We previously proved the following statements about modular forms and Hecke operators in the context of $\Gamma = SL_2(\mathbb{Z})$

1. $M(\Gamma) = \bigoplus_k M_k(\Gamma) = \mathbb{C}[E_4, E_6]$ which implies in particular that each $M_k$ is finite dimensional and:

$$M(\Gamma, \mathbb{Q}) = \bigoplus_k M_k(\Gamma, \mathbb{Q}) = \mathbb{Q}[E_4, E_6]$$

where $M_k(\Gamma, \mathbb{Q})$ is the space of modular forms with rational Fourier coefficients.

2. $M_k(\Gamma)$ has extra structures:
   - Hermitian inner product
   - Hecke Operators $(T_n)_{n \geq 1}$

3. 

$$M_k(\Gamma) = \bigoplus_{\phi \in \text{Hom}_{\text{Alg}}(T, \mathbb{C})} M_k(\Gamma)^\phi$$

where $T = \mathbb{C}[T_1, T_2, ...] \subset \text{End}_\mathbb{C}(M_k(\Gamma))$ and $\text{dim}_\mathbb{C}(M_k(\Gamma))^\phi = 1$

$$= \bigoplus_{\phi} \mathbb{C}f_\phi$$

where $f_\phi$ is the normalized eigenform attached to $\phi$, completely characterized by:
- $T_n(f_\phi) = \phi(T_n)f_\phi$
- $a_1(f_\phi) = 1$

4. If $f \in S_k(\Gamma)$, $f = \sum_{n=1}^{\infty} a_n q^n$, $L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}$ then $L(f, s)$ has a functional equation relating $s$ and $k - s$.
   Moreover, if $f$ is a normalized eigenform then:

$$L(f, s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$$

In particular, $L(f, s) \neq 0$ when $\Re(s) > 1 + k/2$
What about other congruence groups?
We will discuss the four statements above in the case of $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$

1. We still have $M(\Gamma) = \bigoplus_k M_k(\Gamma)$ and the $M_k(\Gamma)$’s are still finite dimensional but they need not be generated by Eisenstein Series.

In particular, this is still the case that

$$M(\Gamma, \mathbb{Q}) = \bigoplus_k M_k(\Gamma, \mathbb{Q})$$

and

$$M_k(\Gamma) = M_k(\Gamma, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

One can produce elements of $M_k(\Gamma)$ using the following tricks:

- Any $f \in M_k(SL_2(\mathbb{Z}))$ belongs to $M_k(\Gamma)$
- If $f \in M_k(SL_2(\mathbb{Z}))$ then $f(dz) \in M_k(\Gamma_0(d))$ (Exercise)
- $E_l(d_1, z)E_m(d_2, z) \in M_{l+m}(\Gamma_0(lcm(d_1, d_2)))$
- Hecke translates

In fact, one can prove that if $\Gamma = \Gamma_0(N)$ and $k$ is large enough, then these basic tricks are enough to generate all of $M_k(\Gamma_0(N))$

2. $M_k(\Gamma)$ is still a Hilbert space with Hecke operators but we have to consider different cases:

- If $n$ is prime to $N$,

$$T_nf(z) = n^{k-1} \sum_{\gamma \in \Gamma_1(N) \backslash \Gamma_0(N)} f|_{\gamma}(z)$$

where $M_n(N)$ are upper triangular unipotent matrices modulo $N$ of determinant $n$ and

$$T_nf(z) = n^{k-1} \sum_{\gamma \in \Gamma_0(N) \backslash \Gamma_0(N)} f|_{\gamma}(z)$$

where $M_n(N)$ are upper triangular matrices modulo $N$ of determinant $n$

- If $l$ is prime and $l \nmid N$,

$$T_lf(q) = \sum_{n=1}^{\infty} a_n lq^n + l^{k-1} \sum_{n=1}^{\infty} a_n < l > fql$$

where $f \in M_k(\Gamma_0(N))$ and $< l >$ is the Diamond Operator defined below. These are called **good Hecke Operators**.
If \( l | N \), we still define some kind of Hecke Operators:

\[ T_l f(q) = \sum_{n=1}^{\infty} a_{nl} q^n \]

These are called **bad Hecke Operators**.

**Diamond Operators** < \( a \) > for \( a \in (\mathbb{Z}/N\mathbb{Z})^* \):

\[ < a > f(z) := f|_k \gamma_a(z) \]

where \( \gamma_a \in \Gamma_0(N) \) and \( \gamma_a \equiv \begin{pmatrix} a & * \\ 0 & a^{-1} \end{pmatrix} \mod(N) \)

**Remark 1.** \( T_l \) is not self adjoint in general

\[ T_l^* = < l > T_l \ ( \text{if} \ l \nmid N) \]

**Remark 2.** The bad Hecke operators \( T_l \) do not commute with their adjoints.

Let \( \mathbb{T} = \mathbb{C}(T_l, < a >) \) where \( l \nmid N \) and \( (a, N) = 1 \).

Then \( \mathbb{T} \subset \text{End}_{\mathbb{C}}(M_k(\Gamma)) \) and \( \mathbb{T} \) is an algebra of commuting operators. We have:

\[ M_k(\Gamma) = \bigoplus_{\phi \in \text{Hom}_{\text{Alg}}(\mathbb{T}, \mathbb{C})} M_k(\Gamma)^{\phi} \]

but \( M_k(\Gamma)^{\phi} \) is not necessary 1-dimensional.

Note that if \( f_1, f_2 \in M_k(\Gamma)^{\phi} \) are both normalized then

\[ \phi(T_n) = a_n(f_1) = a_n(f_2) \quad (\forall(n, N) = 1) \]

**Example 1.** Construction of 2 such functions:

Let \( f_1 \in S_k(\Gamma_1(M)) \) be normalized, with \( M|N, M \neq N \) and define:

\[ f_2(z) = f_1(z) + \lambda f_1(dz) = \sum_{n=1}^{\infty} a_n q^n + \lambda \sum_{n=1}^{\infty} a_n q^{nd} \]

where \( d|\frac{M}{N} \)

This example motivates the following definition:
**Definition 2.** A modular form in $M_k(\Gamma_1(N))$ which is a linear combination of forms of type 

$$g(dz)$$

where $g \in M_k(\Gamma_1(M))$ with $M|N$, $M \neq N$ and $d \frac{M}{N}$ is called an **old form**.

Considering the space generated by old forms, we have:

**Definition 3.** $S_k(\Gamma_1(N))^{old} = \text{Space of old forms}$

$S_k(\Gamma_1(N))^{new} = (S_k(\Gamma_1(N))^{old})^\perp = \text{Space of new forms}$

where the orthogonality is relative to the Petersson inner product.

**Theorem 4.** (Atkin-Lehner)

The space $S_k(\Gamma_1(N))^{new}$ decomposes as:

$$S_k(\Gamma_1(N))^{new} = \bigoplus_{\phi \in \text{Hom}_{\text{Alg}}(\mathbb{T}_{\text{new}}, \mathbb{C})} S_k(\Gamma_1(N))^\phi$$

where $\text{dim}_{\mathbb{C}} S_k(\Gamma_1(N))^\phi = 1$ and $\mathbb{T}_{\text{new}} = \mathbb{T}|_{S_k(\Gamma_1(N))^{new}}$

**Definition 5.** A normalized eigenform $f \in S_k(\Gamma_1(N))^{new}$ is also called a **new form** of weight $k$ and level $N$

It remains to discuss $L(f, s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in this context.

Can we find a functional equation?

We note that there is an extra symmetry on $S_k(\Gamma_1(N))$:

**Fricke or Atkin–Lehner involution**: $w_N \leftrightarrow \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$

**Fact**: $w_N$ normalizes $\Gamma_0(N)$ and $\Gamma_1(N)$:

$$w_N \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} w_N^{-1} = \begin{pmatrix} d & -c \\ -Nb & a \end{pmatrix}$$

**Remark 3.** $w_N$ does not commute in general with the action of $\mathbb{T}$:

$$w_N T_l =< l > T_l w_N \quad (\forall l \nmid N)$$

But if $\Gamma = \Gamma_0(N)$ then $w_N$ does commute.

In particular, $w_N f = w f$ for $w \in \{\pm 1\}$
Theorem 6. Let \( f \in S_k(\Gamma_0(N)) \) and \( w_N f = w f \) with \( w \in \{ \pm 1 \} \) then:

\[
A(f, s) := (2\pi)^{-1} N_s^{1/2} \Gamma(s) L(f, s) \\
= (-1)^{k/2} w A(f, k - s)
\]

\textit{proof: Assignment 2} \( \Box \)