189-726B: Modular Forms II
Assignment 9

Due: Friday, March 28

1. Let k be a field of characteristic different from 2 and 3. Fill in the details of the proof sketched in class of the following "key lemma" on elliptic curves:

Let E be an elliptic curve defined over k and let $\mathcal{O}$ be its origin. Let ω be a regular differential on E over k. Then there are *unique* functions X and Y on E regular outside $\mathcal{O}$, and scalars $g_2, g_3 \in k$ such that

$$Y^2 = X^3 + g_2 X + g_3, \qquad \omega = \frac{dX}{Y}.$$

2. Show that the assignments $(E, \omega) \mapsto g_2$ and $(E, \omega) \mapsto g_3$ are algebraic modular forms over $\mathbf{Z}[1/6]$ of weight 4 and 6 respectively.

3. Prove the formula

$$\delta_k(f|_k\gamma) = (\delta_k f)|_{k+2}\gamma,$$

where $|_k$ is the slash operator on forms of weight k and γ is any element of $\mathrm{SL}_2(\mathbf{Q})$.

4. Suppose that f and g are modular forms of weights ℓ and m respectively, with $\ell + m = k$. Prove the Leibiz formula:

$$\delta_k^r(fg) = \sum_{j=0^r} \binom{r}{j} (\delta_\ell^j f)(\delta_m^{r-j} g).$$