189-346/377B: Number Theory

Assignment 1

Due: Monday, January 17

1. Show that the cubic equation $x^3 - 3x + 1 = 0$ has three real roots. Write down an explicit formula for the roots of this equation in terms of nested square and cube roots of rational numbers, using the Cardano formula proved in class. Evaluate the resulting expression to 8 digits of decimal accuracy (in all possible ways!) and check that you indeed obtain, among the resulting expressions, real approximations to the three roots of the equation.

2. Make a table of the factorisations of the integers $2^k + 1$ for $1 \leq k \leq 16$. For which values of k is this integer a prime? Formulate a conjecture about the values of k for which $2^k + 1$ is prime, based on your calculations.

377: Prove your conjecture.

3. Let $li(x) = \int_2^x \frac{dt}{\log(t)}$ be the function that occurs in Gauss's statement of the Prime Number Theorem. Show that

$$\lim_{x \longrightarrow \infty} \frac{li(x)}{x/\log(x)} = 1,$$

and conclude that $li(x)$ can be replaced by the simpler function $x/log(x)$ in the statement of the PNT.

4. Show that 55 can be written as a difference of two perfect integer squares in exactly two different ways, and write down those expressions.

The following question will require you to use a computer algebra system.

5. Compute $e^{\pi\sqrt{163}}$ with 30 significant digits on Pari. (For this, enter the Pari command $\backslash p$ 30.) What do you observe? Repeat the calculation with 40

significant digits. (This exercise is meant to get you familiar with using Pari, and also as a cautionary tale about drawing conclusions too hastily based on experimental data.)