Quadratic spaces and Orthogonal groups

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In this note, we give a brief overview of the theory of quadratic spaces following chapter 2 of [BvdGHZ08] as well as chapter 0 of [KR99], culminating in the determination of the special orthogonal groups attached to certain rational quadratic spaces of dimension 4. Throughout the text, R will denote a commutative ring and k will denote a field of characteristic 0. The letters $\mathbf{Q}$,$\mathbf{R}$,$\mathbf{C}$ will respectively denote the fields of rational, real and complex numbers.

1 Quadratic Spaces

Definition 1.1. Let V be an R-module, a *quadratic form* on V is a function $Q : V \to R$ which satisfies:

(i) For all $r \in R$ and $x \in V$,
$$Q(r \cdot x) = r^2 \cdot Q(x)$$

(ii) The function $B : V \times V \to R$ defined by
$$B(x,y) = Q(x+y) - Q(x) - Q(y)$$
is a bilinear form on V.

We call the pair (V,Q) a *quadratic space over R.*

Definition 1.2. Let (V,Q) be a quadratic space over R.

(i) Two elements $x,y \in V$ are *orthogonal*, denoted $x \perp y$ if, $B(x,y) = 0$. We denote by $x^\perp$ the set of all elements of V which are orthogonal to x.

(ii) An element $x \in V$ is *isotropic* (cont.*anisotropic*) if $Q(x) = 0$ (cont. $Q(x) \neq 0$)

(iii) The quadratic space (V,Q) is said to be *degenerate* if there exits $v \in V$ such that for all $w \in V$, $B(v,w) = 0$.

Throughout this paper, we will primarily consider the case where $R = k$ is a field such that V is a vector space. Notably, if $\mathrm{char}(k) \neq 2$, then degeneracy of (V,Q) implies the existence of an isotropic vector.

Definition 1.3. Let $(V,Q)$and(V',Q') be quadratic spaces over R and $\sigma : V \to V'$ an injective homomorphism of modules. We say σ is an *isometry* if for all $v \in V$,
$$Q'(\sigma(v)) = Q(v).$$
If in addition σ is surjective, we say that the quadratic spaces (V,Q) and $(V,'Q')$ are *isometric.*

Definition 1.4. We respectively define the *orthogonal group* and *special orthogonal group* of a quadratic space (V,Q) as the groups
$$\mathbf{O}_V = \{\sigma \in \mathrm{Aut}(V) \mid \sigma \text{ is an isometry}\}$$
$$\mathbf{SO_V} = \{\sigma \in \mathbf{O_V} \mid \det(\sigma) = 1\}.$$

Example 1.5. For any quadratic space (V,Q) over R containing an anisotropic element x such that $Q(x) \in R^*$, we define the *reflection along* $x^\perp$, $\tau_x : V \to V$ as

$$\tau_x(y) = y - B(x,y)Q(x)^{-1}x$$

The reflection τ_x is an isometry of V. Indeed as x is anisotropic, we obtain a decomposition of V as

$$V = \operatorname{span}_k(x) \oplus x^\perp.$$

A vector $y \in V$ may therefore be expressed as $y = y_x + y_{x^\perp}$ according to the previous decomposition. We then have

$$Q(\tau_x(y)) = Q(\tau_x(y_x) + \tau_x(y_{x^\perp})) = Q(\tau_x(y_x)) + Q(\tau_x(y_{x^\perp}))$$

since the quadratic form is additive for orthogonal vectors. Then, since $y_x = \alpha x$ for some $a \in k$,

$$\begin{aligned} Q(\tau_x(y_x)) + Q(\tau_x(y_{x^\perp})) &= Q(\alpha x - B(\alpha x, x)Q(x)^{-1}x) + Q(y_{x^\perp} - B(y_{x^\perp}, x)Q(x)^{-1}x) \\ &= Q(\alpha x - \alpha 2x) + Q(y_{x^\perp}) \\ &= Q(\alpha x) + Q(y_{x^\perp}) \\ &= Q(y). \end{aligned}$$

The following properties follow from explicit computations:

(i) $\tau_x(x) = -x$

(ii) $\tau^2 = \mathrm{Id}$

(iii) $\tau_x(y) = y, \forall y \in x^\perp$

(iv) $\det(\tau_x) = -1$

Reflections are in some sense the most important isometries as if V is finitely generated as an R-module, then the group $\mathbf{O}_V$ is generated by reflections; the group $\mathbf{SO_V}$ is generated by the products of two reflections.

Example 1.6. Let p,q be non-negative integers and consider the vector space $\mathbf{R}^{p+q}$ with standard basis $\{v_1, \dots v_p, \dots, v_{p+q}\}$. We denote by $\mathbf{R}^{p,q}$ the quadratic space $(\mathbf{R}^{p+q}, Q_{p,q})$, where for a vector $v = \alpha_1 v_1 + \cdots + \alpha_p v_p + \cdots + \alpha_{p+q} v_{p+q}$,

$$Q_{p,q}(v) = \alpha_1^2 + \cdots + \alpha_p^2 - \alpha_{p+1}^2 - \cdots - \alpha_{p+q}^2.$$

We denote by $O(p,q)$ and $SO(p,q)$ the orthogonal and special orthogonal groups of $\mathbf{R}^{p,q}$

The quadratic spaces $\mathbf{R}^{p,q}$ are of central importance due to the following theorem

Theorem 1.7. *Let (V,Q) be a finite dimensional real non-degenerate quadratic space, then V is isometric to some unique vector space $\mathbf{R}^{p,q}$. We call the pair (p,q) the signature of V and denote it by sigV.*

Proof. Let $(v_i)_{i=1}^n$ be a basis of V and let $B_{ij} := B(v_i, v_j)$. We call the matrix $A := (B_{ij})_{i,j=1}^n \in \mathrm{M}_n(\mathbf{R})$ the *Gram matrix* associated to (V,Q) and its basis. For $x \in V$, a direct computation shows that

$$Q(x) = x^T A x.$$

Furthermore, as the form B is symmetric, so is the matrix A. Since every real symmetric matrix is diagnosable over $\mathbf{R}$, there exists an orthonormal basis of V with respect to which Q may be expressed in the form $Q_{p,q}$ which completes the proof. □

Therefore, the classification of real finite dimensional quadratic spaces and their orthogonal groups amounts to the classification of the spaces $\mathbf{R}^{p,q}$ and the groups $O(p,q), SO(p,q)$.

2 Clifford Algebras

There is a natural geometric structure on a quadratic space; one can speak of the norm of a vector, as well as orthogonality relations between vectors. Admittedly not as natural, the symmetries of a quadratic space also carry a geometric structure induced by the map $v \mapsto \tau_v$ sending a vector to the reflection along its orthogonal complement. These considerations suggest that there should be some object which "unifies" the quadratic space (V,Q) with its symmetry groups $\mathbf{O}_V$ and $\mathbf{SO_V}$. The Clifford algebra associated to a quadratic space is in some sense the universal construction of that object. In turn, this construction serves as a useful tool to determine the symmetry groups of our quadratic space. We begin by explicitly constructing the Clifford algebra of a space. Let (V,Q) be a non-degenerate quadratic spaces over a field k[1] and consider its tensor algebra

$$T_V = \bigoplus_{m=0}^{\infty} V^{\otimes m} = k \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

The tensor algebra T_V is a large universal associative k-algebra containing our original vector space. In itself, it does not carry any of the additional quadratic structure of V. In order to restore it we first consider the ideal I_V generated by all elements of the form $v \otimes v - Q(v) \in T_V$ and then the quotient

$$C_V := {}^{T_V}\!/_{I}.$$

We call C_V the *Clifford algebra* associated to the quadratic space (V,Q). It is again quite a large algebra but it now satisfies the following identities:

(i) $v \otimes v = Q(v)$ for all $v \in V$.

(ii) $u \otimes v - v \otimes u = B(v,u)$ for all $u,v \in V$. In particular, if $u \perp v$, then $u \otimes v = -v \otimes u$.

To make the notation clearer, it is customary to denote tensor products $v_i \otimes v_j \otimes v_k \otimes \dots$ as $v_i v_j v_k \cdots$. The Clifford algebra admits a $\mathbf{Z}/2\mathbf{Z}$ grading given by

$$C_V = C_V^0 \oplus C_V^1$$

where C_V^0 is the algebra generated by all products of an *even* number of elements of V and C_V^1 is the module[2] generated by all products of an *odd* number of elements of V; we respectively refer to C_V^0 and C_V^1 as the *even* and *odd* Clifford algebras.
We give some examples of Clifford algebras in low-dimension.

Example 2.1. Let $V = \mathbf{R} = \text{span}_{\mathbb{R}}\{v\}$ with the quadratic form induced from $Q(v) = -1$. The tensor algebra of V is given by

$$T_V = \mathbf{R} \oplus \text{span}_{\mathbf{R}}\{v\} \oplus (\text{span}_{\mathbf{R}}\{v\} \otimes \text{span}_{\mathbf{R}}\{v\}) \otimes \dots$$

After passing to the quotient by I_V to obtain the Clifford algebra of V we notice the following. If $x \in C_V^0$, it is given by a sum $x = \sum_i a_i \cdot (v_1 v_2 \cdots v_{2i})$ where $a_i \in \mathbf{R}$ and equal to zero except for a finite set of indices. As V is one-dimensional $v_j = v_k \forall\ j,k$ such that

$$x = \sum_i a_i \cdot (v_1 v_2 \cdots v_{2i}) = \sum_i (-1)^i \cdot a_i \in \mathbf{R}.$$

If $x \in C_V^1$, then it is given by a sum $\sum_i b_i \cdot (v_1 v_2 \cdots v_{2i+1})$. Similarly, we have

$$x = \sum_i b_i \cdot (v_1 v_2 \cdots v_{2i+1}) = \sum_i (-1)^i \cdot b_i \cdot v \in \text{span}_{\mathbf{R}}\{v\}.$$

Therefore $C_V = \mathbf{R} \oplus \text{span}_{\mathbf{R}}\{v\}$ with the property that $v^2 = -1$. Therefore, as an $\mathbf{R}$-algebra, C_V is isomorphic to the complex numbers $\mathbf{C}$.

[1] The same construction can be for a module over a ring R but the examples we will consider will come from vector spaces.

[2] We emphasize that C_V^0 is an *algebra* while C_V^1 is simply a module. Indeed the product of 2 elements of C_V^1 is an element of C_V^0.

Example 2.2. Let k be a field and n a positive integer. Consider the vector space $V = k^n$ with the zero-quadratic form $Q \equiv 0$. Let $\{v_1,\dots,v_n\}$ be a basis of V and consider its tensor algebra, T_V. For $m > n$, the vector space $V^{\otimes m}$ is generated by all products of the form $v_{i_1}v_{i_2}\cdots v_{i_m}$. By the pigeonhole principle, we must have two indices j,l such that $v_{i_j} = v_{i_l}$. Therefore, in the Clifford algebra, which satisfies the identities $v^2 = 0$ and $vu = -uv$ for vectors $u,v \in V$, the elements of $V^{\otimes m}$ are annihilated. We then have that

$$C_V = \bigoplus_{i=0}^{n} V^{\otimes m}$$

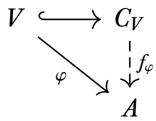
subject to the relations:

(i) $v_i^2 = 0$

(ii) $v_i v_j = -v_j v_i$

The Clifford algebra associated to V is thus isomorphic to the k'th exterior power $\bigwedge^k(V)$ of V.

As previously mentioned, the Clifford algebra C_V associated to a quadratic space (V,Q) is a universal algebraic construction containing both V and, as will be discussed later, some analogue of its symmetry groups. We make precise what we mean by "universal". If A is a unitary k-algebra and $\varphi : V \to A$ is a linear map such that for all $v \in V$ $\varphi(v)^2 = Q(v)\cdot 1_A$, then there exists a unique k-algebra morphism f_φ such that the following diagram commutes.



There are two canonical automorphisms on C_V. The first of which is given by the map $J : V \to V$, $J(v) = -v$ which we then extend linearly and multiplicatively to all of C_V. From this map we get the following characterization

$$C_V^0 = \{x \in C_V : J(x) = x\}.$$

The second map is an *anti-involution* $(\ )^t : C_V \to C_V$ generated by the function $(v_{i_1}\cdots v_{i_n})^t = (v_{i_n}\cdots v_{i_1})$ which is then extended to the whole of C_V. We note that $k \oplus V$ remains fixed by $(\ )^t$. From this map, we can define the *Clifford norm* fo an element $x \in C_V$ to be $N(x) = x \cdot x^t \in C_V$. We emphasize that unlike the usual uses of the word "norm" in mathematics, the norm function we've defined takes values in the Clifford algebra C_V and not simply in the field k. However, as V is fixed by $(\ )^t$ we have that $N_{|V} = Q$, where Q is the quadratic form on V. In this sense, we may view the norm on C_V as an extension of the quadratic form on the original vector space.
An important sub-algebra of C_V is its center $Z(C_V)$. In the case where (V,Q) is an n-dimensional quadratic space with orthogonal basis $\{v_1,\dots,v_n\}$, if we let δ denote the element $v_1\cdots v_n \in C_V$ we find that the center of C_V is given by [BvdGHZ08, p. 123]

$$Z(C_V) = \begin{cases} k & \text{if } n \text{ is even,} \\ k \oplus k\delta & \text{if } n \text{ is odd.} \end{cases}$$

and the center of C_V^0 is given by

$$Z(C_V^0) = \begin{cases} k \oplus k\delta & \text{if } n \text{ is even,} \\ k & \text{if } n \text{ is odd.} \end{cases}$$

The notions we've developed can be applied to the problem of determining the groups $\mathbf{O}_V$ and $\mathbf{SO}_V$ associated to a quadratic space. We do so by noticing that, through the multiplication in the Clifford Algebra, some elements of C_V act on V. We consider the following group (under multiplication)

$$\Gamma_V = \{x \in C_V \mid x \text{ is invertible and } xVJ(x)^{-1} = V\}$$

These are the elements of C_V which preserve the vector space V under this "twisted" conjugation action. Since multiplication in C_V is distributive, the action of an element $x \in \Gamma_V$ is linear, giving us a representation $\rho : \Gamma_V \to \mathrm{Aut}(V)$. We refer to the image $\rho_x := \rho(x)$ as the *vector representation* of x.

Proposition 2.3. *The kernel of the map $\rho : \Gamma_V \to \mathrm{Aut}(V)$ is the field k^* and the norm map $N : \Gamma_V \to k^*$ is a homomorphism of groups.*

Proof. It is clear that $k^* \in \ker(\rho)$ so we prove the opposite inclusion. Suppose that $x \in \ker(\rho)$, then it may be written as a sum $x = x_0 + x_1$ where $x_i \in C_V^i$. Since ρ_x is the identity on V we have that for all vectors Let $v \in V$. Then:

$$v = xvJ(x)^{-1} = (x_0 + x_1)v(J(x_0) + J(x_1))^{-1} = (x_0 + x_1)v(x_0 - x_1)^{-1}$$

which implies

$$(x_0 + x_1)v = v(x_0 - x_1).$$

and thus $x_0 v = vx_0$ and $x_1 v = -vx_1$. Therefore, x_0 commutes with all elements of V and V generates C_V as an algebra, $x_0 \in Z(C_V)$ and thus $x \in (C_V^0)^* \cap Z(C_V) = k^*$. Additionally, by linearity, the identity involving x_1 must hold for all its individual components of the form $\alpha_i \cdot v_{i_1} \cdots v_{i_j} \in C_V^1$ where j is odd. If we then let x_1 act on v_{i_1} we get that

$$0 = v_{i_1} \cdot \alpha_i \cdot v_{i_1} \cdots v_{i_j} + \alpha_i \cdot v_{i_1} \cdots v_{i_j} \cdot v_{i_1} = \alpha_i \cdot Q(v_{i_1}) \cdot v_{i_2} \cdots v_{i_j} + (-1)^{j-1} \alpha_i \cdot Q(v_{i_1}) \cdot v_{i_2} \cdots v_{i_j}.$$

As $j-1$ is even we get that

$$0 = \alpha_i \cdot Q(v_{i_1}) \cdot v_{i_2} \cdots v_{i_j} + \alpha_i \cdot Q(v_{i_1}) \cdot v_{i_2} \cdots v_{i_j}$$

As k has characteristic zero, this implies that $\alpha_i = 0$ and thus that $x_1 = 0$. Therefore, $x = x_0 \in k^*$.
We now prove that for $x \in \Gamma_V$,the norm $N(x) \in k^*$ by showing that $N(x)$ acts trivially on V. For this, we let $v \in V$ and let $w := \rho_x(v)$ be the image of v under the transformation ρ_x. We note that since $w \in V$, $-J(w)^t = w$ and thus

$$w = xvJ(x)^{-1} = -J(xvJ(x)^{-1})^t = (x^t)^{-1}vJ(x^t).$$

By multiplying by x^t on the left and $J(x^t)^{-1}$ on the right we obtain that

$$v = x^t xvJ(x)^{-1}J(x^t)^{-1} = N(x)vJ(N(x))^{-1} = \rho_{N(x)}(v)$$

and so $N(x)$ acts as the identity on V which implies that $N(x) \in k^*$. □

In fact, one can say more about the vector representation of elements of Γ_V.

Proposition 2.4. *Let $x \in \Gamma_V$, then the vector representation $\rho_x : V \to V$ is an isometry of the quadratic space (V,Q).*

Proof. Let $v \in V$, $x \in \Gamma_V$ and $w = \rho_x(v)$. As $w \in V$, we have that $Q(w) = N(w)$ and thus

$$\begin{aligned}
Q(w) &= N(w) \\
&= N(xvJ(x)^{-1}) \\
&= (xvJ(x)^{-1})^t(xvJ(x)^{-1}) \\
&= J(x^{-1})^t v x^t x v J(x^{-1}) \\
&= N(x) \cdot Q(v) \cdot N(x)^{-1} \\
&= Q(v)
\end{aligned}$$

Therefore, $\rho_x \in \mathbf{O_V}$ □

Corollary 2.5. *For all anisotropic vectors $v \in V \subset \Gamma_V$, the vector representation ρ_v is the reflection τ_v.*

Proof. We first note that $J(v)^-1 = -v^{-1} = -vQ(v)^{-1}$ and recall that for two vectors $u_1, u_2 \in C_V, u_1 u_2 = B(u_1, u_2) - u_1 u_2$. If we let $w \in V$ we have that

$$vwJ(v)^{-1} = vw(-v)Q(v)^{-1} = -v(B(v,w) - vw)Q(v)^{-1} = w - B(v,w)Q(v)^{-1} = \tau_v(w)$$

□

As the reflections τ_x generate $\mathbf{O}_V$, the map $\rho : \Gamma_V \to \mathbf{O_V}$ is a surjection and we have the following exact sequence.

$$1 \longrightarrow k^* \longrightarrow \Gamma_V \xrightarrow{\rho} \mathbf{O}_V \longrightarrow 1$$

Recalling that $\mathbf{SO_V}$ is the subgroup of $\mathbf{O}_V$ generated by pairs of reflections we define the *general spin group* and *spin group* as the following subgroups of Γ_V:

$$\mathrm{GSpin}_V = \Gamma_V \cap C_V^0$$

$$\mathrm{Spin}_V = \{x \in \mathrm{GSpin}_V \mid N(x) = 1\}.$$

These two groups verify the following exact sequences

$$1 \longrightarrow k^* \longrightarrow \mathrm{GSpin}_V \xrightarrow{\rho} \mathbf{SO_V} \longrightarrow 1$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathrm{Spin}_V \xrightarrow{\rho} \mathbf{SO_V} \xrightarrow{\vartheta} {k^*}/{(k^*)^2}$$

The map ϑ is known as the *Spinor norm* which for an element $x \in \mathbf{SO_V}$, $\vartheta(x)$ is the Clifford norm of its preimage in Spin_V which is well defined up to a square of the field k. We note that for $k = \mathbf{C}, {k^*}/{(k^*)^2} = \{1\}$ and for $k = \mathbf{R}, {k^*}/{(k^*)^2} = \{\pm 1\}$. Viewed as topological groups, $\mathrm{GSpin_V}$ is the universal covering space of $\mathbf{SO_V}$ which itself is not simply connected. If we let $\mathbf{SO_V}^+ := \ker(\vartheta) \subset \mathbf{SO_V}$ be the component of $\mathbf{SO_V}$ containing the identity, then Spin_V is the double cover of $\mathbf{SO_V}$.

We've now reduced the task of computing the special orthogonal group (up to a small kernel) of a quadratic space $(V,Q)_{/k}$ to the computation of the group Spin_V generated by its Clifford algebra. In practice, If V has dimension ≤ 4, the latter can be computed effectively as the even Clifford algebra C_V^0 is at its largest, a quaternion algebra over k.

Definition 2.6. A unitary k-algebra A is said to be a *quaternion algebra* if it has a basis of the form $\{1, x_1, x_2, x_3\}$ as a k-vector space and is subject to the relations

$$x_1^2, x_2^2, \text{ and } k^*, x_1 x_2 = -x_2 x_1 = x_3.$$

A is entirely determined by the values x_1^2 and x_2^2 and is thus denoted by $(x_1^2, x_2^2)_{/k}$. For an element $x = \alpha_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 \in A$, we define its conjugate to be $x^* = \alpha_0 - \alpha_1 x_1 - \alpha_2 x_2 - \alpha_3 x_3 \in A$.

Proposition 2.7. *[BvdGHZ08, p. 133] Let (V,Q) be a non-degenerate quadratic space of dimension 4 over k with orthogonal basis $\{v_1, v_2, v_3, v_4\}$. If we let $q_i = Q(v_i)$, then C_V^0 is isomorphic to a quaternion algebra of the form*

$$C_V^0 = (-q_1 q_2, -q_2 q_3)_{/Z(C_V^0)} = (k + k\delta) \oplus (k + k\delta) v_1 v_2 \oplus (k + k\delta) v_2 v_3 \oplus (k + k\delta) v_1 v_3.$$

Under this assignment, the anti-involution $(\ )^t$ of C_V corresponds to the conjuagation $(\ )^*$ in the quaternion algebra.

The dimension of V being 4 also gives the following result

Proposition 2.8. *If* $\dim(V) \leq 4$, *then* $\mathrm{GSpin}_V = \{x \in C_V^0 \mid N(x) \in k^\star\}$ *and thus* $\mathrm{Spin}_V = \{x \in C_V^0 \mid N(x) = 1\}$.

Proof. By definition, $\mathrm{GSpin}_V \subset C_V^0$ so it is left to show that if $x \in C_V^0$ such that $N(x) \in k^*$, then $x \in \mathrm{GSpin}_V$. First as $N(x) \in k^*$, we have that $x \cdot N(x)^{-1} x^t = 1$ making x invertible. We are then left to check that $xVJ(x)^{-1} = xVx^{-1} \subset V$. For a given $v \in V$ we let $w := xvx^{-1} \in C_V^1$. As $\dim_V \leq 4$, we get the following characterization [3]

$$V = \{x \in C_V^1 \mid x^t = x\}$$

and so we compute:

$$w^t = (xvx^{-1})^t = (x^{-1})^t v x^t = (x^{-1})^t N(x)^{-1} v N(x) x^t = xvx^{-1} = w$$

since $x^t = N(x)x^{-1}$. Therefore x preserves V which proves the result. □

We've shown that for $\dim V = 4$, C_V^0 is a quaternion algebra over its center $Z(C_V^0)$. It is possible and quite useful to express the vector representation of Spin_V intrinsically inside of C_V^0 by identifying an isometric copy $\tilde{V} \subset C_V^0$ of V upon which the action of Spin_V is given by the multiplication in the quaternion algebra. To construct $\tilde{V}$, we let $v_0 \in V$ be an anisotropic vector, $q_0 := Q(v_0)$ and consider the adjoint operator $\mathrm{Ad(v_0)\colon C_V^0 \to C_V^0}$ which sends x to $x^\sigma = v_0 x v_0^{-1}$. We consider the vector space

$$\tilde{V} = \{x \in C_V^0 \mid x^t = x^\sigma\}$$

equipped with the quadratic form

$$\tilde{Q}(x) = q_0 \cdot N(x)$$

Analogously to the action of Spin_V on V we define the vector representation $\tilde{\alpha} : \mathrm{Spin}_V \to \mathrm{Aut(\tilde{V})}$ which for $g \in \mathrm{Spin}_V$ is given by

$$\tilde{\alpha}_g(\tilde{v}) = gxg^{-\sigma}.$$

Under this action, we have that

$$\tilde{Q}(gxg^{-\sigma}) = q_0 \cdot (gxg^{-\sigma})^t (gxg^{-\sigma}) = q_0 \cdot (g^{-\sigma})^t x^t g^t gxg^{-\sigma} = q_0 \cdot N(g) \cdot N(x) \cdot N(g)^{-1} = \tilde{Q}(x)$$

such that the quadratic form $\tilde{Q}$ is preserved.

Proposition 2.9. (V,Q) *is isometric to* $(\tilde{V},\tilde{Q})$.

Proof. See [KR99, p. 11] □

2.1 Accidental isomorphisms

The theory developed throughout this paper can be applied to the following important example. We let $D \in \mathbf{Z}$ be square-free and $F = \mathbf{Q}(\sqrt{D})$ denote the field extension of degree 2 of the rational numbers. We consider the following 4-dimensional $\mathbf{Q}$-vector-space

$$V = \mathbf{Q}^2 \oplus F.$$

For an element $x = (a,b,\omega) \in V$ where $a,b \in \mathbf{Q}$and $\omega \in F$ we define the quadratic form

$$Q(x) = \omega\bar{\omega} - ab$$

[3] If V were of dimension ≥ 5 we could have an element of the form $x = v_1 v_2 v_3 v_4 v_5$ where v_i are orthogonal. In that case, one can obtain x^t from x by performing $\frac{n^2-n}{2} = 10$ transpositions such that $v_1 v_2 v_3 v_4 v_5 = (-1)^{10} v_5 v_4 v_3 v_2 v_1 = v_5 v_4 v_3 v_2 v_1$.

where $\bar{\omega}$ is ω's Galois conjugate. One checks the vectors

$$v_1 = (1,1,0), v_2 = (1,-1,0), v_3 = (0,0,1), v_4 = (0,0,\sqrt{D})$$

form an orthogonal basis of V. Under this assignment, for $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4$ the quadratic form is given by

$$Q(v) = -\alpha_1^2 + \alpha_2^2 + \alpha_3^2 - D\alpha_4^2.$$

Thus, letting $V(\mathbf{R}) := V \otimes_{\mathbf{Q}} \mathbf{R}$ denote the extension of scalars to $\mathbf{R}$, we have that $V(\mathbf{R})$ is a quadratic space of type $(2,2)$ if $D > 0$ and of type $(3,1)$ if $D < 0$. In order to determine $\mathbf{SO_V}$, we note that since $\dim V = 4$ the even Clifford algebra is given by

$$C_V^0 = (-q_1 q_2, -q_2 q_3)_{/Z(C_V^0)} = (1,-1)_{/Z(C_V^0)}$$

where $Z(C_V^0) = \mathbf{Q} \oplus \mathbf{Q} v_1 v_2 v_3 v_4 = \mathbf{Q} \oplus \mathbf{Q}\delta$. We notice that $\delta^2 = q_1 \cdot q_2 \cdot q_3 \cdot q_4 = D$ and therefore $Z(C_V^0) = \mathbf{Q} \oplus \mathbf{Q}\sqrt{D} \cong F$. Therefore the even Clifford algebra C_V^0 is a split quaternion algebra over F which is isomorphic to the matrix algebra $\mathrm{M}_2(F)$. An explicit isomorphism is given by the map

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; v_1 v_2 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; v_2 v_3 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; v_1 v_3 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By writing a general matrix M as a linear combination of the matrices above one checks that the conjugation in $\mathrm{M}_2(F)$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and the norm by

$$N\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = \begin{pmatrix} \det(M) & 0 \\ 0 & \det(M) \end{pmatrix}.$$

We can then compute the group Spin_V as

$$\begin{aligned} \mathrm{Spin}_V &= \{x \in C_V^0 \mid N(x) = 1\} \\ &\cong \{M \in \mathrm{M}_2(F) \mid N(M) = \mathrm{Id}\} \\ &= \{M \in \mathrm{M}_2(F) \mid \det(M) = 1\} \\ &= \mathbf{SL}_2(F). \end{aligned}$$

We can also determine the spin group of the extension of scalars $V(\mathbf{R})$ as follows:

$$\mathrm{Spin}_V(\mathbf{R}) = \mathbf{SL}_2(F \otimes_{\mathbf{Q}} \mathbf{R}) = \begin{cases} \mathbf{SL}_2(\mathbf{R}) \times \mathbf{SL}_2(\mathbf{R}) & \text{if } D > 0, \\ \mathbf{SL}_2(\mathbf{C}) & \text{if } D < 0. \end{cases}$$

We then have the following exact sequences

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{SL}_2(\mathbf{R}) \times \mathbf{SL}_2(\mathbf{R}) \xrightarrow{\rho} \mathbf{SO}(2,2) \xrightarrow{\vartheta} \{\pm 1\} \tag{1}$$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbf{SL}_2(\mathbf{C}) \xrightarrow{\rho} \mathbf{SO}(3,1) \xrightarrow{\vartheta} \{\pm 1\} \tag{2}$$

We compute the twisted vector space $\tilde{V}$ by letting $v_0 = v_1 = (1,1,0)$ and considering the adjoint operator Ad_{v0} acting on $C_V^0 \cong \mathrm{M}_2(F)$ On the basis vectors of C_V^0, the action is given by

$$1^\sigma = 1, \; (v_1 v_2)^\sigma = v_2 v_1, \; (v_2 v_3)^\sigma = v_2 v_3, \; (v_1 v_3)^\sigma = v_3 v_1.$$

By linearity, we find that for a general matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(F)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sigma} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}.$$

Therefore,

$$\begin{aligned} \tilde{V} &= \{x \in C_V^0 \mid x^t = x^{\sigma}\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{SL}_2(F) \middle| \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{d} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & \omega \\ \bar{\omega} & b \end{pmatrix} \middle| \; a,b \in \mathbf{Q},\ \omega \in F \right\} \end{aligned}$$

For $M \in \tilde{V}$, the quadratic form $\tilde{Q}$ is given by

$$\tilde{Q}(M) = -\det(M)$$

and the induced bilinear form by

$$\tilde{B}(M_1, M_2) = -\operatorname{Tr}(M_1 \cdot M_2^{\sigma})$$

In the case where $D > 0$ so that $\operatorname{sig} V(\mathbf{R}) = (2,2)$,

$$\tilde{V}_{(2,2)}(\mathbf{R}) \cong \mathrm{M}_2(\mathbf{R}).$$

For $X \in V_{(2,2)}(\mathbf{R})$,the action of $(M,N) \in \mathrm{Spin}_V(\mathbf{R}) = \mathbf{SL}_2(\mathbf{R}) \times \mathbf{SL}_2(\mathbf{R})$ is the obvious one given by

$$X \mapsto MXN^{-1}.$$

$V_{(2,2)}(\mathbf{R})$ is trivially preserved by the action of $\mathbf{SL}_2(\mathbf{R}) \times \mathbf{SL}_2(\mathbf{R})$ and as the determinant is multiplicative, the quadratic form $\tilde{Q}$ is also preserved.

In the case where $D < 0$ so that $\operatorname{sig} V(\mathbf{R}) = (3,1)$,

$$\tilde{V}_{(3,1)}(\mathbf{R}) = \left\{ \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix} \middle| a,b \in \mathbf{R},\ z \in \mathbf{C} \right\}$$

where $\bar{z}$ denotes the complex conjugate of z. For $Y \in V_{(3,1)}(\mathbf{R})$, and $P \in \mathbf{SL}_2(\mathbf{C})$, the action of P on Y is given by

$$Y \mapsto P \cdot Y \cdot P^{\dagger}$$

where $P^{\dagger}$ denotes P's conjugate transpose. We can also describe $\tilde{V}_{(3,1)}(\mathbf{R})$ as the set $\{M \in \mathrm{M}_2(\mathbf{C}) \mid M^{\dagger} = M\}$. Since

$$(PMP^{\dagger})^{\dagger} = PM^{\dagger}P^{\dagger} = PMP^{\dagger}$$

$V_{(3,1)}(\mathbf{R})$ is indeed preserved by the action of $\mathbf{SL}_2(\mathbf{C})$.

Altogether, we've computed the identity components of the groups $\mathbf{SO}(2,2)$ and $\mathbf{SO}(3,1)$ using Clifford algebras associated with certain rational quadratic spaces of dimension 4. A similar approach can be applied to other such spaces—for instance, those arising from biquadratic extensions $V = \mathbf{Q}(\sqrt{D_1}) \oplus \mathbf{Q}(\sqrt{D_2})$, or even from the standard space $\mathbf{Q}^4$. In some of these cases, however, the even Clifford algebra C_V^0 may no longer be split over its center $Z(C_V^0)$, and as a result, Spin_V might not admit a description as a matrix group over $Z(C_V^0)$. Furthermore, the center $Z(C_V^0)$ need not be a field extension of $\mathbf{Q}$; in general, it is an *étale algebra* over $\mathbf{Q}$, meaning a finite product of finite field extensions. However, after extending scalars to $\mathbf{R}$, all cases devolve into the two presented above depending on the signature of V.

References

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