April 14, Tuesday

Last time we discussed some rudiments of spectral Graph Theory.

- Let $G = (V, E)$ be a graph
- $L^2(V) = \mathbb{C}$ valued functions on V.
- $A : L^2(V) \to L^2(V)$ adjacency operator.

Spectral graph theory studies the rich interplay between the geometry/combinatorics of G and spectral properties of A.

<u>Theorem</u> Let G be a connected d-regular graph. Then

$$\mathrm{Spectrum}(T) \subseteq [-d, d].$$

The largest eigenvalue is $\lambda_1 = d$, and its associated eigenspace is one-dimensional, spanned by the constant function $\mathbb{1}_V$ $\boxtimes$

We proved this last wednesday by direct estimates.

$$\mathrm{Spectrum}(T) = \{\lambda_1, \dots, \lambda_t\}$$

$$-d \leq \lambda_t < \lambda_{t-1} < \cdots < \lambda_2 < \lambda_1 = d$$

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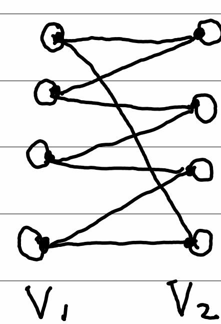
We will discuss an application of the fact that the "largest eigenvalue" λ_1 is d, and has a one-dimensional eigenspace.

<u>Q</u>: When is $-d \in \mathrm{Spec}(A)$?

<u>Definition</u>: A graph G is bipartite if $V(G) = V_1 \sqcup V_2$ and

$$E(G) \subseteq V_1 \times V_2$$

<u>Eg</u>:



Theorem: If G is bipartite, then $-d \in \text{Spectrum}(A)$

Proof: Let $f(v) = \begin{cases} 1 & \text{if } v \in V_1 \\ -1 & \text{if } v \in V_2 \end{cases}$

Clearly $Af = -df$. $\square$

Fact: The converse is also true (We will not prove this.)

The averaging operator

On a d-regular graph, it is

$T = \frac{1}{d} A \qquad Tf(v) = \frac{1}{d} \sum_{w \sim v} f(w)$

For example, $V(G)$ could represent the members of a population, with edges describing the interactions between them. A function $f \in L^2(V(G))$ could represent the distribution of a certain ressource (or virus!) being shared across the population.

$$T^k f \quad (k = 1, 2, 3, \ldots)$$

is a simple model for how this distribution might evolve over time.

Theorem: Let G be a connected d-regular graph which is not bipartite. If $f \in L^2(V(G))$

$$\lim_{k \to \infty} (T^k f)(v) = \frac{1}{N} \left(\sum_{w \in V(G)} f(w) \right)$$

where $N = \# V(G)$.

Proof: Let $\phi_1, \ldots, \phi_N$ be eigenfunctions of $T = \frac{1}{d} A$, with eigenvalues $1 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N > -1$

By the spectral theorem, the eigenvectors ϕ_j of T

can be chosen to be an orthonormal system for $L^2(G)$. Furthermore,

$$\phi_1(v) = \frac{1}{\sqrt{N}} \quad (\forall v \in V(G))$$

(i.e., ϕ_1 is the appropriate multiple of the constant function, normalised so that $\langle \phi_1, \phi_1 \rangle = 1$.)

Now, write

$$f = \alpha_1 \phi_1 + \alpha_2 \phi_2 + \cdots + \alpha_N \phi_N$$

with $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{C}$.

⑧

$$T^k f = \alpha_1 \lambda_1^k \phi_1 + \alpha_2 \lambda_2^k \phi_2 + \cdots + \alpha_N \lambda_N^k \phi_N$$

but $\lambda_1 = 1$, while $|\lambda_2|, \ldots, |\lambda_N| < 1$

$\Rightarrow$ $T^k f \longrightarrow \alpha_1 \phi_1$ as $k \to \infty$

$\alpha_1 \phi_1$ is the orthogonal projection of f onto $\mathbb{C} \cdot \phi_1$. Hence

$$\alpha_1 = \langle f, \phi_1 \rangle = \frac{1}{\sqrt{N}} \sum_{v \in V(G)} f(v)$$

$$\alpha_1 \phi_1 = \frac{1}{\sqrt{N}} \left(\sum_{v \in V(G)} f(v) \right) \times \frac{1}{\sqrt{N}} \mathbb{1}_G$$

constant function $\mathbb{1}$ on $V(G)$

$$\alpha_1 \phi_1 (v) = \frac{1}{N} \sum_{v \in V(G)} f(v) \qquad \boxtimes$$

The size of the eigenvalues $\lambda_2, \ldots, \lambda_N$ controls the rate at which $T^k f$ converges to the uniform distribution $\alpha_1 \phi_1 \ldots$

Of course, this is a very crude, simplistic model with which to capture the spread of a disease, but it illustrates how spectral methods arise in questions of this sort. $\square$

Assignment 10

A few comments about question 1.

(a) $\overline{T} = \operatorname{Trace}(T) - T$

In matrix form, if $T \sim \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\overline{T} = \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$T\overline{T} = \overline{T}T = ad - bc = \det(T)$$

So $\overline{T} = \det(T)\, T^{-1}$, if T is invertible.

(b) $\langle S, T \rangle = \operatorname{Trace}(S\,\overline{T})$

In the basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
$= (e_1, e_2, e_3, e_4)$

$e_1 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ $e_2 = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ $e_3 = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$ $e_4 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

$M = \left(\langle e_i, e_j \rangle \right)$

$$M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

(c) $G = \{ \underset{\in \mathrm{Aut}(W)^2}{(a,b)}$ s.t. $\det(a) = \det(b) \neq 0 \}$

G acts on $V = \mathrm{End}(W)$ by

$$(a,b) * S = a S b^{-1}$$

For all $S, T \in V$,

$\langle (a,b)*S, (a,b)*T \rangle = \langle a S b^{-1}, a T b^{-1} \rangle$

$= \mathrm{Trace}(a S b^{-1} \overline{a T b^{-1}} \rangle = \mathrm{Trace}(a S b^{-1} \bar{b}^{-1} \bar{T} \bar{a})$

$= \mathrm{Trace}(a S \det(b)^{-1} \bar{T} \bar{a}) = \mathrm{Trace}(\det(a) \det(b) S \bar{T})$

$= \operatorname{Trace}(S\bar{T}) = \langle S, T \rangle$

So (a,b) operates on V as an orthogonal transformation for the bilinear form $\operatorname{Trace}(S\bar{T})$.

We get a homomorphism

$$G \longrightarrow O(V)$$

$$(a,b) \longmapsto S \mapsto aSb^{-1}.$$

Last topics

* The Jordan canonical form

Let T be a linear transformation $V \to V$. If F is algebraically closed, or $f_T(x) = (x-\lambda_1)^{e_1} \cdots (x-\lambda_t)^{e_t}$,

$$V = \bigoplus_{i=1}^{t} V_{\lambda_i}$$

where

$$V_{\lambda_i} = \ker (T - \lambda_i)^{e_i}$$

$=$ generalised eigenspace.

<u>Question</u>: How does T, or, equivalently, the nilpotent operator $N = T - \lambda$, operate on the generalised eigenspace V_λ?

<u>Equivalently</u>, let $Nil_n(F)$ be the set of nilpotent $n \times n$ matrices with coefficients in F.

The group $GL_n(F)$ operates on $Nil_n(F)$ by conjugation.

<u>Q</u> What is $\# Nil_n(F) / GL_n(F)$?

<u>Definition</u>: A Jordan subspace of V_λ is a space with a basis of the form $(N^{k-1}v, N^{k-2}v, \ldots, Nv, v)$

<u>Theorem</u>: There is a unique sequence of integers $k_1 \geq k_2 \geq \cdots \geq k_t$ with

- $k_1 + \cdots + k_t = \dim V_\lambda$
- V_λ is non-canonically isomorphic to a direct sum

$$V_\lambda = W_1 \oplus \cdots \oplus W_t, \quad \text{where } W_j$$

is a Jordan subspace of dimension k_j.

Abstract formulation Let R be a ring. A **module** over R is an abelian group M, endowed with a "scalar multiplication" $R \times M \to M$ satisfying:

- $1 \cdot m = m \quad \forall\, m \in M$
- $\lambda_1(\lambda_2 m) = (\lambda_1 \lambda_2) m \quad \forall\, \lambda_1, \lambda_2 \in R$
- $(\lambda_1 + \lambda_2) m = \lambda_1 m + \lambda_2 m$
- $0_R \cdot m = 0_M \quad \forall\, m \in M$

A module is just a "vector space over a ring". Because rings have non trivial ideals, the structure theory of modules is more complicated than for vector spaces.

Definition: A commutative ring R is a **principal ideal domain** if every ideal of R is principal.

$\forall\, I \trianglelefteq R,\ \exists\, a \in R$ s.t. $I = (a) = aR$

Examples: $\mathbb{Z}$, $F[x]$

Key remark A vector space V equipped with a $T \in \mathrm{End}_F(V)$ is "equivalent" to a module over $F[x]$

Rule $p(x) \cdot v = p(T)(v)$

<u>Cyclic modules:</u> A cyclic module over R is a module of the form R/I where $I \triangleleft R$. Equivalently M is cyclic if it is generated by a single element.

<u>Structure Theorem for finitely generated modules over a PID</u>

If M is finitely generated, there is a <u>unique</u> sequence $I_1 \subseteq \cdots \subseteq I_t$ of ideals such that

$$M \cong R/I_1 \oplus \cdots \oplus R/I_t$$

Most important application

$N : V \to V \qquad N^e = 0$

V is an $F[x]/(x^e)$- module

What are the cyclic modules

over $R = F[x]/(x^e)$?

Ideals of R are of the

form $x^j \qquad 1 \leq j \leq e$.

T acts on $R/(x^j) = F[x]/(x^j)$

as multiplication by x.

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Relative to the basis

$(x^{j-1}, x^{j-2}, \dots, x, 1)$, of $W = F[x]/(x^j)$,

T acts via the <u>Jordan Matrix</u>

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \quad (j \times j \text{ matrix})$$

and is a Jordan subspace.

The Jordan decomposition follows from this. $\boxtimes$

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Office hours?

Thursday 10~12 AM?