometry of $l_0^\perp$ and thus an element of $\mathbf{SO}(3)^+ \times \mathbf{SO}(1)^+ \cong \mathbf{SO}(3)$. Therefore, we identify $\mathrm{Gr}^-(V)$ with the quotient ${\mathbf{SO}(3,1)^+}/{\mathbf{SO}(3)}$. We will take for granted that $\mathbf{SO}(3)$ is a maximal compact subgroup of $\mathbf{SO}(3,1)^+$, allowing us to realize the symmetric space of our group as the negative Grassmannian $\mathrm{Gr}^-(V)$.
From (3) have the following chain of isomorphisms

$$\mathrm{Gr}^-(V) \cong {\mathbf{SO}(3,1)^+}/{\mathbf{SO}(3)} \cong {\mathbf{PSL}_2(\mathbf{C})}/{K}$$

where K is some undetermined subgroup of $\mathbf{PSL}_2(\mathbf{C})$ corresponding to the image of the stabilizer of a fixed line. To determine K, we consider the one-dimensional subspace $\mathrm{span}_{\mathbf{R}}(I) \in \mathrm{Gr}^-(V)$ generated by the identity matrix and compute its stabilizer in $\mathbf{PSL}_2(\mathbf{C})$. By definition, an element $P \in \mathbf{PSL}_2(\mathbf{C})$ fixes I if and only if $PIP^\dagger = I \iff PP^\dagger = I$. Therefore P fixes I if and only if P is unitary and thus $K = \mathrm{Stab}(\mathrm{span}_{\mathbf{R}}(I)) = \mathbf{PSU}_2(\mathbf{C})$. Combining this result with previous ones we get

$$\mathrm{Gr}^-(V) \cong {\mathbf{SO}(3,1)^+}/{\mathbf{SO}(3)} \cong {\mathbf{PSL}_2(\mathbf{C})}/{\mathbf{PSU}_2(\mathbf{C})} \cong \mathcal{H}_3$$

which gives the identification between the symmetric space attached to $\mathbf{SO}(3,1)$ and the hyperbolic upper-half plane. $\mathcal{H}_3$.
Explicitly, this isomorphism is given by mapping mapping a point $p = z + jv \in \mathcal{H}_3$ to the class of the matrix g_p in ${\mathbf{PSL}_2(\mathbf{C})}/{\mathbf{PSU}_2(\mathbf{C})}$ defined in proposition 1.1. The class of g_p is then sent to the class of $g_p I g_p^\dagger = g_p g_p^\dagger$ in ${\mathbf{SO}(3,1)^+}/{\mathbf{SO}(3)}$ which corresponds to some line $l_p \in \mathrm{Gr}_1^-(V)$. Under this assignment, the matrix $g_p g_p^\dagger$ is sent to the vector $\begin{pmatrix} v + \frac{|z|^2}{v} & \frac{z}{v} \\ \frac{\bar{z}}{v} & \frac{1}{v} \end{pmatrix} \in \mathrm{span}_{\mathbf{R}}(l_p)$ which corresponds to the oriented basis element of l_p of norm one. While we will not prove this statement, at each step, the maps between spaces are diffeomorphisms.

This completes the identification of the symmetric space $\mathfrak{D} = {\mathbf{SO}(3,1)^+}/{\mathbf{SO}(3)^+}$ with the hyperbolic upper half-space $\mathcal{H}_3$. This example illustrates a special case of a more general result about orthogonal groups of signature $(n,1)$ or $(1,n)$: their associated symmetric spaces ${\mathbf{SO}(n,1)^+}/{\mathbf{SO}(n)}$ can be identified with higher-dimensional analogs of the hyperbolic upper half-spaces

$$\mathscr{H}_n = \{(x_1,\ldots,x_{n-1},t) \mid x_i \in \mathbf{R},\ t \in \mathbf{R}^{>0}\}. \qquad \text{[Liv16]}$$

. In general, one can associate symmetric spaces to a broader class of Lie groups such as the groups $\mathbf{O}(p,q)$, the pseudo-unitary groups $\mathbf{U}(p,q)$ or the sympletic groups $\mathrm{Sp}(n:\mathbf{R})$.

References

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