The Chowla–Selberg Formula

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MATH726

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Complex Multiplication

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- An elliptic curve E has *complex multiplication* or *CM* by $\mathcal{O}$ if $End(E) \cong \mathcal{O}$.
- A *CM* point of $\mathbb{H}$ is a point $\tau \in \mathbb{H}$ which satisfies a quadratic equation over $\mathbb{Q}$, so that $\tau = a + b\sqrt{d}$ for some $a, b, d \in \mathbb{Q}$ with $d < 0$ and $b > 0$.

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- The discriminant of a *CM* point is the smallest discriminant of a quadratic polynomial over $\mathbb{Z}$ of which it is a root. Let $\mathfrak{Z}_D \subset \mathbb{H}$ be the set of *CM* points of discriminant D. In particular, the cardinality of $\Gamma_1 \backslash \mathfrak{Z}_D$ is $h(D)$, the class number of D.

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- Elliptic curves with *CM* by $\mathcal{O}$ correspond to the elliptic curves $\mathbb{C}/\langle 1, \tau \rangle$ for $\tau \in \mathbb{H} \cap K$.

Complex Multiplication

Theorem 1

Let $E/\mathbb{C}$ be an elliptic curve with CM by $\mathcal{O}$. Then $j(E) \in \bar{\mathbb{Q}}$.

In fact, $j(E) \in H_K$, where H_K is the Hilbert class field of K.

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Theorem 2

Let $\tau \in \mathbb{H} \cap K$, and let f be a modular function with rational or algebraic Fourier coefficients. Then $f(\tau)$ is algebraic.

Periods

- If $f \in M_k(1)$ and $g \in M_n(1)$ then f^n/g^k has weight 0, and by **Theorem 2** it is algebraic at τ. Therefore $f(\tau)^{\frac{1}{k}}$ and $g(\tau)^{\frac{1}{n}}$ is algebraically proportional.

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- **Theorem 2** implies that for any modular form of weight k for the modular group Γ_1, the value of $f(\tau)$ is an algebraic multiple of Ω_τ^k, where Ω_τ depends on τ only.

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Proposition 1

Let $G_{2k}(z)$ be the Eisenstein series of weight k on the modular group Γ_1, and let τ be a *CM* point in $\mathbb{H}$. Then there is a transcendental number Ω_τ such that

$$\frac{G_{2k}(\tau)}{\Omega_\tau^{2k}} \in \bar{\mathbb{Q}}.$$

Moreover, Ω_τ can be viewed as a fundamental period of a *CM* elliptic curve defined over the Hilbert class field of $\mathbb{Q}(\tau)$.

Periods

Since any two *CM* points which generate the same imaginary quadratic field are related by some $M \in GL(2, \mathbb{Z})$, we can show that:

Proposition 2

For each imaginary quadratic field K there is a number $\Omega_K \in \mathbb{C}^*$ such that $f(\tau) \in \bar{\mathbb{Q}} \cdot \Omega_K^k$ for all $\tau \in K \cap \mathbb{H}$, for all $k \in \mathbb{Z}$, and all modular forms f of weight k (of the modular group Γ_1) with algebraic coefficient.

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- A natural choice of f is the modular form $\Delta(z)$, since it never vanishes. Recall that $\Delta(z) = (2\pi)^{12}\eta(z)^{24}$.
- A better choice would be $\eta(z)^2$ in order to achieve weight 1. As $F(z) = \text{Im}(z)|\eta(z)|^4$ is Γ_1-invariant, we can choose our f to be $F(z)$.

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- Later we can take the $h(D)$-th root of the product to obtain Ω_K. In fact, it is reasonable to take the $2h(D)/w(D)$-th root, as the elliptic fixed points i and ρ are always to be counted with multiplicity $\frac{1}{2}$ and $\frac{1}{3}$, respectively. In other words,

$$h'(D) = \frac{2h(D)}{w(D)} = \frac{1}{3}, \frac{1}{2}, \text{ or } h(K) \quad \text{for} \quad D = -3, D = -4, \text{ or } D < -4.$$

Chowla–Selberg formula

Let χ_D be the quadratic character associated to K, and $\Gamma(x)$ be the *Euler gamma function*. Then the product of the invariants $F(\tau)$ over $\tau \in \Gamma_1 \backslash \mathfrak{Z}_D$ can be evaluated as a product of $\Gamma(r)$s, where $r \in \mathbb{Q}$:

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Theorem [S.Chowla, A. Selberg (1949)]

Let K be an imaginary quadratic field of discriminant D. Then

$$\prod_{\tau \in \Gamma_1 \backslash \mathfrak{Z}_D} \left(4\pi \sqrt{|D|} F(\tau) \right)^{\frac{2}{w(D)}} = \prod_{m=1}^{|D|-1} \Gamma\left(\frac{m}{|D|}\right)^{\chi_D(m)}, \tag{1}$$

where χ_D and $\Gamma(x)$ are defined as above.

Recall that $|\Gamma_1 \backslash \mathfrak{Z}_D| = h(D) < \infty$, and therefore the product on the left is well-defined.

Chowla–Selberg formula

As a corollary to the Theorem, the constant Ω_K in *Proposition 2* can be chosen to be:

$$\Omega_K = \frac{1}{\sqrt{(2\pi|D|)}} \left(\prod_{m=1}^{|D|-1} \Gamma\left(\frac{m}{|D|}\right)^{\chi_D(m)} \right)^{\frac{1}{2h'(D)}}. \tag{2}$$

Chowla–Selberg formula

Let $\mathcal{O}_K$ be the ring of integers of K, and let $\mathfrak{a}_i$ be ideals of $\mathcal{O}_K$ which represent the distinct ideal classes. We fix an embedding $K \hookrightarrow \mathbb{C}$, so the ideals $\mathfrak{a}_i$ give lattices and the quotients $\mathbb{C}/\mathfrak{a}_i$ corresponds to the $h(K)$ distinct complex elliptic curves with CM by $\mathcal{O}_K$. Then (1) can also be expressed as:

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$$\prod_{i=1}^{h(K)} \Delta(\mathfrak{a}_i)\Delta(\mathfrak{a}_i^{-1}) = \left(\frac{2\pi}{|D|}\right)^{12h(K)} \prod_{\substack{0<a<|D| \\ (a,|D|)=1}} \Gamma\left(\frac{a}{|D|}\right)^{6w(D)\chi_D(a)}, \qquad (3)$$

where the product on the left $\Delta(\mathfrak{a})\Delta(\mathfrak{a}^{-1})$ depends only on the ideal class of $\mathfrak{a}$.

Examples

- Hurwitz was able to show using elliptic functions that

$$\sum_{\substack{\lambda \in \mathbb{Z}[i] \\ \lambda \neq 0}} \frac{1}{\lambda^k} = \frac{H_k}{k!} \omega^k \qquad \text{for all } k \geq 3, \tag{4}$$

for certain rational numbers H_k, where

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- Note that the sum on the left of (4) is the special value of the modular form $2G_k(z)$ at $z = i$.

Examples Contd.

- Using *Proposition 1* and (2) for $K = \mathbb{Q}(i)$ we get

$$\Omega_K = \frac{1}{\sqrt{(8\pi)}} \left(\prod_{m=1}^{3} \Gamma\left(\frac{m}{|D|}\right)^{\chi_D(m)} \right) = \frac{1}{2\sqrt{2\pi}} \frac{\Gamma(1/4)}{\Gamma(3/4)} = 2\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)^2,$$

where $D = -4, h'(D) = 1$, and the last equality follows from the functional equation $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$.

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- A similar argument for $\mathcal{O}_{\mathbb{Q}(\rho)} = \mathbb{Z}[\rho]$ shows that

$$G_{6k}(\rho) = r'\Omega_{\mathbb{Q}(\rho)}^{6k} = r'\left(\Gamma(1/3)^3/\pi\right)^{6k},$$

  where $r' \in \bar{\mathbb{Q}}$.

Sketch of the proof of (3)

The analytic proof involves the computation of the logarithmic derivative of the zeta function of K at the point $s = 0$ in two different ways, where the zeta function of K is given by

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K, \mathfrak{a} \neq 0} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s},$$

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Let $\zeta_K(\mathfrak{a}_i, s)$ be the partial zeta functions which are defined by taking the partial sum over the ideals $\mathfrak{a}$ of $\mathcal{O}_K$ in the same class as $\mathfrak{a}_i$, for $i = 1, \ldots, h(K)$. We will also denote $h(K) = h, w(D) = w$, and $\chi_D(\cdot) = \chi(\cdot)$ for a fixed K.

Proof Contd.

- Kronecker's limit formula gives the first two terms in the Taylor expansion of $\zeta_K(\mathfrak{a}_i, s)$ at $s = 0$, and summing them over the ideal classes yields:

$$\zeta_k(s) = -\frac{h}{w} - \frac{1}{12w} \log\big(\Delta(\mathfrak{a}_i)\Delta(\mathfrak{a}_i^{-1})\big) s + O(s^2).$$

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- Therefore,

$$\left.\frac{d\log(\zeta_k(s))}{ds}\right|_{s=0} = \frac{1}{12h}\sum_{i=1}^{h}\log\bigl(\Delta(\mathfrak{a}_i)\Delta(\mathfrak{a}_i^{-1})\bigr).$$

Proof Contd.

On the other hand:

- For $\mathrm{Re}(s) > 1$ we may also write:

$$\zeta_K(s) = \zeta(s)L(\chi, s);$$

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- It follows that $d\log(\zeta_K(s)) = d\log\zeta(s) + d\log L(\chi, s)$. These logarithmic derivative st $s = 0$ can be calculated from Lerch's expansion of the Hurwitz zeta function (with $0 < x \leq 1$):

$$H(x, s) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} = \left(\frac{1}{2} - x\right) + \log\left(\frac{\Gamma(x)}{\sqrt{2\pi}}\right) s + O(s^2).$$

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- We also obtain by summing over $0 < a < |D|$ with $(a, |D|) = 1$:

$$d\log L(\chi, s)|_{s=0} = \frac{w}{2h} \sum_{0<a<|D|} \chi(a) \log \Gamma\left(\frac{a}{|D|}\right) - \log|D|.$$

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- Comparing this with the one obtained using Kronecker's limit formula and exponentiation gives the Chowla-Selberg formula (3).

Proof Contd.

- Summing over $0 < a < |D|$ with $(a, |D|) = 1$ we find

$$L(\chi, s) = d^{-s} \sum \chi(a) H\left(\frac{a}{|D|}, s\right),$$

and $L(\chi, 0) = -\sum \chi(a) \frac{a}{|D|}$, $\zeta_K(0) = \frac{1}{2} \sum \chi(a) \frac{a}{|D|}$.

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- Comparing the last formula with the formula for $\zeta_K(0)$ obtained by summing the partial zeta functions gives the *Dirichlet's Class Number formula*:

$$h = -\frac{w}{2} \sum_{0<a<d} \chi(a) \frac{a}{|D|}.$$

Thank you for your attention!