189-726B: Modular Forms II
Assignment 2

Due: Wednesday, January 23

1. Prove the recurrence formula

$$\tilde{T}_{p^{r-1}}\tilde{T}_p([\Lambda]) = \tilde{T}_p\tilde{T}_{p^{r-1}}([\Lambda]) = \tilde{T}_{p^r}([\Lambda]) + p\tilde{T}_{p^{r-2}}([p\Lambda])$$

for the abstract Hecke operators on lattices that was used in class on Monday.

2. Show that the Hecke operators T_n acting on the space $S_k(\Gamma_0(N))$ are self-adjoint with respect to the Petersson scalar product, when $\gcd(n, N) = 1$. You may eventually have to extend some definitions that were only given for $\mathrm{SL}_2(\mathbf{Z})$. Try to use only your notes, and avoid peeking into a textbook!

3. Show that the modular forms $f(\tau)$ and $f(d'\tau)$ belong to $S_k(\Gamma_0(N))$ when dd' divides N and f belongs to $S_k(\Gamma_0((d))$. Show that if f is an eigenvector for T_n with $\gcd(n, N) = 1$, then the same holds for $f(d'\tau)$, and that the eigenvalues are equal.

4. Let $\mathbf{T}$ be the $\mathbf{Q}$-algebra of Hecke operators T_n acting on $S_{24}(\mathrm{SL}_2(\mathbf{Z}))$. Express $\mathbf{T}$ as a product of specific number fields, and write down a basis of eigenforms for $S_{24}(\mathrm{SL}_2(\mathbf{Z}))$. (This fun calculation was done by Hecke.)