Wednesday, April 1

On Monday, we discussed orthogonal projections and orthogonal complements.

Theorem If V is an inner product space, and W is a finite dimensional subspace of V, then
$$V = W \oplus W^{\perp}$$

Proof Let $(e_1, \dots, e_n)$ be an ON basis of W. If $v \in V$
$$v = \left(\langle v, e_1\rangle e_1 + \dots + \langle v, e_n\rangle e_n\right) + w'$$

Question: What if $\dim W = \infty$?

②

Then the statement is

- Still true?

- False?

⋮

③

<u>Counterexample:</u> $V = \ell_2(\mathbb{R})$

$=$ set of $\mathbb{R}$-valued sequences $(a_1, a_2, \dots)$ such that $\sum_{i=1}^{N} |a_i|^2$ converges.

V contains elements like

$$e_1 = (1, 0, 0, \dots)$$
$$e_2 = (0, 1, 0, \dots)$$
$$e_3 = (0, 0, 1, \dots)$$
$$\vdots$$
$$e_n = (0, \dots, 0, 1, 0, \dots)$$
$$\vdots$$

Let $W = \operatorname{span}(e_1, e_2, \dots, e_n, \dots)$

$W \neq V$

Why not?

4

The space W is the space of finite sequences, of the form

$$(a_1, a_2, \dots, a_N, 0, 0, 0, \dots)$$

Hence, for example,

$$v = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right) \notin W$$

$\left(v \in V \text{ because } 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} < \infty. \right)$

$$W^\perp = \{ v \in V \mid \langle v, e_i \rangle = 0 \}$$

But: if $v = (a_1, \dots, a_n, \dots)$, $\langle v, e_i \rangle = a_i$

So $W^\perp = \{ (a_1, \dots, a_n, \dots) \text{ st } a_j = 0 \ \forall j \}$
$= 0.$

(5)

Problem: The vector space W is not complete for the metric induced by $\langle \, , \rangle$. E.g., $(1, 0, \ldots)$, $(1, \frac{1}{2}, 0, \ldots)$, $(1, \frac{1}{2}, \frac{1}{3}, 0, \ldots)$, $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$, ... is a Cauchy sequence in W which does not have a limit in W.

Definition A Hilbert space is an inner product space that is complete for $\| \ \|$.

Remark: An infinite dimensional Hilbert space is always of uncountable dimension!

6

Back to $\dim V < \infty$.

Recall that on Monday we defined the adjoint of $T: V \to V$ as the unique operator $T^*: V \to V$ satisfying

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

Example of the adjoint:

$V = L^2(\mathbb{R})$ = space of square integrable functions on $\mathbb{R}$

$= \{ f: \mathbb{R} \to \mathbb{R}$ s.t. $\int_{-\infty}^{\infty} f(x)^2 \, dx < \infty \}$

and $\int_{-\infty}^{\infty} f^{(n)}(x)^2 < \infty$

7

$T : V \to V$

$T(f) = \frac{d}{dx} f(x)$

$T^* = ?$

What is the adjoint of T?

8

$$\langle Tf, g \rangle = \int_{-\infty}^{\infty} f'(x)\, g(x)\, dx$$

$$= -\int_{-\infty}^{\infty} f(x)\, g'(x)\, dx$$

(Integration by parts!)

Hence $T^* = -T$.

We want to single out those linear transformations that enjoy a special relation with their adjoint.

<u>Definition</u>: A linear transformation $T: V \to V$ is <u>self adjoint</u> if $T^* = T$, i.e., if

$$\langle Tv, w \rangle = \langle v, Tw \rangle \ \forall v, w \in V. \ \boxtimes$$

<u>Remark</u> A matrix $M \in M_n(\mathbb{R})$ (resp. in $M_n(\mathbb{C})$) is said to be symmetric (resp. hermitian) if $M^t = M$ (resp $M^t = \overline{M}$).

<u>Lemma</u> T is self-adjoint if and only if its matrix <u>in</u> <u>an</u>

orthonormal basis is symmetric ($F = \mathbb{R}$) or hermitian ($F = \mathbb{C}$).

Examples:

1. Let $V = F^n$ with the standard dot product. Then multiplication by a hermitian symmetric matrix is self-adjoint. ($M = (m_{ij})$, $m_{ij} = \overline{m_{ji}}$)

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

Char poly of T $= x^2 - 2ax + (a^2 - b^2)$
$= (x - (a-b))(x - (a+b))$

So T is diagonalisable:
$f_T(x)$ has distinct roots in F, if $b \neq 0$
$T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ if $b = 0$.

2. If $W \subseteq V$, then $P_W : V \to V$ is self-adjoint.

Proof Let $v = v_1 + v_2$, $v_1 \in W$, $v_2 \in W^{\perp}$
$u = u_1 + u_2$ $u_1 \in W$, $u_2 \in W^{\perp}$

$\langle P_W(v), u \rangle = \langle v_1, u_1 + u_2 \rangle = \langle v_1, u_1 \rangle$

$\langle v, p(u) \rangle = \langle v_1 + v_2, u_1 \rangle = \langle v_1, u_1 \rangle$

3. $V = L^2(\mathbb{C})$ $\quad u(x) \in L^2(\mathbb{C})$

$T(f) = u(x) \cdot f(x)$ is linear

$$\langle Tf, g \rangle = \int_{-\infty}^{\infty} u(x) f(x) \cdot \overline{g(x)}\, dx$$

$$= \int_{-\infty}^{\infty} f(x)\, \overline{\overline{u(x)}\, g(x)}\, dx = \langle f, T^* g \rangle$$

where $T^*(g) = \overline{u(x)}\, g(x)$

Proposition: If T is self-adjoint, then $\langle Tv, v\rangle \in \mathbb{R}$. In particular, any eigenvalue of T is real.

Proof $\langle Tv, v\rangle = \langle v, Tv\rangle = \overline{\langle Tv, v\rangle}$ $\boxtimes$

Remark: This Proposition is trivial when $F = \mathbb{R}$.

Proposition If $F = \mathbb{C}$, and $\langle Tv, v\rangle \in \mathbb{R}$ for all v, then T is self adjoint.

Proof If $\langle Tv, v\rangle \in \mathbb{R}$, then

$$\langle Tv, v\rangle = \langle v, T^* v\rangle = \langle T^* v, v\rangle$$

$$\therefore \langle (T - T^*) v, v\rangle = 0 \quad \forall v.$$

The proposition will now follow from:

13

Lemma Suppose that $\langle T(v), v\rangle = 0$ $\forall\ v$. Then $T = 0$.

Proof $\langle T(u+v), u+v\rangle = \langle T(u), v\rangle + \langle T(v), u\rangle$

$0 = \langle T(u+iv), u+iv\rangle$
$= -i\langle T(u), v\rangle + i\langle T(v), u\rangle$

$$\Rightarrow \begin{cases} \langle T(u), v\rangle + \langle T(v), u\rangle = 0 \\ \langle T(u), v\rangle - \langle T(v), u\rangle = 0 \end{cases}$$

$\Rightarrow \langle T(u), v\rangle = 0 \quad \forall\ u, v$

$\Rightarrow T(u) = 0 \quad \forall\ u.$ $\square$

Remark: The hypothesis $F = \mathbb{C}$ is crucial for this lemma to hold.

Why?

14

For example, the transformation

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (y,-x)$ (rotation by a $90°$ angle) satisfies

$\langle T(v), v \rangle = 0 \quad \forall v$

but $T \neq 0$.

Orthogonal and Unitary Transformations

Theorem: The following are equivalent:

1. $TT^* = T^*T = id$
2. T sends one ON basis to another
3. T carries any ON basis to another
4. $\langle T(v), T(w) \rangle = \langle v, w \rangle \quad \forall\, v, w \in V$
5. $\|T(v)\| = \|v\|$

$1 \Rightarrow 4$ $\quad \langle T(v), T(w) \rangle = \langle v, T^*Tw \rangle = \langle v, w \rangle$

$4 \Rightarrow 5$ $\quad$ set $v = w$

$5 \Rightarrow 1$ $\quad \langle Tv, Tv \rangle = \langle v, v \rangle \quad \forall\, v \in V$

$$\Rightarrow \langle v, (id - T^*T)v \rangle = 0 \quad \forall\, v \in V$$

$$\Rightarrow (id - T^*T) = 0 .$$

Rest is left as an exercise.

<u>Definition:</u> A transformation satisfying these equivalent properties is called <u>orthogonal</u> (if $F = \mathbb{R}$) or <u>unitary</u> (if $F = \mathbb{C}$).

T is orthogonal / unitary if its matrix relative to an orthonormal basis has columns that form an orthonormal system in F^n.

The set of orthogonal / unitary matrices forms a group

$O(V)$ = group of orthogonal transformations on V $(F = \mathbb{R})$

$\mathcal{U}(V)$ = group of unitary transf. $(F = \mathbb{C})$.

$SO(V) :=$ "special orthogonal group"
$=$ orthogonal ~~tr~~ transformations of determinant one.

$SU(V) :=$ "special unitary group."

Examples ① $V = \mathbb{R}^2$

$$SO(V) = \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \right\}$$

$$= \mathbb{R}/2\pi\mathbb{Z} = \{e^{i\theta}\}$$

$$= S^1 \quad \text{(circle)}$$

② $V = \mathbb{C}^2$

$$SU(V) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \right\} \quad |a|^2 + |b|^2 = 1$$

$= S^3$ (three-dimensional sphere in $\mathbb{R}^4$.

$\boxtimes$

18

Groups like $SO(V)$, $SU(V)$, $O(V)$, $U(V)$ are a rich source of examples of real Lie Groups. $\boxtimes$

.

We are now ready to state the spectral theorem for self-adjoint operators.

<u>Theorem</u> Let $T : V \to V$ be a self-adjoint operator on a finite-dimensional inner product space. Then V admits an orthonormal basis of eigenvectors for T. In particular, T is diagonalisable.

20

Spectral theorem, concrete formulation

$\boxed{F = \mathbb{R}}$. If M is a symmetric matrix, it can be written as $M = PDP^{-1}$, where P is orthogonal and D is diagonal.

$\boxed{F = \mathbb{C}}$ If M is a hermitian matrix, then it can be written as $M = UDU^{-1}$, where U is unitary and D is diagonal.

On Friday, we will prove this theorem.