April 6, Monday

This week, we will meet on Monday & Wednesday.

There is no class on Friday (Easter)

Next week, we will have our last meeting on Tuesday, 9:30-10:30.

On Tuesday, we will go over previous material by asking a set of questions.

You are encouraged to mail in requests and suggestions!

Assignment 10 is due this Wednesday.

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Last week, we stated and proved various forms of the spectral theorem.

Theorem 1: If $T : V \to V$, where V is a finite dimensional inner product space, is self-adjoint, then T is diagonalisable, relative to an orthonormal basis.

Theorem 2: If $F = \mathbb{C}$, and T is normal, then T is diagonalisable relative to an orthonormal basis.

Remarks

1. T is normal if it commutes with its adjoint.

2. When $F = \mathbb{C}$, one can always write a linear transformation T as
$T = T^{+} + iT^{-}$, with $T = (T + T^{*})/2$ and $T^{-} = (T - T^{*})/2i$ self-adjoint.
T is normal iff T^{+} and T^{-} commute.

Corollary If T is unitary, then it admits an orthonormal basis of eigenvectors.

We will now describible several applications of the spectral theorem.

$\boxed{\text{Application \#1:}}$ every positive-semi-definite operator which is self adjoint has a unique positive semi-definite square root.

$\boxed{\text{Application \#2}}$: Polar decomposition

<u>Theorem:</u> Any invertible linear transformation $T: V \to V$ can be factored <u>uniquely</u> as $T = UP$, where U is orthogonal/unitary ($F = \mathbb{R}/\mathbb{C}$) and P is positive semidefinite, (and self adjoint).

(5)

<u>Proof</u>: Let us first examine <u>uniqueness.</u> Assume

$$T = UP = U'P'$$

Consider the positive semi-definite operator $T^*T = P^*U^*UP = P^*P = P^2$.

(It is positive beacause $\langle T^*Tv, v\rangle = \langle Tv, Tv\rangle \geq 0$.)

By application # 1, P and P' are the unique semi-positive square roots of T^*T, hence $P = P'$

If T is invertible, so is P, and hence $U = TP^{-1} = U'$.

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Existence: Given T, let P be the unique positive semidefinite operator satisfying $P^2 = T^*T$. Then $U = TP^{-1}$ satisfies

$$U^*U = (P^{-1})^* T^* T P^{-1}$$
$$= P^{-1} P^2 P^{-1} = 1$$

So U is unitary / orthogonal $\boxtimes$

Application #3: Fourier analysis

This is a vast part of analysis and one of the historical motivations for the spectral theorem.

Algebraic Set-Up

Let G be a finite abelian group.

$V = L^2(G) =$ space of $\mathbb{C}$-valued functions on G, with the inner product

$$\langle f_1, f_2 \rangle = \sum_{g \in G} f_1(g)\, \overline{f_2(g)}$$

The space $L^2(G)$ is equipped with a family of "translation operators"

$$T_g(f)(x) = f(g^{-1}x)$$

Key properties:

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(1) $T_g^* = T_{g^{-1}}$

Proof $\langle T_g f_1, f_2 \rangle = \sum_{x \in G} T_g f_1(x) \overline{f_2(x)}$

$= \sum_{x \in G} f_1(g^{-1}x) \overline{f_2(x)} = \sum_{y \in G} f_1(y) \overline{f_2(gy)}$

$(y = g^{-1}x)$

$= \langle f_1, T_{g^{-1}} f_2 \rangle$ $\boxed{\text{QED}}$

(2) The collection $\{T_g\}_{g \in G}$ are a commuting family of linear transformations of $L^2(G)$. $T_{g_1} \circ T_{g_2} = T_{g_1 g_2}$

(1) + (2) $\Rightarrow$ $\{T_g\}_{g \in G}$ are normal operators.

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Spectral theorem for commuting families of normal operators $\Rightarrow$ there is an eigenbasis of $L^2(G)$.

If $\phi \in L^2(G)$ is a simultaneous eigenvector, then

$$T_g \phi = \lambda_g \phi \qquad \forall g \in G$$

The function $g \mapsto \lambda_g$ is multiplicative:

namely, $\lambda_{g_1 g_2} = \lambda_{g_1} \lambda_{g_2}$

$$T_{g_1 g_2} \phi = \lambda_{g_1 g_2} \phi$$
$$= T_{g_1} \circ T_{g_2} \phi \qquad \|$$
$$= T_{g_1} (\lambda_{g_2} \phi)$$
$$= \lambda_{g_2} T_{g_1}(\phi) = \lambda_{g_2} \lambda_{g_1} \phi$$

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Lemma: If ϕ is a simultaneous eigenvector for $\{T_g\}$, then $\phi(1)\neq 0$.

Proof $T_g\phi = \lambda_g \phi$

$\Rightarrow T_g\phi(1) = \lambda_g \phi(1)$

$\Rightarrow \phi(g^{-1}) = \lambda_g \phi(1), \quad \forall g \in G$

If $\phi(1) = 0$, we'd have $\phi(g)=0 \ \forall g \in G$
$\Rightarrow \phi = 0$. $\square$

Hence we can normalise ϕ so that $\phi(1) = 1$.

We then have $\phi(g^{-1}) = \lambda_g$

$\therefore \phi(g_1 g_2) = \phi(g_1)\phi(g_2) \quad \forall g_1, g_2 \in G$.

<u>Definition:</u> A homomorphism $\phi: G \to \mathbb{C}^\times$ is called a <u>character</u> of the abelian group G.

<u>Corollary:</u> The characters of G form an orthogonal system of eigenvectors for $\{T_g\}_{g \in G}$.

What is the length of ϕ?

$$\langle \phi, \phi \rangle = \sum_{g \in G} \phi(g) \overline{\phi(g)} = \sum_{g \in G} |\phi(g)|^2$$

$$= \sum_{g \in G} 1 = \# G.$$

<u>Corollary:</u> Let $\hat{G} :=$ the set of characters of G. Then

$$\# \hat{G} = \# G = \dim_{\mathbb{C}} L^2(G), = N$$

and $\left(\frac{1}{\sqrt{N}} \phi_1, \ldots, \frac{1}{\sqrt{N}} \phi_N \right)$

form an orthonormal basis of $L^2(G)$.

<u>Fourier expansion:</u> If $f \in L^2(G)$ is any $\mathbb{C}$-valued function on G, we may write:

$$f = \sum_{\phi \in \hat{G}} \lambda_\phi \cdot \phi$$

<u>Notation:</u> $\lambda_\phi = \hat{f}(\phi)$. $f = \sum_{\phi \in \hat{G}} \hat{f}(\phi) \phi$

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Definition: The function

$\hat{f} : \hat{G} \longrightarrow \mathbb{C}$ is called the

fourier transform of f.

Defining equation:

$$f = \sum_{\phi \in \hat{G}} \hat{f}(\phi)\, \phi$$

The fourier transform is itself a

linear map $f \mapsto \hat{f}$, from $L^2(G)$

to $L^2(\hat{G})$. We equip the latter

with the inner product $\langle f, h \rangle_{\hat{G}} = N \sum_{\phi \in \hat{G}} f(\phi) \overline{h(\phi)}$

Theorem: The Fourier transform $f \mapsto \hat{f}$ defines an isomorphism of inner product spaces from $L^2(G)$ to $L^2(\hat{G})$.

Proof For all $f_1, f_2 \in L^2(G)$,

$$\langle f_1, f_2 \rangle_G = \left\langle \sum_{\phi \in \hat{G}} \hat{f}_1(\phi)\,\phi,\ \sum_{\eta \in \hat{G}} \hat{f}_2(\eta)\cdot\eta \right\rangle$$

$$= \sum_{(\phi,\eta) \in \hat{G} \times \hat{G}} \hat{f}_1(\phi)\,\overline{\hat{f}_2(\eta)}\,\langle \phi, \eta \rangle$$

$$= \left(\sum_{\phi \in \hat{G}} \hat{f}_1(\phi)\,\overline{\hat{f}_2(\phi)} \right) \cdot N$$

$$= \langle \hat{f}_1, \hat{f}_2 \rangle_{\hat{G}} \qquad \boxed{X}$$

<u>Remark</u> One can, more generally, be interested in settings where G is an infinite (locally compact) group.

- $G = \mathbb{R}/\mathbb{Z}$,

  $\hat{G} = \{ e^{2\pi i n x} \mid n \in \mathbb{Z} \} = \mathbb{Z}$

<u>Fourier series</u>: $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n z}$

- $G = \mathbb{R}$

  $\hat{G} = \{ e^{2\pi i \lambda x} \mid \lambda \in \mathbb{R} \} \simeq \mathbb{R}$ ⊠

  (Fourier-Transform)