189-726B: Modular Forms II
Assignment 1

Due: Wednesday, January 16

1. Let F be a number field and let

$$\zeta(F,s) = \sum_{a} (\mathrm{N}a)^{-s}$$

be its Dedekind zeta-function, where the sum is taken over all integral ideals of F.

If λ is an integral ideal, define $\mathrm{N}(\lambda)$ to be the index of λ in $\mathcal{O}_F$, and extend this definition to fractional ideals in the obvious way. To any fractional ideal λ we can associate the zeta function

$$\zeta_\lambda(s) = \sum_{x \in \lambda/\mathcal{O}_F^\times} |Nx|^{-s},$$

and to an ideal class $\mathcal{C}$, the zeta-function

$$\zeta(\mathcal{C},s) = (\mathrm{N}\lambda)^s \zeta_\lambda(s),$$

where λ is any representative ideal in $\mathcal{C}$. Show that

$$\zeta(F,s) = \sum_{\mathcal{C}} \zeta(\mathcal{C},s),$$

where the sum is taken over the distinct ideal classes of F.

2. Fill in the details that are needed to prove the integral representation for the L-function of a cusp form f of weight k on $\Gamma_0(N)$:

$$\Lambda(f,s) = \int_{1/\sqrt{N}}^{\infty} f(it) t^s \frac{dt}{t} + i^k N^{k-s} \int_{1/\sqrt{N}}^{\infty} g(it) t^{k-s} \frac{dt}{t},$$

where $g := f|_k S_N$.

3. Use the modular transformation formula for the Riemann theta-function

$$\theta(\tau) = \sum_{n=1}^{\infty} e^{2\pi i n^2 \tau}$$

which you saw last semester to derive the functional equation for the Riemann zeta-function.