ASSIGNMENT 6: SOLUTIONS

Question 1.

Solution. Assume that $\Re(s) > 1$. Then using the hint and the Euler product expansion of $\zeta(s)$, we get that

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\left(\prod_{p \text{ prime}}(1-p^{-s})^{-1}\right) = \sum_{p \text{ prime}}\frac{d}{ds}\log(1-p^{-s}) = \sum_{p \text{ prime}}\frac{\log(p)p^{-s}}{1-p^{-s}}.$$

Re-writing this using the geometric series exapansion of $\frac{1}{1-p^{-s}}$, we get that

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \text{ prime}}\log(p)p^{-s}\sum_{m=0}^{\infty}(p^{-s})^m = \sum_{p \text{ prime}}\sum_{m=1}^{\infty}\frac{\Lambda(p^m)}{(p^m)^s} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

where the last equality is due to the fact that $\Lambda(n) = 0$ for all n that are not prime powers. $\square$

Question 2.

Solution. Again take s such that $\Re(s) > 1$. In order to "kill two birds with one stone", let $\chi : \mathbb{N} \to \mathbb{C}^\times$ be any multiplicative character (*i.e.* . $\chi(nm) = \chi(n)\chi(m)$ and $|\chi(n)| = 1$ for all $n \in \mathbb{N}$). Then since $|\chi(p)p^{-s}| = |p^{-s}| < 1$, and the series $\sum_{m=0}^{\infty}\chi(p)^m p^{-ms}$ converges absolutely for all primes. This means that we can proceed as in the case of proving the Euler product expansion for the zeta function:

$$\begin{aligned}\prod_{p \text{ prime}<x}\left(1-\chi(p)p^{-s}\right)^{-1} &= \prod_{p \text{ prime}<x}\sum_{m=0}^{\infty}(\chi(p)p^{-s})^m = \prod_{p \text{ prime}<x}\sum_{m=0}^{\infty}\chi(p^m)(p^m)^{-s}\\ &= \sum_{n \text{ is } x\text{-smooth}}\chi(n)n^{-s}.\end{aligned}$$

Taking the limit as x goes to infinity gives that

$$\prod_{p \text{ prime}}\left(1-\chi(p)p^{-s}\right)^{-1} = \sum_{n=1}^{\infty}\chi(n)n^{-s}.$$

Observing that the Legendre symbol is a multiplicative character completes the proof. $\square$

Question 3.

Solution. Write $S_k = \frac{1}{(5k+1)^s} - \frac{1}{(5k+2)^s} - \frac{1}{(5k+3)^s} + \frac{1}{(5k+4)^s}$. By the mean value theorem applied to $f(x) = 1/x^s$, there is a $c_k \in [5k+1, 5k+2]$ such that

$$\frac{1}{(5k+1)^s} - \frac{1}{(5k+2)^s} = -f'(c_k) = \frac{s}{c_k^{s+1}}$$

and likewise, a $d_k \in [5k+3, 5k+4]$ such that

$$\frac{1}{(5k+3)^s} - \frac{1}{(5k+4)^s} = \frac{s}{{d_k}^{s+1}}.$$

Therefore, $S_k = s\left(\frac{1}{c_k^{s+1}} - \frac{1}{d_k^{s+1}}\right)$ and since $1 < c_k < d_k$ and $c_k \geq 5k+1$ for all $k \geq 0$,

$$0 < S_k \leq \frac{s}{c_k^{s+1}} \leq \frac{s}{(5k+1)^{s+1}}.$$

We now apply our results to $L(s)$ when $q = 5$ as in the previous question. Since $L(s)$ is absolutely convergent for $s > 1$, it follows that

$$L(s) = \sum_{k=0}^{\infty}S_k.$$

By what we have just shown about S_k, this means that for all $1 < s < 2$,

$$0 < S_0 = 1 - \frac{1}{2^s} - \frac{1}{3^s} + \frac{1}{4^s} \leq L(s) \leq \sum_{k=0}^{\infty}\frac{s}{(5k+1)^{s+1}} \leq s\sum_{n=0}^{\infty}\frac{1}{n^2} = s\zeta(2).$$

Therefore, the limit as $s \to 1^+$ of $L(s)$ converges to a non-zero number. $\square$

Question 4.

Solution. By our work in question 2, we only need to show that χ and $\bar{\chi}$ are multiplicative characters in order to prove the Euler product expansions for $L(\chi, s)$ and $L(\bar{\chi}, s)$. This is easy to check.

Now, for $s \in \mathbb{R}$, we want to show that $\lim_{s\to 1^+} L(\chi, s)$ converges and is non-vanishing (similarly for $L(\bar{\chi}, s)$). Since the series $L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ converges absolutely for $s > 1$, we can rearrange and find that

$$\Re(L(\chi, s)) = \sum_{k=0}^{\infty} \frac{1}{(5k+1)^s} - \frac{1}{(5k+4)^s}$$

and

$$\Im(L(\chi, s)) = \sum_{k=0}^{\infty} \frac{1}{(5k+2)^s} - \frac{1}{(5k+3)^s}.$$

Again, by the mean value theorem applied to the function $1/x^s$, we get that for every k, there exists $c_k \in [5k+1, 5k+4]$ and $d_k \in [5k+2, 5k+3]$ such that

$$\frac{1}{(5k+1)^s} - \frac{1}{(5k+4)^s} = \frac{s}{c_k^{s+1}}, \qquad \frac{1}{(5k+2)^s} - \frac{1}{(5k+3)^s} = \frac{s}{d_k^{s+1}}.$$

Therefore, for $1 < s < 2$

$$0 \neq \frac{s}{4^{s+1}} \leq \sum_{k=0}^{\infty} \frac{s}{(5k+4)^{s+1}} \leq \Re(L(\chi, s)) = \sum_{k=0}^{\infty} \frac{s}{c_k^{s+1}} \leq \sum_{k=0}^{\infty} \frac{s}{(5k+1)^{s+1}} \leq s \sum_{n=0}^{\infty} \frac{1}{n^2} = s\zeta(2)$$

and similarly,

$$0 \neq \frac{s}{3^{s+1}} \leq \Im(L(\chi, s)) \leq s\zeta(2)$$

and as $s \to 1^+$, $L(\chi, s)$ converges to a non-zero complex number. The argument for $L(\bar{\chi}, s)$ is similar. □

Question 5.

Solution. Let $\rho : \mathbb{N} \to \mathbb{C}^{\times}$ denote any of $\left(\frac{\cdot}{5}\right), \chi, \bar{\chi}$ or the trivial character and let $L(\rho, s) = \sum_{n=1}^{\infty} \rho(n) n^s$ (note that if ρ is the trivial character, $L(\rho, s) = \zeta(s)$. Then it follows from the Euler product expansions of these L-functions (which we proved in Questions 2 and 4) that

$$\log L(\rho, s) = \sum_{p \text{ prime}} \sum_{n=1}^{\infty} \frac{\rho(n)}{np^{ns}} = \sum_{p \text{ prime}} \frac{1}{p^s} + \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{\rho(n)}{np^{ns}}.$$

Now,

$$\begin{aligned}\left| \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{\rho(n)}{np^{ns}} \right| &\leq \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{|\rho(n)|}{np^{ns}} \leq \sum_{p \text{ prime}} \sum_{n=2}^{\infty} \frac{1}{p^{ns}} \\ &\leq \sum_{p \text{ prime}} \frac{1}{p^{2s}} \left(\sum_{n=0}^{\infty} \frac{1}{p^{ns}} \right) \leq \sum_{p \text{ prime}} \frac{1}{p^{2s}} \left(\frac{1}{1-p^{-s}} \right) \\ &\leq 2 \sum_{p \text{ prime}} \frac{1}{p^{2s}} \leq 2\zeta(2s)\end{aligned}$$

since $\left(\frac{1}{1-p^{-s}}\right) \leq 2$ for all primes and $s \geq 1$.

Consider the following for p prime:

$$1 + \left(\frac{p}{5}\right) + \chi(p) + \bar{\chi}(p) = \begin{cases} 1 & p \equiv 0 \pmod 5 \\ 4 & p \equiv 1 \pmod 5 \\ 0 & p \equiv 2 \pmod 5 \\ 0 & p \equiv 3 \pmod 5 \\ 0 & p \equiv 4 \pmod 5. \end{cases}$$

Thus,

$$\sum_{p \text{ prime}} \frac{1 + \left(\frac{p}{5}\right) + \chi(p) + \bar{\chi}(p)}{p^s} = \frac{1}{5^s} + 4 \sum_{p \equiv 1(5)} \frac{1}{p^s}.$$

It follows that

$$\begin{aligned}
&\left|\log \zeta(s) + \log L(s) + \log L(\chi, s) + \log L(\bar{\chi}, s) - 4 \sum_{p\equiv 1 \pmod 5} \frac{1}{p^s}\right| \\
&\leq \left|\sum_{p \text{ prime}} \frac{1 + \left(\frac{p}{5}\right) + \chi(p) + \bar{\chi}(p)}{p^s} - 4 \sum_{p\equiv 1(5)} \frac{1}{p^s}\right| + 8\zeta(2s) \\
&= \frac{1}{5^s} + 8\zeta(2s).
\end{aligned}$$

Therefore, as we take the limit as $s \to 1$, we get that the absolute value of the sum is $1/5 + 8\zeta(2)$. From previous work in this assignment, the limit as $s \in \mathbb{R}$ approaches 1 from the right is finite for $\log L(\rho, s)$ where ρ is $\chi, \bar{\chi}, \left(\frac{\cdot}{5}\right)$; however, we know that the limit as $s \to 1$ of $\zeta(s)$ goes to infinity and so does its logarithm. Since the absolute value of the above sum converges, it must be that $\sum_{p\equiv 1(5)} \frac{1}{p^s}$ diverges and there infinitely many primes in this equivalence class. The proof is essentially the same for the other equivalence classes modulo 5. □

Question 6.

Solution. The main observation here should be that the second series converges to π faster than the first one does. □

Question 7.

Solution. The continued fraction expansion of $\sqrt{7}$ is $[2; \overline{1, 1, 1, 4}]$. The first few convergents p_n/q_n are

$$2, 3, \frac{5}{2}, \frac{8}{3}, \frac{37}{14}, \frac{45}{17}$$

and in order to find a minimal (with respect to the size of x) solution to the Pell equation, we can successively test (p_n, q_n) for $n = 1, 2, 3, \ldots$ until we get a solution. The first such solution is $(8, 3)$. □

Question 8.

Solution. Fix $m \in \mathbb{Z}$ and write $\mathrm{ord}(\alpha - m) = b$. Then $\alpha \equiv m \pmod{p^b}$ and

$$f(m) \equiv f(\alpha) \equiv 0 \pmod{p^b}$$

so $\mathrm{ord}(\alpha - m) \leq \mathrm{ord}(f(m))$.

Let a denote the coefficient of f that is largest in abolute value. Then for any $m \in \mathbb{Z}$,

$$\log|f(m)| \leq \log|akm^k| = \log|a|k + k\log|m|.$$

Therefore,

$$\mathrm{ord}(\alpha - m) \leq \mathrm{ord}(f(m)) \leq \frac{\log(|f(m)|)}{\log(p)} < \frac{\log(|a|k) + k\log(|m|)}{\log(p)} = \frac{\log(|a|k)}{\log(p)} + k\frac{\log(|m|)}{\log(p)}$$

and since a, k, p do not depend on m, we are done. □

Question 9.

Solution. Let $\alpha = \sum_{m=0}^{\infty} p^{m!} \in \mathbb{Z}_p$ and let $\alpha_n = \sum_{m=0}^{n} p^{m!}$, and suppose that α satisfies an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ where $k = \deg(f) \geq 2$. Observe that for all $n \in \mathbb{N}$ we have that

$$\mathrm{ord}(\alpha - \alpha_n) = \mathrm{ord}(\sum_{m=n+1}^{\infty} p^{m!}) \geq (n+1)!.$$

By Question 8, there exists a constant C such that

$$(n+1)! < C + k\frac{\log(|\alpha_n|)}{\log(p)} \leq C + k\frac{\log n + n!\log(p)}{\log(p)}.$$

In other words,

$$n \leq \frac{C}{n!} + \frac{k\log n}{n!\log(p)} + k - 1$$

for all n. However, for $n >> 0$ the LHS is bigger than the RHS. Therefore, α cannot satisfy any such $f(x) \in \mathbb{Z}[x]$. □