189-726B: Modular Forms II
Assignment 7

Due: Wednesday, March 12

Note: Because this assignment is given over a two-week break, it will be longer than the usual assignments, and designed to make you learn new things. Allow enough time to do it well! A few of the questions are designed to get you to read the relevant portions of Serre's paper "Formes modulaires et Fonctions zeta p-adiques" (of which an English translation can be found on Cameron Franc's web page...) I advise you to try to get as far as you can without *looking at Serre's article — you'll learn a lot more that way. Only consult Serre for hints after you've been stuck for a while.*

1. Write down the Hasse polynomial $A(Q, R) \in \mathbf{F}_{13}[Q, R]$ for $p = 13$.

2. Let $f \in M_k \otimes \mathbf{Q}$, $g \in M_\ell \otimes \mathbf{Q}$, with $(p-1)|k$ and $t = \mathrm{ord}_p(k-\ell)+1$. Assume that $t_0 := \mathrm{ord}_p(k)+1 < t$. and $a_n(f) - a_n(g) \in p^t\mathbf{Z}_{(p)}$ for all $n \geq 1$. In class, we sketched a proof of the fact that $p^{t_0}(a_n(f) - a_n(g)) \in p^t\mathbf{Z}_{(p)}$. Fill in the omitted details of the argument.

3. In class we explained that, if K is a totally real field of degree r, then the special value $\zeta_K(1-k)$ ($k \geq 2$ even) can be written as the constant term of a modular form of weight rk on $\mathrm{SL}_2(\mathbf{Z})$ all of whose other Fourier coefficients are *rational*. Use this fact to prove that $\zeta_K(1-k)$ is rational. (Hint: Do not be misled by the confusing answer I gave to a question about this in my last lecture: your proof should be short and conceptual and only use what you already know well!)

4. Let U and V be the operations on formal q-expansions defined by

$$U(\sum a_n q^n) = \sum a_{np} q^n, \qquad V(\sum a_n q^n) = \sum a_n q^{np}.$$

Show that, if f is a p-adic modular form of weight k, then the same is true of $U(f)$ and $V(f)$.

5. Let $f \in \bar{M}_\alpha$ be a modular form mod p, of weight $\alpha \in \mathbf{Z}/(p-1)\mathbf{Z}$ and filtration $w(f) = k$. What is $w(Vf)$?

6. Show that the weight two Eistenstein series $P = E_2$ is a p-adic modular form of weight 2.

7. Show that if f is a p-adic modular form of weight $k \in \mathbf{Z}/(p-1)\mathbf{Z} \times \mathbf{Z}_p$, then θf is a p-adic modular form of weight $k+2$.

8. If $f = \sum_n a_n q^n$ is a p-adic modular form of weight k, show that the same holds for $\tilde{f} = \sum_{(p,n)=1} a_n q^n$.

9. Read Chapter 3 of Serre's article. Although I will not be assuming this material in the subsequent lectures, familiarity with it will provide you with motivation and clarify those lectures. So, although this will not be graded, I encourage you to take this last part of the assignment seriously!