1

Monday, November 2

Last week, we described the rigid analytic structure on

$$\mathcal{H}_p = \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$$

<u>Proposition</u> There is a unique <u>reduction map</u> $r : \mathcal{H}_p \longrightarrow \mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ satisfying:

(1) $r(A^*) = v_*$; $r(W_j) = e_j$

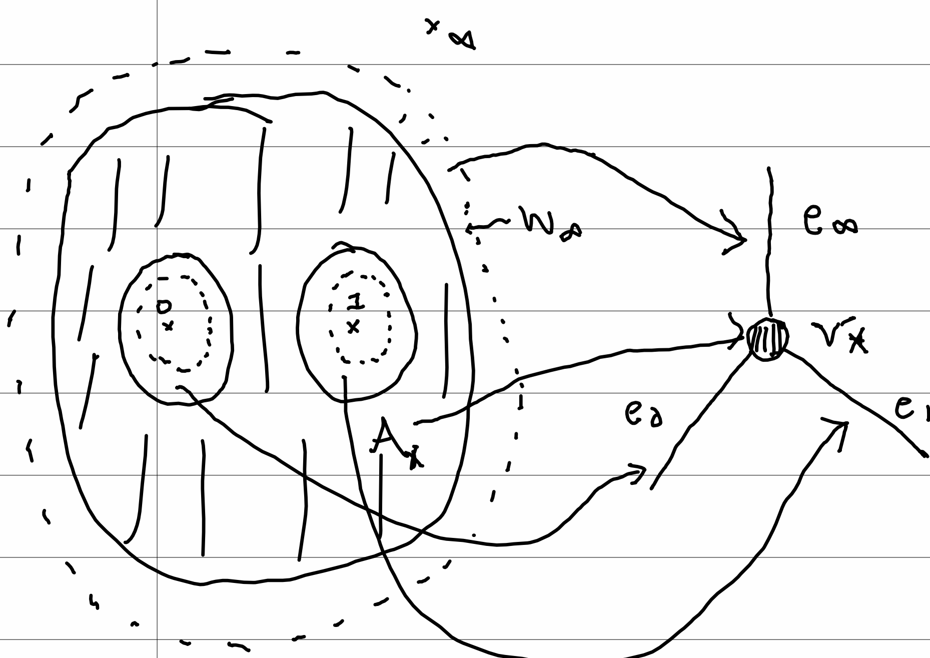
(2) $r(\gamma z) = \gamma r(z) \quad \forall\ \gamma \in PGL_2(\mathbb{Q}_p),\ z \in \mathcal{H}_p$

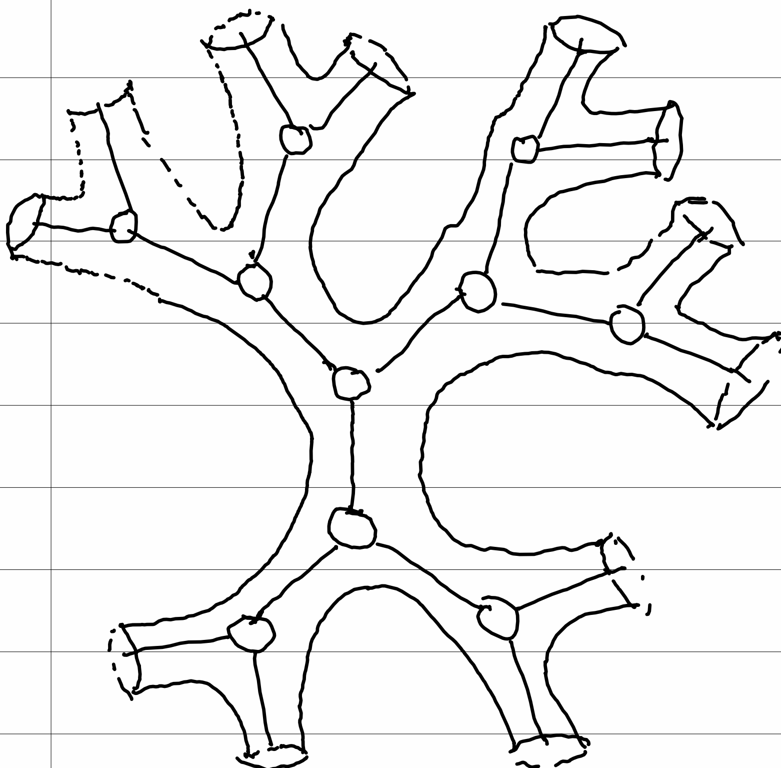
Here - $A_* := \mathbb{P}_1(\mathbb{C}_p) - \mathrm{red}^{-1}(\mathbb{P}_1(\mathbb{F}_p))$

- $v_* = [\mathbb{Z}_p^2]$ $\quad\quad \mathrm{Stab}(v_*) = SL_2(\mathbb{Z}_p)$

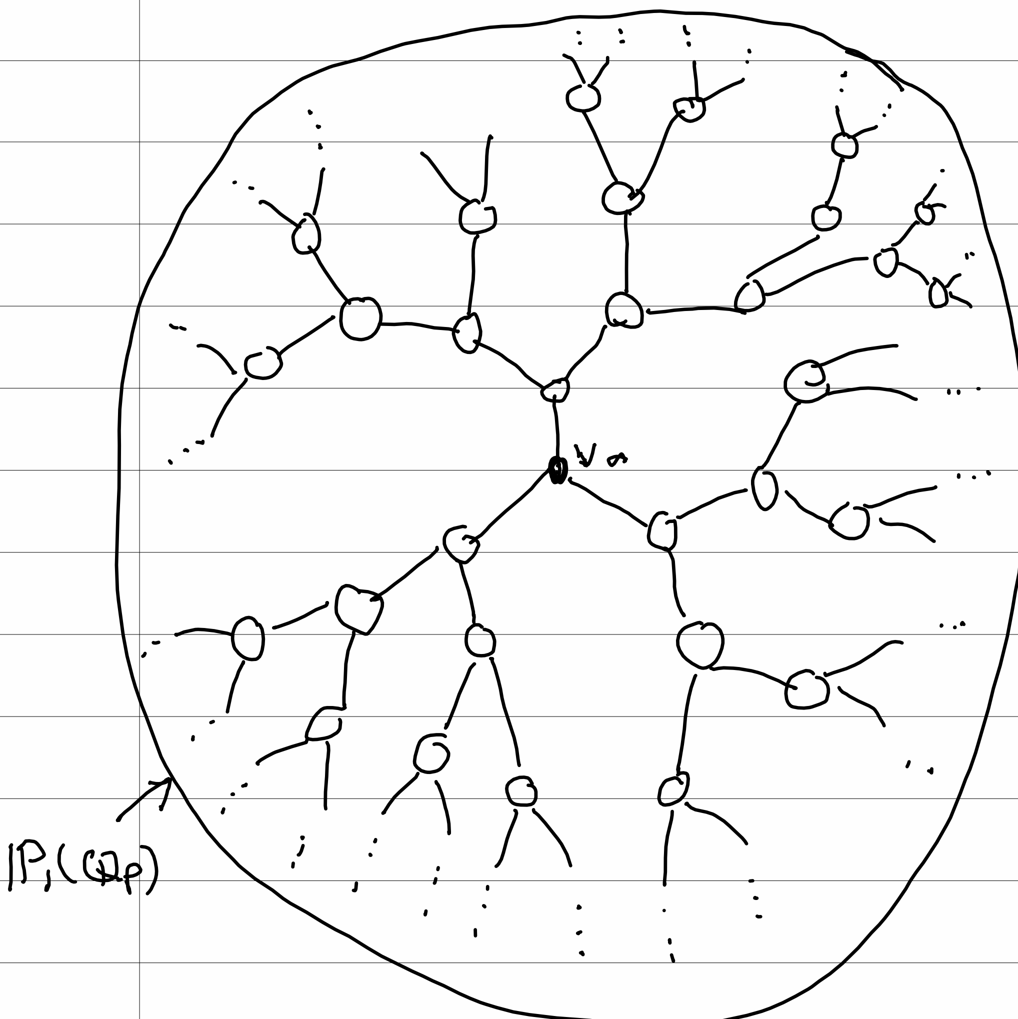
- $W_0, W_1, \ldots, W_\infty$ are the annuli "connected" to A_*

- $e_0, e_1, \ldots, e_\infty$ are the edges containing v_*.





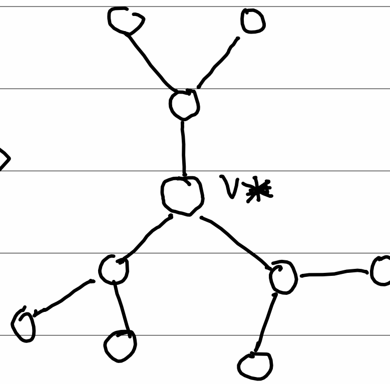
$\mathcal{H}_p$ can be envisaged as a "tubular neighhbourhood" of $\mathcal{T}$ (cf Isabella's presentation.)



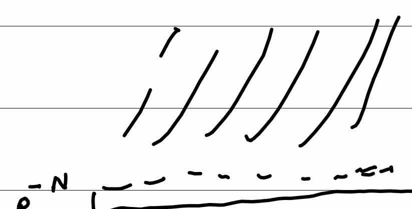
The ends of $\mathcal{T}$ are in bijection with the "boundary" $\mathbb{P}_1(\mathbb{Q}_p)$

$$\mathcal{H}_p^{\le N} := \left\{ z = [z_1 : z_2] \in \mathbb{P}_1(\mathcal{O}_{\mathbb{C}_p}) \text{ s.t. } \left|\det\begin{vmatrix} z_1 & z_2 \\ a & b \end{vmatrix}\right|_p \ge p^{-N} \quad \forall\, [a:b] \in \mathbb{P}_1(\mathbb{Z}_p) \right\}$$

$$\mathcal{H}_p^{\le 2} \longrightarrow$$



$$\mathcal{H}_p = \bigcup_{N \ge 1} \mathcal{H}_p^{\le N}$$



$$SL_2(\mathbb{Z}_p) \circlearrowright \mathcal{H}_p^{\le N}$$

$\mathcal{H}_p^{\le N}$ is the complement of $(p+1)p^{N-1}$ mod p^N "residue discs" centered at $\mathbb{P}_1(\mathbb{Q}_p)$.

<u>Definition</u> A rigid analytic function on $\mathcal{H}_p$ is a function $f : \mathcal{H}_p \longrightarrow \mathbb{C}_p$ such that $f|_A$ is a uniform limit of rational functions having poles outside A, for all affinoids $A \subseteq \mathcal{H}_p$.

(Enough to require this for the affinoids $\mathcal{H}_p^{\leq N}$.)

A <u>rigid meromorphic</u> function on $\mathcal{H}_p$ is a ratio of rigid analytic functions.

<u>Goal</u>: To produce Γ-invariant rigid analytic (or meromorphic) functions. (p-adic automorphic forms.)

$$\Gamma = (R_{S \to p})^{\times}_{1} \circlearrowright C_B(\mathbb{C}_p) - C_B(\mathbb{Q}_p) \simeq \mathcal{H}_p$$

Action of Γ on $\mathcal{H}_p$

Let $R = R_{S,p}$ = maximal $\mathbb{Z}[\frac{1}{p}]$-order in $B_{S-\{p\}}$.

If v is a vertex of $\mathcal{T}$, then

$$R_v = \{ x \in R \text{ s.t. } x \cdot v = v \} \cup \{0\}$$

is a maximal $\mathbb{Z}$-order in R

Recall: $B_{S-\{p\}}$ is a <u>definite</u> quaternion algebra, i.e., $B_{S-\{p\}} \otimes \mathbb{R} \cong \mathbb{H}$

Lemma 1: For all vertices $v \in \mathcal{T}_0$, $\mathrm{stab}_\Gamma(v)$ is finite.

Proof: $\mathrm{Stab}_\Gamma = (R_v)_1^\times$, where

where R_v is a maximal $\mathbb{Z}$-order in B. But we know that maximal $\mathbb{Z}$-orders in a definite quaternion alg. have finite unit group $\boxtimes$

Lemma 2: If A_1, A_2 are any two affinoids in $\mathcal{H}_p$, then

$$\#\{\gamma \in \Gamma \text{ s.t. } \gamma A_1 \cap A_2 \neq \emptyset\} < \infty.$$

Proof: Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the finite subgraphs of $\mathcal{T}$ with $r(A_1) = \mathcal{G}_1$, $r(A_2) = \mathcal{G}_2$. Then

$$\{\gamma \in \Gamma \mid \gamma A_1 \cap A_2 \neq \emptyset\}$$

$$= \{\gamma \in \Gamma \mid \gamma \mathcal{G}_1 \cap \mathcal{G}_2 \neq \emptyset\}$$

$$= \bigcup_{\substack{v_1 \in \mathcal{G}_1 \cap \mathcal{T}_0 \\ v_2 \in \mathcal{G}_2 \cap \mathcal{T}_0}} \{\gamma \in \Gamma \mid \gamma v_1 = v_2\}$$

But these sets are finite, by Lemma 1, and $\mathcal{G}_1 \times \mathcal{G}_2$ is also finite. QED

The Weil Symbol

Given $\mathcal{D} \in \mathrm{Div}^0(\mathbb{P}_1(\mathbb{C}_p))$

Let $f_{\mathcal{D}}$ be a rational function satisfying $\mathrm{div}(f_{\mathcal{D}}) = \mathcal{D}$.

(It is unique up to a multiplicative constant.)

<u>Definition</u> The <u>Weil symbol</u> attached to $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Div}^0(\mathbb{P}_1)$ is $[\mathcal{D}_1; \mathcal{D}_2] := f_{\mathcal{D}_1}(\mathcal{D}_2)$

where $f\left(\sum_{x \in \mathbb{P}_1} n_x \cdot x\right) := \prod f(x)^{n_x}$ $\boxtimes$

Properties of the Weil symbol

① It is bilinear:

$$[\mathcal{D}_1 + \mathcal{D}_2 ; \mathcal{D}_3] = [\mathcal{D}_1 , \mathcal{D}_3] \times [\mathcal{D}_2 ; \mathcal{D}_3]$$

$$[\mathcal{D}_1 ; \mathcal{D}_2 + \mathcal{D}_3] = [\mathcal{D}_1 , \mathcal{D}_2] \times [\mathcal{D}_1 ; \mathcal{D}_3]$$

② It is symmetric: (Weil reciprocity)

$$[\mathcal{D}_1 ; \mathcal{D}_2] = [\mathcal{D}_2 ; \mathcal{D}$$

③ It is $SL_2(\mathbb{Q}_p)$ - equivariant, i.e.

$$[\gamma \mathcal{D}_1 ; \gamma \mathcal{D}_2] = [\mathcal{D}_1 ; \mathcal{D}_2]$$

$$\forall \gamma \in SL_2(\mathbb{Q}_p).$$

④ $$[(x_1) - (x_2) ; (y_1) - (y_2)] = \frac{(x_1 - y_1)(x_2 - y_2)}{(x_1 - y_2)(x_2 - y_1)}$$

(cross-ratio).

<u>Lemma</u> Suppose there is a $t \in \mathbb{P}_1(\mathbb{Q}_p)$ satisfying $d(x,t) \leq p^{-2N}$ for all $x \in$ Support$(\mathcal{D}_1)$, and $d(y,t) \geq p^{-N}$ for all $y \in$ Support$(\mathcal{D}_2)$. Then

$$\left| [\mathcal{D}_1 ; \mathcal{D}_2] - 1 \right|_p \leq p^{-N}$$

<u>Proof</u> Since $SL_2(\mathbb{Z}_p)$ acts transitively of $\mathbb{P}_1(\mathbb{Q}_p)$ and preserves $d(x,y)$, we can assume $t = 0$.

By bilinearity, can assume $\mathcal{D}_1 = (x_1) - (x_2)$, $\mathcal{D}_2 = (y_1) - (y_2)$. $[\mathcal{D}_1 ; \mathcal{D}_2] = \dfrac{(x_1 - y_1)(x_2 - y_2)}{(x_1 - y_2)(x_2 - y_1)}$
$\equiv 1 \pmod{p^N}$ ⌧

Corollary: If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are degree zero divisors on $\mathcal{H}_p$, the infinite product

$$[\mathfrak{D}_1; \mathfrak{D}_2]_p := \prod_{\gamma \in \Gamma} [\mathfrak{D}_1; \gamma \mathfrak{D}_2]$$

converges absolutely.

Proof: Let N be chosen large enough so that support$(\mathfrak{D}_1)$, and support$(\mathfrak{D}_2) \subseteq \mathcal{H}_p^{\leq N}$

By lemma 2, for all but finitely many $\gamma \in \Gamma$,

$$\gamma \mathcal{H}_p^{\leq N} \cap \mathcal{H}_p^{\leq 2N} = \emptyset$$

Hence, $\tilde{\forall} \gamma$, there is $t_\gamma \in \mathbb{P}_1(\mathbb{Q}_p)$

with $d(\gamma z, t_\gamma) \leq p^{-2N}$ $\forall z \in \mathcal{H}_p^{\leq N}$

while $d(z, t_\gamma) \geq p^{-N}$ $\forall z \in \mathcal{H}_p^{\leq N}$

In particular,

- $d(z, t_\gamma) \geq p^{-N}$ $\forall z \in$ support$(\mathcal{D}_1)$
- $d(z, t_\gamma) \leq p^{-2N}$ $\forall z \in$ support$(\gamma \mathcal{D}_2)$

$\Rightarrow$ $[\mathcal{D}_1 ; \gamma \mathcal{D}_2] \equiv 1 \pmod{p^N \mathcal{O}_{\mathbb{C}_p}}$

$\tilde{\forall}$ $\gamma \in \Gamma$,

$\Rightarrow$ $[\mathcal{D}_1 ; \mathcal{D}_2]_\Gamma$ converges absolutely. $\boxtimes$

Definition: The quantity $[\mathcal{D}_1 ; \mathcal{D}_2]_\Gamma$ is called the **Γ-weil symbol** attached to $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Div}^0(\mathcal{H}_p)$.

The p-adic period pairing:

Given $\gamma_1, \gamma_2 \in \Gamma$, set

$$\langle \gamma_1 ; \gamma_2 \rangle := [(\gamma_1 z_1) - (z_1) ; (\gamma_2 z_2) - (z_2)]_\Gamma$$

Some simple facts:

I. $\langle \gamma_1 ; \gamma_2 \rangle$ does not depend on the choice of z_1 & z_2

Proof:

$$[(\gamma z)-(z);\mathcal{D}]_\Gamma \div [(\gamma z')-(z');\mathcal{D}]_\Gamma$$

$$= [(\gamma z)-(\gamma z');\mathcal{D}]_\Gamma \div [(z)-(z');\mathcal{D}]_\Gamma$$

$$= [(z)-(z');\mathcal{D}]_\Gamma \div [(z)-(z');\mathcal{D}]_\Gamma$$

$$= 1.$$

II. $\langle \; ; \; \rangle$ takes values in $\mathbb{Q}_p^\times$

Proof: Embed $B \otimes \mathbb{Q}_p$ into $M_2(\mathbb{Q}_p)$ and hence Γ into $SL_2(\mathbb{Q}_p)$.

Then, $\langle \gamma_1, \gamma_2 \rangle^\sigma = \langle \gamma_1 \tau_1^\sigma - \tau_1^\sigma ; \gamma_2 \tau_2^\sigma - \tau_2^\sigma \rangle$

$$= \langle \gamma_1, \gamma_2 \rangle \qquad \forall \sigma \in \mathrm{Aut}(\mathbb{C}_p/\mathbb{Q}_p)$$

III. $\langle\ ,\ \rangle$ is a homomorphism in each variable.

Proof: $\langle \gamma_1\gamma_2, \gamma_3 \rangle$

$= [\gamma_1\gamma_2 z) - (z);\ \mathcal{D}]_\Gamma$, $\quad \mathcal{D} = (\gamma_3 z') - (z')$

$= [(\gamma_1\gamma_2 z) - (\gamma_2 z);\ \mathcal{D}]_\Gamma \times [(\gamma_2 z) - z;\ \mathcal{D}]_\Gamma$

$= \langle \gamma_1, \gamma_3 \rangle \times \langle \gamma_2, \gamma_3 \rangle$ $\boxtimes$

Hence, $\langle\ ,\ \rangle$ induces a symmetric bilinear pairing:

$$\langle\ ,\ \rangle : \Gamma_{ab} \times \Gamma_{ab} \longrightarrow \mathbb{Q}_p^\times$$

Theorem: The function

$$-\mathrm{ord}_p \circ \langle \ , \ \rangle : \Gamma_{ab} \times \Gamma_{ab} \longrightarrow \mathbb{Z}$$

is positive definite, i.e.,

$\mathrm{ord}_p \langle \gamma, \gamma \rangle \leq 0$, with equality iff $\gamma = 1$.

Proof: Gerritzen, Van der Put, Chapter VI, §2.

Corollary: $\langle \ , \ \rangle$ induces a map $j : \Gamma \longrightarrow \mathrm{Hom}(\Gamma, \mathbb{Q}_p^\times)$ and $\mathrm{ord}_p \circ j : \Gamma \longrightarrow \mathrm{Hom}(\Gamma, \mathbb{Z})$ is injective.

As we will see shortly, the quotient $\mathrm{Hom}(\Gamma, \mathbb{Q}_p)/j(\Gamma)$ is identified with $J_S(\mathbb{Q}_p)$, where

$$J_S = \mathrm{Jac}(\Gamma \backslash \mathcal{H}_p).$$

p-adic θ-functions

Suppose $\Delta \in \mathrm{Div}^0(\Gamma \backslash \mathcal{H}_p)$

Problem: Produce a rigid meromorphic function on $\Gamma \backslash \mathcal{H}_p$, such that

$$\mathrm{div}(F_\Delta) = \Delta \quad \boxtimes$$

Choose $\mathcal{D} \in \mathrm{Div}^0(\mathcal{H}_p)$ with $\mathcal{D} \equiv \Delta \pmod{\Gamma}$

Definition The p-adic Θ-function attached to the divisor $\mathcal{D}$ is

$$\theta_{\mathcal{D}}(z) = [z - \eta ; \mathcal{D}]_{\Gamma},$$

where η is a base point.

Properties of $\theta_{\mathcal{D}}$

① $\theta_{\mathcal{D}}$ is a rigid meromorphic function of $z \in \mathcal{H}_p$.

② $\theta_{\mathcal{D}}$ is Γ-invariant up to multiplicative scalars, i.e., there is a homomorphism

$$c_{\mathscr{D}} : \Gamma \longrightarrow \mathbb{C}_p^\times$$

satisfying $\theta_{\mathscr{D}}(\gamma z) = c_{\mathscr{D}}(\gamma) \times \theta_{\mathscr{D}}(z)$

for all $\gamma \in \Gamma$.

<u>Proof</u> $\theta_{\mathscr{D}}(\gamma z) = [\gamma z - \eta ; \mathscr{D}]_\Gamma$

$= [z - \gamma^{-1}\eta ; \mathscr{D}]_\Gamma$

$= [z - \eta ; \mathscr{D}]_\Gamma \times [\eta - \gamma^{-1}\eta ; \mathscr{D}]_\Gamma$

$= \theta_{\mathscr{D}}(z) \times [\gamma\eta - \eta ; \mathscr{D}]$

so $c_{\mathscr{D}}(\gamma) = [\gamma\eta - \eta ; \mathscr{D}]$

for any $\eta \in \mathcal{H}_p$.

The function $c_{\mathfrak{D}} \in \mathrm{Hom}(\Gamma, \mathbb{C}_p^\times)$ is called the <u>factor of automorphy</u> associated to $\theta_{\mathfrak{D}}$.

<u>Theorem</u>: Assume that $c_{\mathfrak{D}}$ belongs to $j(\Gamma)$. Then Δ is a principal divisor, i.e., there is a rigid meromorphic function F_Δ on $\Gamma \backslash \mathcal{H}_p$ with $\mathrm{Div}(F_\Delta) = \Delta$.

Proof If $c_{\mathfrak{D}} \in j(\Gamma)$, then $\exists\ \alpha \in \Gamma$ s.t. $c_{\mathfrak{D}}(\gamma) = \langle \alpha, \gamma \rangle$

Replacing $\mathfrak{D}$ by $\mathfrak{D} - [(\alpha\eta) - (\eta)]$ we get $c_{\mathfrak{D}}(\gamma) = 1$ for all $\gamma \in \Gamma$, and $\mathfrak{D} = \Delta \pmod{\Gamma}$, i.e.,

$\pi_*(\mathfrak{D}) = \Delta$, where $\pi : \mathcal{H}_p \to \Gamma \backslash \mathcal{H}_p$.

But then $\theta_{\mathfrak{D}}(z)$ is Γ invariant, hence descends to a function on $\Gamma \backslash \mathcal{H}_p$. $F_\Delta = \theta_{\mathfrak{D}}$ has divisor equal to Δ. QED

<u>Conclusion</u> The image of $c_{\mathscr{D}}$ in $\mathrm{Hom}(\Gamma, \mathbb{C}_p^\times)/j(\Gamma)$, for any $\mathscr{D} \in \mathrm{Div}^0(\mathcal{H}_p)$ for which

$$\pi_*(\mathscr{D}) = \Delta,$$

encodes the image of Δ in

$$J_S(\mathbb{C}_p) = \mathrm{Hom}(\Gamma, \mathbb{C}_p^\times)/j(\Gamma),$$

$$J_S = \mathrm{Jac}(\Gamma \backslash \mathcal{H}_p).$$

Cohomological formulation

Let $M^\times$ = multiplicative group of non-zero rigid meromorphic functions on $\mathcal{H}_p$.

It is a Γ-module under

$$\gamma \cdot f(z) := f(\gamma^{-1} z)$$

Given $\mathcal{D} \in \mathrm{Div}^0(\mathcal{H}_p)$

$$\theta_{\mathcal{D}} \in H^0(\Gamma, M^\times / \mathbb{C}_p^\times)$$

$$0 \to \mathbb{C}_p^\times \to M^\times \longrightarrow M^\times/\mathbb{C}_p^\times \to 0$$

$$0 \to \mathbb{C}_p^\times \longrightarrow H^0(\Gamma, M^\times) \to H^0(\Gamma, M^\times/\mathbb{C}_p^\times) \to H^1(\Gamma, \mathbb{C}_p^\times).$$

The automorphy factor $c_{\mathscr{D}}$ represents the obstruction to lifting $\theta_{\mathscr{D}}$ to a Γ-invariant element of $M^\times$.