189-235A: Basic Algebra I
Assignment 9

Due: Wednesday, November 17

1. Show that the intersection of two prime ideals need not be a prime ideal.

2. Show that a ring R is an integral domain if and only if $\{0\}$ is a prime ideal.

3. If P is a prime ideal in a commutative ring R, is the ideal $P \times P$ a prime ideal of $R \times R$?

4. Find an ideal of $\mathbf{Z} \times \mathbf{Z}$ that is prime but not maximal.

5. Let R be a commutative ring, let $F[x]$ be the ring of polynomials with coefficients in a field F, and let $f : R \longrightarrow F[x]$ be a ring homomorphism. Show that the kernel of f is a prime ideal of R.

6. Let F be a field, and define a binary composition law on $G = F - \{1\}$ by the rule

$$a * b = a + b - ab.$$

Show that G, with this operation, is a group. (In particular, write down the neutral element for $*$, and give a formula for the inverse of $a \in G$.

7. List all the elements of order 3 in S_3. How many are there?

8. List all the elements of order 6 in S_5. How many such elements are there?

9. Given an example of non-abelian groups of order 12 and 30.

10. **Extra credit question.** Suppose that G is a group in which $x^2 = 1$, for all $x \in G$. Show that G is abelian. Give an example of a **non-abelian** group G of order 27 in which $x^3 = 1$ for all $x \in G$.