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Algebra 251 B

Logistical Questions

- Classes will meet on-line from 9:30 – 10:30 MWF

- Last lecture: Monday April 14

- Assignment 9: due this Wednesday. Your solutions can be uploaded to MyCourses, as a single pdf file.

- Assignment 10 will be posted before Wednesday.

- Final exam: take-home, will be posted on Friday, April 17 at 9 AM. To be returned before midnight, Monday.

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Technical Matters:

* Do not hesitate to interrupt, and ask questions! (set your microphones)
* I will try to upload the recordings of the lectures, but can not guarantee this: so please do not count on it!!
* Please make use of the Mycourses discussion group, and other ressources on the course blog.

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<u>Brief Review</u>:

- $F = \mathbb{R}$ or $\mathbb{C}$
- An inner product space is a vector space V over F equipped with a bilinear pairing

$$V \times V \longrightarrow F$$

s.t.
- $\langle \lambda v_1 + v_2, w \rangle = \lambda \langle v_1, w \rangle + \langle v_2, w \rangle$
- $\langle v, \lambda w \rangle = \bar{\lambda} \langle v, w \rangle$
- $\langle v, w \rangle = \overline{\langle w, v \rangle}$
- $\langle v, v \rangle \geq 0$, with $=$ iff $v = 0$.
(Positivity axiom)

The pairing $\langle , \rangle$ is non degenerate and induces a map

$$\varphi : V \hookrightarrow \mathrm{Hom}(V, F) = V^*$$
$$v \longmapsto w \mapsto \langle w, v \rangle$$

<u>Caveat:</u> φ is not F-linear, but rather hermitian-linear:

$$\varphi(\lambda v) = \bar{\lambda}\, \varphi(v).$$

If $\dim_F V < \infty$, then the injective map φ is also surjective,

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and identifies V with its dual.

Orthogonality and projections

If $W \subseteq V$,

$$W^{\perp} = \{ v \in V \text{ s.t. } \langle v, w \rangle = 0 \ \forall w \in W \}$$

$$= \ker \left(V \xrightarrow{\varphi} V^* \xrightarrow{\mathrm{res}_W} W^* \right)$$

res_W = restriction to W

$V^* \twoheadrightarrow W^*$

(dual to inclusion $W \hookrightarrow V$.)

Rank-nullity theorem $\Rightarrow$

$$\dim W^{\perp} = \dim V - \dim W^*$$

$$\therefore \boxed{\dim W + \dim W^{\perp} = \dim V}$$

In fact, more is true. Both W and $W^\perp$ are subspaces of V, of complementary dimensions.

<u>Lemma:</u> $W \cap W^\perp = \{0\}$.

<u>Proof</u>

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Indeed, if $v \in W \cap W^\perp$, then

$v \perp v$, $\langle v, v \rangle = 0 \Rightarrow v = 0$ $\boxtimes$

$\Rightarrow \boxed{V = W \oplus W^\perp}$.

This is very important. In general, a subspace of a vector space has a complement, but it is far from unique or canonical.

<u>Orthogonal</u> <u>projection</u>: Every $v \in V$ can therefore be written uniquely as $v = w + w'$, $w \in W$, $w' \in W^\perp$

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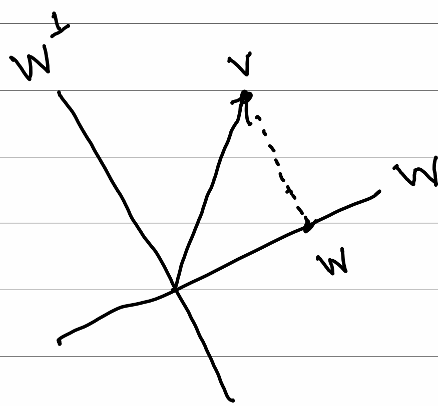
Definition: The vector w is called the orthogonal projection of v onto W

$w := P_W(v)$



Theorem: The vector $w_0 = P_W(v)$ is the unique vector in W which minimises $\|v-w\|^2$.

Proof: Let w be any vector in W

$$v - w = \underbrace{(v - w_0)}_{\in W^\perp} + \underbrace{(w_0 - w)}_{\in W}$$

$$\|v-w\|^2 = \|v-w_0\|^2 + \|w_0-w\|^2$$

$\therefore$ $\|v-w\|^2 \geq \|v-w_0\|^2$ with $=$ iff $\|w-w_0\|=0$ ($\Rightarrow w = w_0$) $\boxtimes$

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Orthonormal bases

Definition $(e_1, \dots, e_n)$ is an orthonormal basis if $\langle e_i, e_j \rangle = \delta_{ij}$.

Theorem (Gram-Schmidt' orthogonalisation)

If $(v_1, \dots, v_n)$ is a basis of V, there is an orthonormal basis $(e_1, \dots, e_n)$ satisfying $\operatorname{span}(e_1, \dots, e_k) = \operatorname{span}(v_1, \dots, v_k)$

for $k = 1, \dots, n$.

Proof: We did this on the last lecture. $\boxtimes$

Corollary: If $W \subset V$, any orthonormal basis $(e_1, \dots, e_k)$ for W can be completed to an ON basis for V.

Proof: Complete $(e_1, \dots, e_k)$ to a basis $(e_1, \dots, e_k, v_{k+1}, \dots, v_n)$ for V. Now, apply Gramm-Schmidt to $(e_1, \dots, e_k, v_{k+1}, \dots, v_n)$. $\square$

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Formula for orthogonal projection onto W:

Let $(e_1, \dots, e_n)$ = orthonormal basis of V, with $W = \operatorname{span}(e_1, \dots, e_k)$

Proposition $P_W(v) = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_k \rangle e_k$

Proof $v = \lambda_1 e_1 + \cdots + \lambda_n e_n$

$\Rightarrow \langle v, e_j \rangle = \lambda_j$

$$\Rightarrow v = \underbrace{\left(\langle v, e_1 \rangle e_1 + \cdots + \langle v, e_k \rangle e_k\right)}_{\in W} + \underbrace{\left(\langle v, e_{k+1} \rangle e_{k+1} + \cdots + \langle v, e_n \rangle e_n\right)}_{\in W^\perp}$$

$\boxed{\text{QED}}$

Adjoints Let $T : V \to V$ be a linear transformation of inner product spaces.

Goal: to formulate various ways in which T might be required to be "compatible" with $\langle , \rangle$.

Theorem: If $\ell \in V^*$, then $\exists$ unique $w \in V$ s.t. $\ell(v) = \langle v, w \rangle$

Proof Recall the isom. $\varphi : V \xrightarrow{\sim} V^*$.
Let w be the unique vector s.t. $\varphi(w) = \ell$ $\blacksquare$

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<u>Theorem</u> For each $w \in V$, there is a unique vector $T^*(w) \in V$ s.t. $\langle T(v), w\rangle = \langle v, T^*(w)\rangle \ \forall v \in V$

<u>Proof</u> Define $\ell \in V^*$ by

$$\ell(v) = \langle T(v), w\rangle$$

Riesz rep theorem $\Rightarrow$ $\exists$ a vector $\tilde{w}$ (depending on T, w) s.t.

$$\ell(v) = \langle v, \tilde{w}\rangle$$

Set $\tilde{w} = T^*(w)$

Then $\cancel{\ell}'$ $\langle T(v), w\rangle = \langle v, T^*(w)\rangle \ \forall v, w$ $\boxtimes$

Definition T^* is called the adjoint of T.

Proposition

1. The function T^* is a linear function from V to V.

2. The map $T \mapsto T^*$ $\quad Hom_F(V,V) \to Hom_F(V,V)$ is skew-linear:
$$(T_1+T_2)^* = T_1^* + T_2^*$$
$$(\lambda T)^* = \bar{\lambda} T^*$$

3. $(T_1 T_2)^* = T_2^* \cdot T_1^*$

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Proof All properties follow from the definition.

$$\langle Tv, w_1 + w_2 \rangle = \langle v, T^*(w_1 + w_2) \rangle$$

OTOH

$$\langle Tv, w_1 + w_2 \rangle = \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle$$
$$= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle$$
$$= \langle v, T^* w_1 + T^* w_2 \rangle$$

Hence $\langle v, T^*(w_1 + w_2) \rangle = \langle v, T^* w_1 + T^* w_2 \rangle$ $\forall v$

$$\Rightarrow \quad T^*(w_1 + w_2) = T^*(w_1) + T^*(w_2)$$

$$\langle v, T^*(\lambda w) \rangle = \langle Tv, \lambda w \rangle = \bar{\lambda} \langle Tv, w \rangle$$
$$= \bar{\lambda} \langle v, T^* w \rangle$$
$$= \langle v, \lambda T^*(w) \rangle \quad \forall v$$

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- $(\lambda T)^* = \bar{\lambda} T^*$?

$$\begin{aligned}\langle (\lambda T)v, w\rangle &= \lambda\langle Tv, w\rangle \\ &= \lambda\langle v, T^* w\rangle \\ &= \langle v, (\bar{\lambda} T^*) w\rangle \quad \boxtimes\end{aligned}$$

$\langle (\lambda T)v, w\rangle = \langle v, (\lambda T)^* w\rangle$

$$\langle T_1 T_2 v, w\rangle = \langle v, (T_1 T_2)^* w\rangle$$

$$\langle T_1 T_2 v, w\rangle = \langle T_2 v, T_1^* w\rangle = \langle v, T_2^* T_1^* w\rangle \quad \boxtimes$$

The matrix of T^*

Lemma Let $(e_1, \dots, e_n)$ be an orthonormal basis for V, and let M = matrix of T in this basis. Then $M = (m_{ij})$, $m_{ij} = \langle Te_j, e_i\rangle$

Proof $Te_j = m_{1j} e_1 + \cdots + m_{nj} e_n$

$\langle Te_j, e_i \rangle = m_{ij}$ $\boxtimes$

Proposition

Now, let $M^* = (m^*_{ij})$ be the matrix of T^*.

$$m^*_{ij} = \overline{m_{ji}}$$

So

$M^* = \overline{M}^t$ (The conjugate transpose of M).

Proof $m^*_{ij} = \langle T^* e_j, e_i \rangle$

$= \overline{\langle e_i, T^* e_j \rangle}$

$= \overline{\langle Te_i, e_j \rangle}$

$= \overline{m_{ji}}$ $\blacksquare$

So the notion of adjoint is an abstract version of the (conjugate) transpose of a matrix.