189-235A: Basic Algebra I

Practice Final Exam

Fall 2004

This mock exam has twelve questions, worth 9 points each. The final grade will be taken out of 100, although it is possible to achieve a maximum grade of 108.

1. Let $(u_n)_{n\geq 0}$ be the Lucas sequence, defined recursively by the rule

$$u_0 = 0, \quad u_1 = 1, \quad u_{n+1} = 2u_n + u_{n-1}, \text{ for } n \geq 1.$$

Show that $u_n \leq 3^n$ for all $n \geq 0$.

2. Solve the congruence equation

$$4x = 11 \pmod{55}.$$

3. Let a be an element of a group G, and suppose that $a^m = a^n = 1$, for some integers m and n. Let $d = \gcd(m, n)$. Show that $a^d = 1$. (You may use any of the properties of the gcd that were shown in class.)

4. Let $\mathbf{Z}_3$ denote the field with 3 elements. Show that the greatest common divisor of the polynomials $x^3 - 1$ and $x^2 + 1$ in $\mathbf{Z}_3[x]$ is equal to 1, and express 1 as a linear combination of these two polynomials.

5. Show that the ring $\mathbf{Z}_3[x]/(x^2 + 1)$ is a field. Write down the multiplicative inverse of the element $[x^3 - 1]$ in this field.

6. Give an example of three rings R_1, R_2 and R_3 of cardinality 25 which are not isomorphic to each other. (You should justify your assertion.)

7. Show that the ring $\mathbf{Q}[x]/((x-1)^2)$ is *not* isomorphic to the ring $\mathbf{Q}[x]/((x^2 - 1)$.

8. Construct an injective group homomorphism from $\mathbf{GL}_2(\mathbf{Z}_3)$ to S_8 (the symmetric group on 8 elements.)

9. Give an example of a ring R and an ideal I which is prime but not maximal. Give a maximal ideal J of R which contains I.

10. Write down two non-isomorphic groups of order 10.

11. Let F be a field, and let G be the set of functions $f : F \longrightarrow F$ of the form $f(x) = ax + b$, with $a \in F - \{0\}$ and $b \in F$. Show that G is a group under the operation of composition of functions. Let H be the subgroup of translations, i.e. functions of the form $f(x) = x + b$ for some $b \in F$. Show that H is a normal subgroup of G and that G/H is isomorphic to the multiplicative group $F^\times$ of non-zero elements of F.

12. Let F be a field. State (without proof) whether the following assertions are true or false.
a. The set I of polynomials $p(x)$ in the ring $F[x]$ satisfying $p(1) = p(2) = 0$ is an ideal of $F[x]$.
b. The set I of polynomials of degree at most 3 is an ideal of $F[x]$.
c. Every ideal in $\mathbf{Q}[x]$ is principal.
d. Every ideal in $\mathbf{Z}$ is principal.
e. The symmetric group S_n has cardinality n.
f. The group S_5 contains an element of order 6.
g. The group S_6 contains an element of order 7.