Recall from last lectures

\[ \forall n \in \mathbb{N}, \zeta(-n) \in \mathbb{Q} \]

The proof is to consider the Mellin transform:

\[ L(f, s) := \frac{1}{F(s)} \int_0^\infty f(t)t^{-s} \, dt. \]

Then one can show that

\[ L(f, -n) = (-1)^n f^{(n)}(0). \]

On the other hand, by a direct computation,

\[ \zeta(s) = \frac{1}{s-1}L(f, s) \text{ where } f(t) = \frac{t}{e^t-1}. \]

Since \( \frac{t}{e^t-1} \) has rational Taylor expansion, we conclude that \( \forall n \in \mathbb{N}, \zeta(-n) \in \mathbb{Q} \).

**Problem 1.** Understand the \( p \)-adic properties of \( \zeta(-n) \):

**Idea:** Express \( \zeta(-n) \) as a \( p \)-adic Mellin transform of \( \frac{t}{e^t-1} \).

**Rudiments of \( p \)-adic integration theory:**

**Definition 1.** A \textbf{\( p \)-adic Banach} space \( B \) is a \( \mathbb{Q}_p \)-vector space, such that there exists a \( \mathbb{Z}_p \)-submodule \( B_0 \) with the following properties:

(i) \( B_0 \to \lim_{\leftarrow} B_0/p^nB_0 \) is an isomorphism

(ii) \( \forall x \in B, \exists n \in \mathbb{Z} \text{ such that } p^n.x \in B_0 \)

We can define the valuation for any element of \( B \), by \( v_p(x) := \min\{n \in \mathbb{Z} | p^n.x \in B_0\} \).

The norm is then defined by \( ||x|| := p^{-v_p(x)} \).

**Remark 1.** The Banach space is complete relative to this norm and \( B_0 = \text{unit ball in } B \).

**Example 2.** \( B = \overline{\mathbb{Q}_p} \) is NOT a Banach space.

Indeed, \( B_0 = \mathcal{O}_{\mathbb{Z}_p} \to \lim_{\leftarrow} B_0/p^nB_0 \) is NOT an isomorphism

**Proof:** Considering elements of the form \( \sum_{i=1}^\infty b_ip^i \) with \( \deg(b_i) \to \infty \). Theses elements are in \( \lim_{\leftarrow} B_0/p^nB_0 \) but one can show that some of thoses elements can NOT have finite degree over \( \mathbb{Q}_p \).

Let’s give an example of this fact. Consider \( \alpha := \sum_{k=0}^\infty p\frac{1}{p^k}.p^k \).
By definition, \( \alpha \) belongs to \( \lim_{n} B_0 / p^n B_0 \).
Suppose that \( \alpha \) has degree \( N \) over \( \mathbb{Q}_p \). Choose \( n \) such that 
\[
\frac{2n^2 N}{2(n+1)^2} < 1 \text{ then } \sum_{k=n+1}^{\infty} p^{\frac{k}{2(n+1)^2}} p^k
\]
belongs to \( \mathbb{Q}_p(p^{\frac{2n^2}{2(n+1)^2}}, x) \) which is an extension of degree \( \leq 2n^2.N \).
So, \( v_p(\sum_{k=n+1}^{\infty} p^{\frac{k}{2(n+1)^2}} p^k).2n^2.N \in \mathbb{Z} \).
But on the other hand:
\[
v_p\left( \sum_{k=n+1}^{\infty} p^{\frac{k}{2(n+1)^2}} p^k \right).2n^2.N = v_p(p^{\frac{1}{2(n+1)^2}} p^{n+1}).2n^2.N
\]
\[
= (n + 1).2n^2.N + \frac{2n^2.N}{2(n+1)^2}
\]
is not an element of \( \mathbb{Z} \), a contradiction. \( \square \)

**Example 3.** We can complete the first example to get a \( p \)-adic Banach space:
\[
B = \mathbb{C}_p := \hat{\mathbb{Q}_p}
\]
\[
B_0 = \mathcal{O}_{\mathbb{C}_p} = \hat{\mathbb{Q}_p}
\]

**Example 4.** \( B = C(\mathbb{Z}_p, \mathbb{Q}_p) = \{ \text{continuous } \mathbb{Q}_p\text{-valued functions on } \mathbb{Z}_p \} \)
\( B_0 = C(\mathbb{Z}_p, \mathbb{Z}_p) \)
The property \((ii)\) for a Banach space, follows from the compactness of \( \mathbb{Z}_p \). \( (f \in C(\mathbb{Z}_p, \mathbb{Q}_p) \)
has bounded valuation)

**Example 5.** \( B = C(\mathbb{Z}_p, \mathbb{C}_p) \)
\( B_0 = C(\mathbb{Z}_p, \mathcal{O}_{\mathbb{C}_p}) \)

**Example 6.** Let \( I \) be an index set:
\( B = \ell_\infty(I, \mathbb{Q}_p) := \{(X_i)_{i \in I} \text{ bounded } \|X_i\| \leq C, \forall i \in I \} \)
\( B_0 = \ell_\infty(I, \mathbb{Z}_p) \)

**Example 7.** Let \( I \) be an index set:
\( B = \ell_1(I, \mathbb{Q}_p) := \{(X_i)_{i \in I}, (X_i) \text{ is } \text{ "summable" } \}
\)
\( := \{(X_i) \parallel \{i \in I | v_p(X_i) \leq C \} < \infty, \forall C > 0 \}. \)
\( B_0 = \ell_1(I, \mathbb{Z}_p) \)
In particular, if \( (X_i) \) belongs to \( \ell_1(I, \mathbb{Q}_p) \), then \( \sum_{i \in I} X_i \in \mathbb{Q}_p \) makes sense.
Definition 8. A **Banach basis** of $B$ is a family of elements $(e_i)_{i \in I}$ in $B_0 \setminus pB_0$, such that $\forall x \in B$ there exists a unique $(x_i)_{i \in I} \in \ell_1(I, \mathbb{Q}_p)$ such that:

$$x = \sum_{i \in I} x_i e_i$$

Theorem 9. $(e_i)_{i \in I}$ is a Banach basis for $B$ if and only if $(\overline{e_i})_{i \in I}$ is a $\mathbb{F}_p$-basis for $B_0/p^n B_0$

**Sketch of the proof:**

$(\Rightarrow)$ Let $\overline{x} \in B_0/p^n B_0$ and choose $x$ lifting $\overline{x}$. Then, $x = \sum_{i \in I} x_i e_i$. So, $\overline{x} = \sum_{i \in I} \overline{x_i} e_i$ and the sum is finite since $(x_i)_{i \in I}$ belongs to $\ell_1(I, \mathbb{Z}_p)$.

$(\Leftarrow)$ Let $\varphi : \ell_1(I, \mathbb{Q}_p) \to B$

$$(x_i)_{i \in I} \to \sum_{i \in I} x_i e_i$$

$\varphi$ is injective:

By multiplying $(x_i)_{i \in I}$ by $v_p((x_i)_{i \in I})$ and by reducting modulo $p$ gives:

$\varphi(x) = 0 \Leftrightarrow v_p(x) = \infty$.

$\varphi$ is surjective:

Let $x \in B$. Without a loss of generality, we can assume that $x \in B_0$. By hypothesis, $\overline{x} = \sum_{i \in I} \overline{x_i} e_i$. Take $x^{(0)} := \sum_{i,j} x_{1,i} e_i$ (Recall that the sum is finite).

$x - x^{(0)} \in p B_0$.

By recurrence, $x = x^{(0)} + p x^{(1)} + \ldots + p^n x^{(n)} + \ldots$ with $x^{(i)}$ linear combination of $e_i$ with coefficients in $\mathbb{Z}_p$.

\[\square\]

Corollary 1. Every Banach space has a Banach basis

**Proof:**

Consider $B_0/pB_0$. Let $\overline{e_i} \in I$ be a basis for $B_0/pB_0$. Then, just take $e_i$ lift for $\overline{e_i}$. \[\square\]

Sometimes, we can easily find an explicit expression of the basis:

**Example 10.** $B = \ell_1(I, \mathbb{Q}_p)$.

A basis is given by $(e_i)_{i \in I}$ such that $e_i = \delta_i$ ($\delta_i(i) = 1$, $\delta_i(j) = 0$ if $i \neq j$).

But, sometimes, the explicit expression is not that clear.

**Example 11.** $B = \ell_\infty(I, \mathbb{Q}_p)$

$B_0/pB_0 = functions(I, \mathbb{F}_p)$

We know that there exists a basis. Its existence relies on the full strenght of the axiom of choice but this basis is not countable.
**Lemma 1.** The functions \( (\mathbb{a}(n) := \frac{x(x-1)...(x-n+1)}{n!} \) belong to \( B_0 \setminus pB_0 \)

**Proof:** (i) \( f_n(x) := (\mathbb{a}(n) \) sends \( \mathbb{Z} \) to \( \mathbb{Z} \). Since \( \mathbb{Z} \) is dense in \( \mathbb{Z}_p \), it maps \( \mathbb{Z}_p \) to \( \mathbb{Z}_p \Rightarrow f_n \in B_0 \).

(ii) \( f_n(n) = 1 \Rightarrow f_n \in B_0 \setminus pB_0 \).

**Theorem 12.** (Mahler)

The functions \((f_n)_{n \in \mathbb{N}}\) are a Banach basis of \( C(\mathbb{Z}_p, \mathbb{Q}_p) \)

**Proof:** Define the ”discrete derivative” \((\delta f)(x) := f(x + 1) - f(x)\) and the Mahler coefficients of \( f \) by \( a_n(f) := (\delta^n f)(0) \)

The proof follows from the lemma

**Lemma 2.** If \( f \) belongs to \( C(\mathbb{Z}_p, \mathbb{Q}_p) \) then:

(i) \((a_n(f))_{n \geq 0} \) belongs to \( \ell_1(1, \mathbb{Q}_p) \).

(ii) \( f(x) = \sum_0^{\infty} a_n(f)(\mathbb{a}(n)) , \forall x \in \mathbb{Z}_p \)

(iii) \( v_p(f) = v_p((a_n(f))_{n \in \mathbb{N}} , \forall f \in C(\mathbb{Z}_p, \mathbb{Q}_p) \)

**Proof of the lemma:**

(i) Let’s assume without a loss of generality that \( f \in C(\mathbb{Z}_p, \mathbb{Z}_p) \setminus pC(\mathbb{Z}_p, \mathbb{Z}_p) \).

\( a_n(f) = (\delta^n f)(0) \)

\( v_p(a_n(f)) \geq v_p(\delta^n f) \) by definition.

The sequence \( v_p(\delta^n f) \) is clearly increasing, since \( \delta \) preserves \( C(\mathbb{Z}_p, \mathbb{Z}_p) \) so it is enough to show that \( v_p(\delta^n f) \) is unbounded as \( n \rightarrow \infty \).

\( f \) is continuous on \( \mathbb{Z}_p \) which is compact, hence \( f \) is uniformly continuous.

Hence, for all \( M, \exists p^k \) satisfying: \( v_p(f(x + p^k) - f(x)) > M, \forall x \in \mathbb{Z}_p \).

Let \( (sf)(x) := f(x + 1), \delta = s - 1 \).

\( (\delta^k f) = (s - 1)^k f = \sum_{j=0}^{p^k} (-1)^j \binom{p^k}{j} s^j f \)

\( (\delta^k f)(x) = f(x) - f(x + p^k) + \sum_{j=1}^{p^k-1} (-1)^j \binom{p^k}{j} f(x + j) \)

which means:

\( v_p(\delta^k f) \geq \min(M, 1 + v_p(f)) \)

So choosing \( M \) such that \( M \geq 1 + v_p(f) \), we get \( v_p(\delta^k f) \geq 1 + v_p(f) \).

Then taking \( \delta^k f \) instead of \( f \) proves by recurrence that the sequence \( v_p(\delta^n f) \) is unbounded.

(ii) Let \( \tilde{f} := \sum_0^{\infty} a_n(f)(\mathbb{a}(n)) \).

\( \delta((\mathbb{a}(n)) = (x+1)(x+2)(x-n) \cdot \frac{x(x-1)...(x-n+1)}{n!} - \frac{x(x+1)(x+2)...(x-n+1)}{n!} = \frac{(x+1)(x-n+1)x(x-n+2)}{n!} = \frac{x}{(n-1)} \)

\( a_n(f) = (\delta^n f)(0) = (\delta^n \sum_0^{\infty} a_n(f)(\mathbb{a}(n)))(0) = a_n(f) \)

The assignment \( f \rightarrow (a_n(f))_{n \in \mathbb{N}} \) is injective. This is because, if \( (\delta^n f)(0) = 0, \forall n \in \mathbb{N} \) then \( f(j) = 0, \forall j \in \mathbb{N} \).
 Indeed, \( f(0) = 0 \) so \( f(1) - f(0) = 0 \Rightarrow f(1) = 0 \), so \( f(2) - 2f(1) + f(0) = 0 \Rightarrow f(2) = 0 \)

Hence \( f = 0 \) since \( \mathbb{N} \) is dense in \( \mathbb{Z}_p \).

Finally, by injectivity, \( \widehat{f} = f \).

(iii) Let \( f \in C(\mathbb{Z}_p, \mathbb{Q}_p) \), we know \( f(x) = \sum_0^\infty a_n(f) \frac{x^n}{n!} \)  

Since \( v_p(\delta^n f) \to \infty \), then \( \exists N \) such that \( v_p(f) = v_p(\sum_{n=0}^N a_n(f) \frac{x^n}{n!}) \).  

But \( v_p(\sum_{n=0}^N a_n(f) \frac{x^n}{n!}) \geq min(v_p(a_n(f) \frac{x^n}{n!}))) \geq min(v_p(a_n(f))) \) since \( \frac{x^n}{n!} \) \( (n \geq 0) \) belongs to \( C(\mathbb{Z}_p, \mathbb{Z}_p) \).  

So, \( v_p(f) \geq v_p((a_n(f))_{n \in \mathbb{N}}) \). 

Reciprocally, if \( p^k f \in B_0 \), then \( \sum_{n=0}^\infty p^k a_n(f) \frac{x^n}{n!} \in B_0 \) and we want to show that \( v_p(p^k(a_n(f))_{n \in \mathbb{N}}) \geq 0 \). 

Let \( j = \min_{n \in \mathbb{N}} \{ n | v_p(p^k(a_n(f)) < 0 \} \). Then, \( v_p(p^k f(j)) \geq 0 \) by hypothesis, but on the other hand \( v_p(\sum_{n=0}^j p^k a_n(f) \frac{x^n}{n!} + p^k a_j(f)) = v_p(p^k a_j(f)) < 0 \) which is absurd. So \( \{ n | v_p(p^k(a_n(f)) < 0 \} \) is empty and \( v_p((a_n(f))_{n \in \mathbb{N}}) \geq v_p(f) \) which finishes the proof. \( \square \)

In conclusion, we understand \( C(\mathbb{Z}_p, \mathbb{Q}_p) \to \ell_1(\mathbb{N}, \mathbb{Q}_p) \). Now, we would like to understand its dual.

**Dual spaces, measures, and integration:**

For all this section, let \( B \) be a Banach space.

**Definition 13.** A **B-valued measure** on \( \mathbb{Z}_p \) is a continuous linear map from \( C(\mathbb{Z}_p, \mathbb{Q}_p) \) to \( B \).

**Notations:** 1/ \( D(\mathbb{Z}_p, \mathbb{Q}_p) \) = space of \( \mathbb{Q}_p \)-valued measures on \( \mathbb{Z}_p \). 

\( D(\mathbb{Z}_p, B) \) = space of \( B \)-valued measures on \( \mathbb{Z}_p \).  

2/ If \( \mu \in D(\mathbb{Z}_p, \mathbb{Q}_p) \), we write \( \mu(f) = \int_{\mathbb{Z}_p} f(x) d\mu(x) \)

**Concrete description of \( D(\mathbb{Z}_p, \mathbb{Q}_p) \)**

**Theorem 14.** The map \( D(\mathbb{Z}_p, \mathbb{Q}_p) \to \ell_\infty(\mathbb{N}, \mathbb{Q}_p) \)  

\[ \mu \to \mu(\frac{x^n}{n!}) \]

is an isomorphism of \( p \)-adic Banach spaces, which identifies:

\[ D(\mathbb{Z}_p, \mathbb{Z}_p) := \{ \mu | \mu(f) \in \mathbb{Z}_p, \forall f \in C(\mathbb{Z}_p, \mathbb{Z}_p) \} \text{ with } \ell_\infty(\mathbb{N}, \mathbb{Z}_p) \]

.
Sketch of the proof: If $\mu \in D(\mathbb{Z}_p, \mathbb{Q}_p)$, $\mu\left(\binom{x}{n}\right) \in \mathbb{Q}_p$ have to be bounded. (indeed $\binom{x}{n} \in C(\mathbb{Z}_p, \mathbb{Z}_p)$, and $C(\mathbb{Z}_p, \mathbb{Z}_p)$ is a compact so $\mu(C(\mathbb{Z}_p, \mathbb{Z}_p)$ is compact).

Conversely, given a sequence $(b_n)_{n \in \mathbb{N}}$ which is bounded, we can define:

$$\mu_b(f) := \sum_{n \in \mathbb{N}} a_n(f)b_n \in \mathbb{Q}_p$$

\[ \square \]

**Definition 15.** Given $\mu \in D(\mathbb{Z}_p, \mathbb{Q}_p)$, the **Amice transform** of $\mu$ is the power serie;

$$A_\mu(T) := \sum_{n=0}^{\infty} \mu\left(\frac{x}{n}\right) T^n$$

$$= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \left(\frac{x}{n}\right) d\mu(x) T^n$$

$$= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \left(\frac{x}{n}\right) d\mu(x) T^n$$

$$= \int_{\mathbb{Z}_p} (1 + T)^x d\mu(x)$$

**Remark 2.** $\mu \rightarrow A_\mu(T)$ gives an isomorphism $D(\mathbb{Z}_p, \mathbb{Q}_p) \rightarrow \mathbb{Q}_p \otimes \mathbb{Z}_p[[T]]$

Let’s give some examples (or counterexamples) of measures

**Example 16.** **Haar measure**

We would like to have a measure invariant by translation (1) and with value 1 for the constant function 1 (2):

It should satisfy $\mu(x + a) = \mu(x)$ by (1), but $\mu(x + a) = \mu(x) + a$ by 2/. That’s absurd, so there exits NO Haar measure on $\mathbb{Z}_p$.

**Example 17.** **Dirac measure**

Let $a \in \mathbb{Z}_p$. The dirac measure associated to $a$ is defined by the evaluation at $a$:

$$\delta_a(f) := f(a)$$

$$A_{\delta_a}(T) = \sum_{n=0}^{\infty} \delta_a\left(\frac{x}{n}\right) T^n = \sum_{n=0}^{\infty} \binom{a}{n} T^n = (1 + T)^a$$