

Week 3, lecture 8: More on Modular Forms

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Recall

We are studying modular forms with respect to a congruence group Γ , i.e., a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ such that $\Gamma \supseteq \Gamma(N)$ for some integer N .

Recall our notations:

$$\begin{array}{ll} \text{modular forms} & M_k(\Gamma) \\ & \cup \\ \text{cusp forms} & S_k(\Gamma). \end{array}$$

More on Modular Forms

CLAIM 1. (to be justified) *Cusp forms (of an appropriate type) play the same role for 2-dimensional representations of $G_{\mathbb{Q}}$ as do Dirichlet characters for 1-dimensional representations of $G_{\mathbb{Q}}$.*

From now on, we assume

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

This implies that, for any $f \in S_k(\Gamma)$,

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.$$

We can then define the L -function attached to f by

$$L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}.$$

QUESTION 1. *Where does this sum converge?*

In order to answer this question, we will introduce some basic results about Möbius transformations and modular forms. In what follows, we assume $f \in S_k(\Gamma)$.

LEMMA 1. $y\left(\frac{az+b}{cz+d}\right) = \frac{y(z)}{(cz+d)(\overline{c\bar{z}+d})}$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

Proof.

$$y \left(\frac{az+b}{cz+d} \right) = \frac{1}{2i} \left[\frac{az+b}{cz+d} - \overline{\left(\frac{az+b}{cz+d} \right)} \right] = \frac{1}{2i} \left[\frac{(ad-bc)(z-\bar{z})}{(cz+d)(c\bar{z}+d)} \right] = \frac{y(z)}{(cz+d)(c\bar{z}+d)}.$$

□

COROLLARY 1. $y^k f(z) \overline{f(z)}$ is invariant under Γ . Hence, so is $y^{k/2} |f(z)|$.

LEMMA 2. There exists a constant C_f (depending on f) such that

$$|f(x+iy)| < C_f y^{-k/2}$$

for all $z = x+iy \in \mathcal{H}$.

Proof. Since f is a cusp form, the values of $y^{k/2} |f(z)|$ are uniformly bounded in a neighborhood of all the cusps. Hence, there exists C_f such that

$$y^{k/2} |f(z)| < C_f.$$

□

PROPOSITION 1. There exists a constant C'_f such that

$$|a_n| < C'_f n^{k/2}.$$

Proof. Let $z = x+iy \in \mathcal{H}$. Then

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n(x+iy)} = \sum_{n=1}^{\infty} a_n e^{-2\pi n y} e^{2\pi i n x}.$$

By Fourier inversion,

$$a_n e^{-2\pi n y} = \int_0^1 f(x+iy) e^{-2\pi i n x} dx.$$

So

$$|a_n| e^{-2\pi n y} \leq \int_0^1 |f(x+iy)| dx \leq C_f y^{-k/2}.$$

And this implies that

$$|a_n| \leq C_f y^{-k/2} e^{2\pi n y}.$$

Notice that for each $y > 0$ we have a bound for $|a_n|$. Choosing $y = \frac{k}{4\pi n}$ will minimize the function on the right hand side and we obtain

$$|a_n| \leq C'_f n^{k/2},$$

where $C'_f = C_f k^{-k/2} (4\pi)^{k/2} e^{k/2}$.

□

COROLLARY 2. $L(f, s)$ converges absolutely for $\operatorname{Re}(s) > 1 + k/2$.

DEFINITION 1. The Mellin transform of f is

$$M(f)(s) = \int_0^\infty f(it)t^s \frac{dt}{t}.$$

PROPOSITION 2. $M(f)(s) = (2\pi)^{-s}\Gamma(s)L(f, s)$.

Proof. Just like in the proof of proposition 1 in lectures 5 and 6, we have that

$$M(f)(s) = \int_0^\infty f(it)t^s \frac{dt}{t} = \int_0^\infty \sum_{n=1}^\infty a_n e^{-2\pi n t} t^s \frac{dt}{t} = \sum_{n=1}^\infty \int_0^\infty a_n e^{-2\pi n t} t^s \frac{dt}{t}.$$

Applying the change of variables $u = 2\pi n t$ yields

$$M(f)(s) = \sum_{n=1}^\infty \int_0^\infty a_n e^{-u} u^s (2\pi n)^{-s} \frac{du}{u}.$$

Finally, we can rearrange terms and obtain

$$M(f)(s) = (2\pi)^{-s} \left(\sum_{n=1}^\infty a_n n^{-s} \right) \int_0^\infty e^{-u} u^s \frac{du}{u}.$$

□

REMARK 1. If $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma$, then $M_k(\Gamma) = \{0\}$ when k is odd.

THEOREM 2. Assume $f \in S_k(\operatorname{SL}_2(\mathbb{Z}))$. Then $L(f, s)$ extends to an analytic function on $\mathbb{C}$ and, if

$$\Lambda(f, s) := (2\pi)^{-s}\Gamma(s)L(f, s),$$

then

$$\Lambda(f, s) = (-1)^{k/2} \Lambda(f, k - s).$$

Proof.

$$\Lambda(f, s) = M(f)(s) = \int_0^\infty f(it)t^s \frac{dt}{t} = \int_0^1 f(it)t^s \frac{dt}{t} + \int_1^\infty f(it)t^s \frac{dt}{t}.$$

Applying the change of variables $t \leftrightarrow 1/t$ for the first integral gives us

$$\int_1^\infty f(i/t)t^{-s} \frac{dt}{t}.$$

Now, since

$$f(i/t) = f(-1/it) = (it)^k f(it),$$

the first integral becomes

$$i^k \int_1^\infty f(it)t^{k-s} \frac{dt}{t} = (-1)^{k/2} \int_1^\infty f(it)t^{k-s} \frac{dt}{t}.$$

Hence

$$\Lambda(f, s) = (-1)^{k/2} \int_1^\infty f(it)t^{k-s} \frac{dt}{t} + \int_1^\infty f(it)t^s \frac{dt}{t},$$

which implies that $\Lambda(f, s)$ has an analytic continuation to $\mathbb{C}$. Moreover, we can also readily see that

$$\Lambda(f, s) = (-1)^{k/2} \Lambda(f, k - s).$$

□

REMARK 2. If $f \in S_k(\mathrm{SL}_2(\mathbb{Z}))$ then the sign in the functional equation for $L(f, s)$ is

$$\begin{cases} 1, & \text{if } k \equiv 0 \pmod{4} \\ -1, & \text{if } k \equiv 2 \pmod{4}. \end{cases}$$

In particular, if $k \equiv 2 \pmod{4}$, then $L(f, k/2) = 0$.

QUESTION 2. *Where are the zeros of $L(f, s)$ on $\mathrm{Re}(s) > 1 + k/2$?*

QUESTIONS 3. (i) *The proof of analytic continuation works well when $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \Gamma$. What about other Γ (e.g. $\Gamma_0(N)$, $\Gamma_1(N)$)?*

(ii) *When does $L(f, s)$ admit an Euler product factorization? And if so, what does it look like?*

The key tool for these questions is

Hecke operators