Overview of Faltings Proof

Goal. Present work of Lawrence - Venkatesh & Faltings

Thm If X smooth (proj. or affine) curve over a number field and $\chi(X) < 0$, then $\# X(\mathcal{O}_{K,S}) < \infty$, where $\chi(X) = 2 - 2g - \# S$, S: finite set of places.

First reduction is to go from points on curves to curves themselves.

Idea. A curve is more tractable, if its points parametrize some geometric object. Eg, the work of Mazur on modular curves $X_1(N)$.

Easy Thm. If $N > 36$ is prime, then $Y_1(N)(\mathbb{Z}[1/N])$ is empty.

* Here we consider the affine curve i.e., we do not have cuspidal points.

Proof. Let $\ell = 5$, a point $x \in Y_1(N)(\mathbb{Z}[1/N])$ gives an elliptic curve E w/ $j(E) \in \mathbb{Z}[1/N]$, then E has potentially good reduction over $\mathbb{F}_5$ and obtains good reduction over $\mathbb{F}_{5^2}$. Since torsion will inject the Hasse bound finishes the arguement //

Rmk. ① Could replace $\mathbb{Z}[1/N]$ w/ $\mathbb{Z}_{(S)}$.
② Could replace $\mathbb{Z}[1/N]$ by $\mathcal{O}_{K,S}$ and $N>36$ by $N \gg 0$.

Mazur classified the rational points on $X_1(N)$, which is much harder due to existence of cusps.

Another natural family of curves are given by Fermat curves
$$\{ X^N + Y^N = Z^N \}_N$$

Wiles. Observed that Mazur $\Longrightarrow$ Fermat

$$\begin{tikzcd}
 & X(2p) \times_{X(2)} F_p \arrow[dl] \arrow[dr] & \\
X(2p) \arrow[r, "SL_2(\mathbb{F}_p)"'] \arrow[d] & X(2) \arrow[d, equal] & \{x^p+y^p=z^p\} \arrow[l, "(\mathbb{Z}/p)^2"'] \\
X(2) \times X_0(p) \arrow[r] & y^2 = x(x-1)(x-\lambda) &
\end{tikzcd}$$

$(x,y,z) \mapsto \lambda = x^p/y^p$

Mazur says no non-cuspidal points.

Kodaira-Parshin Trick. Here is some notation.

$$M_g(\mathcal{O}_{K,S}) := \left\{ \begin{matrix} \text{smooth curves of genus } g \\ \text{over } \mathcal{O}_{K,S} \end{matrix} \right\} / \cong$$

$\mathcal{A}_g(\mathcal{O}_{K,S}) := \left\{ \begin{array}{l} \text{Abelian varieties of dim } g \\ \text{over } \mathcal{O}_{K,S} \end{array} \right\} / \cong$

$\mathcal{I}_g(\mathcal{O}_{K,S}) := \left\{ \begin{array}{l} \text{Isogeny classes of ab var of} \\ \text{dim } g \text{ over } \mathcal{O}_{K,S} \end{array} \right\} / \cong$

Conj (Shafarevich)
$\mathcal{M}_g(\mathcal{O}_{K,S}), \mathcal{A}_g(\mathcal{O}_{K,S}), \mathcal{I}_g(\mathcal{O}_{K,S})$ are finite.

Thm (Kodaira–Parshin)
Shafarevich conj for curves $\Longrightarrow$ Mordell

Sketch. Given $x \in X(K) \rightsquigarrow Y_x \in \mathcal{M}_{g'}(\mathcal{O}_{K',S'})$ //

* We will see this construction explicitly next time

Summary of Faltings strategy

$X(K) \longrightarrow \mathcal{M}_{g'}(\mathcal{O}_{K',S'})$ [Finite-to-one by De Franchis]

$\longrightarrow \mathcal{A}_{g'}(\mathcal{O}_{K',S'})$ [Finite-to-one by Torelli]

$\longrightarrow \mathcal{I}_{g'}(\mathcal{O}_{K',S'})$ [Finite-to-one by Tate conj.]

* A key ingredient in Faltings

$\longrightarrow \mathrm{Rep}_{G_{K',S'}}(2g')$ [Associate rat'l Tate mod].

Isom classes of 2g-dim'l rational Galois repn of $G_{K',S'}$

The image of $I_{g'}(\mathcal{O}_{K',S'})$ in $\mathrm{Rep}_{G_{K',S'}}$ correspond to semi-simple repr'ns which are finite by a lemma of Faltings.

Moreover, we have finiteness of $X(K)$ as all of the above maps are finite-to-one and the image of $I_{g'}(\mathcal{O}_{K',S'})$ in $\mathrm{Rep}_{G_{K',S'}}$ is finite. //

Overview of Lawrence–Venkatesh Method

Setup. $X \longrightarrow Y$ sm., proj. family over # fld K w/ Y sm. K-vars.

Spp this extends to family $\pi: \mathcal{X} \longrightarrow \mathcal{Y}$ over the ring $\mathcal{O}$ of S-int. of K w/ S: finite set of places (containing all Arch. ones).

For $y \in Y(K)$, call $X_y :=$ fiber over y. We want to bound $\mathcal{Y}(\mathcal{O})$ using that if $y \in Y(K)$ extends to $\mathcal{Y}(\mathcal{O})$, then X_y admits a sm. proper model / $\mathcal{O}$.

Pick prime p that is unramified in K and not below any $v \in S$.

Let $G_K := \mathrm{Gal}(\overline{K}/K)$, and write:

$$\rho_y : G_K \longrightarrow \mathrm{Aut}(H^*_{\acute{e}t}(X_y \times_K \overline{K}, \mathbb{Q}_p)).$$

Lemma (Faltings)
As y varies through $\mathcal{Y}(\mathcal{O})$, there are only finitely many possibilities for the semi-simplification

of the G_K-repn ρ_y.

"Proof". This is a consequence of Hermite–Minkowski //

Now, let Y: proj. k-curve. Then the finiteness of $Y(k) = \mathcal{Y}(\mathcal{O})$ follows from deducing:

① The repn ρ_y is semi-simple for all but finitely many $y \in Y(k)$, and

② The map $Y(k) \longrightarrow \{\text{Iso. class of } \rho_y|G_{K_v}\}$ has finite fibers. (*)

For the time being, lets focus on (2).

By p-adic Hodge theory, we have a fully faithful embedding:

$$\left\{\begin{array}{c}\text{crystalline repn} \\ \text{of } G_{K_v}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Filtered } \phi\text{-modules} \\ \text{over } K_v\end{array}\right\}$$

For suitable choice of v, the repn. $\rho_y|G_{K_v} =: \rho_{y,v}$ will be crystalline.

The above map is defined as follows:

$$\rho_{y,v} \longmapsto (\underbrace{H_{dR}(X_y, k_v)}_{\text{de Rham cohomology of } X_y}, \underbrace{Fil^\bullet}_{\text{Hodge filtration inside of } H_{dR}(X_y, k_v)}, \underbrace{\varphi}_{\text{semi-linear automorph. cl of the Frobenius from crystalline cohomology}})$$

To prove (2), it suffices to show that the map:

$$Y(k_v) \longrightarrow \left\{ \begin{matrix} \text{Filtered } \phi\text{-modules} \\ \text{over } k_v \end{matrix} \right\} \quad (**)$$

has finite fibers.

If we think of $H_{dR}(X_y/k_v)$ as the fiber of the relative de Rham cohomology vector bundle over y, then we see this bundle comes equipped w/ a connection, the Gauss–Manin connection, which allow us to identify nearby fibers.

Fix $y_0 \in Y(k_v)$, the GM connectection gives us an identification: $H_{dR}(X_y, k_v) \cong H_{dR}(X_{y_0}, k_v)$ for all $y \in \Omega \subset Y(k_v)$, a small p-adic disk.

⚠ The identification respects Frobenius but not necessarily Hodge filtrations

The variation of the Hodge filtration is described by the p-adic period map:

$$\mathrm{PERIOD}_v : \Omega \longrightarrow \mathcal{F}(k_v)$$
$$y \longmapsto \left(\mathrm{Fil}^\bullet H_{dR}(X_y, k_v) \text{ transported to } H_{dR}(X_{y_0}, k_v) \right)$$

where $\mathcal{F}$: flag variety parametrising certain chains of subspaces of $H_{dR}(X_{y_0}, k_v) =: H$

Two points $y, y' \in \Omega$ have the same image under $(*)$ when

$$(H, \mathrm{PERIOD}_v(y), \varphi) \cong (H, \mathrm{PERIOD}_v(y'), \varphi)$$

which is equivalent to the existence of an element g in the centralizer $Z(\varphi) \subset GL(H)$ such that $\mathrm{PERIOD}_v(y') \in g(\mathrm{PERIOD}_v(y))$

Therefore, we have that the fiber of $(*)$ over $(H, \mathrm{PERIOD}_v(y), \varphi)$ is contained in:

$$\mathrm{PERIOD}_v^{-1}\left(\mathrm{PERIOD}_v(\Omega) \cap Z(\varphi) \cdot \mathrm{PERIOD}_v(y) \right).$$

Recall that we want to show finiteness of fibers so we want to deduce the intersection on the right is finite.

Suppose that:

- $Z(\mathbb{C}) \cdot \mathrm{PERIOD}_v(y)$ is a proper subvariety, and
- $\mathrm{PERIOD}_v(\Omega)$ is Zariski dense.

Then the intersection amounts to the zeros of a non-zero K_v-analytic fcn on Ω, and therefore must be finite!

Let's now briefly discuss the checking of Zariski density. The main point is that it suffices to check the Zariski density of the complex analytic period mapping

$$\widetilde{Y(\mathbb{C})} \longrightarrow \mathbb{C}\text{-pts of flag variety}$$

Why? The p-adic and complex period maps satisfy the same differential eqn. coming from the GM conn and hence are given by the same power series.

The Zariski density of the complex period mapping can be verified via topological methods (i.e, a difficult monodromy computation).

Now let's turn our attention to the centralizer issue. We need to ensure that the centralizer of the crystalline Frobenius φ acting on the cohomology of X_y is not too large. ie, we need to know that

$$Z(\varphi)\cdot \mathrm{PERIOD}_v(y) \subsetneq \mathcal{F}(K_v)$$

<u>EX.</u> $K_v = \mathbb{Q}_p$ Then φ is $\mathbb{Q}_p$-linear map, and we need to show it is not just scalar!

Note that φ can have too large centralizer in simplest setting. Namely, if we consider $Y = \mathbb{P}^1 - \{0,1,\infty\}$, and let $X \longrightarrow Y$ be the Legendre family, so that $X_t : y^2 = x(x-1)(x-t)$. Now the map $(*)$ does not have finite fibers over the points $t \in \mathbb{Z}_p$ such that $t \pmod p \equiv 0,1$.

To fix this, note that Frobenius is a semi-linear operator over an unramified extn $L_w/\mathbb{Q}_p$ and the semi-linearity gives a non-trivial bound on the size of the centralizer, which gets better as $[L_w:\mathbb{Q}_p]$ gets larger.

For the S-unit example, instead of looking at the Legendre family, we need to consider the family w/ fibers

$$X_t = \coprod_{z^{2^k}=t} \{y^2 = x(x-1)(x-t)\}$$

The map $t \longmapsto [\rho_t]$ will have finite fibers, at least on residue disks where $\mathbb{I} \neq \square$.
This is the importance of enlarging the field K_v.

To conclude, let's mention the higher dimensional LV results.

Setup. $\pi: X \longrightarrow Y$ sm. proper morp / $\mathbb{Z}[S^{-1}]$ whose fibers are geom. conn. of relative dimension d.

Pseudo-Thm Suppose that the monodromy repn of Y is large. Then, $Y(\mathbb{Z}[S^{-1}])$ is not Zariski dense. Furthermore, if the monodromy repn of any subvariety $Y' \subset Y$ also has large image, then $Y(\mathbb{Z}[S^{-1}])$ is finite.
$\longrightarrow$ See Thm 10.1 of LV for details
While the verification of the large monodromy is very difficult, the authors make this pseudo thm precise for the set of hypersurfaces.

Thm (LV)

$\exists$ pos. integer n_0 and $d_0(n)$ satisfying:
If $n \geq n_0$ and $d \geq d_0(n)$ and if $X \to Y$ denote the universal family of hypersurfaces in $\mathbb{P}^n$ of degree d, then $Y(\mathbb{Z}[S^{-1}])$ is not Zariski dense in Y for any finite set of primes S.

"The set of hypersurfaces of deg d in $\mathbb{P}^n$ w/ good red away from S are contained in a proper Zariski closed subset of the moduli space of hyp."

Rmk. The classical Bombieri–Lang conjecture posits that if a variety is of general type (i.e., the Kodaira dimension of $X \stackrel{\cdot}{=} \dim X$), then the set of rat'l points is not Zariski dense in X. A variant (due to Lang) states that if the variety X has ample cotangent bundle, then the set of rational points is finite. Note that the first conj. implies the second as having ample cotangent bundle implies that every subvariety is of general type. Therefore, this pseudo-thm and these conjectures seem to suggest a relationship b/w

Large Monodromy Representation $\rightsquigarrow$ Being of general type

For (1) above, we will show that a generic local Galois repn cannot come from a global repn that is not simple, and then utilize the Zariski density of the p-adic period map.