189-726B: Modular Forms II
Assignment 6

Due: Wednesday, February 20

1. Let $P = E_2 = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n$ be the weight two Eisenstein series. Prove the claim made in class that $12\theta P - P^2$ belongs to the space M_4.

2. Let $\Delta = (Q^3 - R^2)/1728$ be Ramanujan's Δ-function. Compute $\partial_{12}(\Delta)$ and conclude that $12P$ is the logarithmic derivative of Δ. Deduce from this the infinite product formula for Δ.

3. Show that

$$\theta R = -\Delta \pmod 5, \qquad \theta Q = 2\Delta \pmod 7,$$

and conclude the well-known congruences for the Ramanujan τ function:

$$\tau(n) \equiv n\sigma_5(n) \pmod 5, \qquad \tau(n) \equiv n\sigma_3(n) \pmod 7.$$

At this point, let me correct a mistake I made in class: the coefficient appearing in the formula for the weight 6 *Eisenstein series* $R = E_6$ *is* 504, *not* 540 *(I miscopied...) So in fact, the only primes that require special treatment are* 2 *and* 3, *everything works for* $p = 5$, *as Shahab said.*

4. Show that the mod 5 modular form $\theta(E_{10})$ has filtration (weight) 12 and relate it to Δ. How does this compare to what you calculated in 3?

5. Prove the assertion made in class that $\partial B = -QA$, where A and B are the homogeneous polynomials of degrees $p-1$ and $p+1$ respectively satisfying $A(Q, R) = E_{p-1}$ and $B(Q, R) = E_{p+1}$.