189-235A: Basic Algebra I
Assignment 10

Due: Wednesday, November 24

1. Show that the conjugate elements a and bab^{-1} in a group G have the same order.

2. Show that the intersection of two subgroups H_1 and H_2 of a group G is a subgroup of G. What about unions of subgroups?

3. If a is an element of a finite group G of cardinality n, show that $a^n = 1$. Apply this general fact to the group $G = \mathbf{Z}_p^\times$ (under multiplication) to give another proof of *Fermat's Little Theorem* that p divides $a^p - a$ for all integers a when p is prime.

4. Let S be a subset of a group G. The centraliser of S, denoted $Z(S)$, is the set of $a \in G$ which commute with every $s \in S$, i.e., such that $as = sa$ for all $s \in S$. Show that $Z(S)$ is a subgroup of G.

5. Let G_1 be the group of strictly positive real numbers, under multiplication, and let G_2 be the group of all real numbers, under addition. Show that G_1 and G_2 are isomorphic.

6. Let $f : G_1 \longrightarrow G_2$ be a homomorphism of groups, and let a be an element of G_1 of finite order. Show that the order of $f(a)$ in G_2 divides the order of a in G_1, and that these two orders are equal if f is injective.

7. Recall the ring $H = \{a + bi + cj + dk, \quad a, b, c, d \in \mathbf{R}\}$ of quaternions defined by the multiplication rules

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = i.$$

Show that the subset $G = \{1, -1, i, -i, j, -j, k, -k\}$ is a subgroup of the multiplicative group of non-zero elements of H. Show that G is *not* isomorphic to the dihedral group D_8 of order 8.

Extra Credit

9. Show that the groups $\mathbf{GL}_2(\mathbf{Z}_2)$ and S_3 are isomorphic, by writing down a specific isomorphism. (Hint: realize each matrix as a permutation on the non-zero column vectors with entries in $\mathbf{Z}_2$).

10. The *conjugacy class* of a in a group G is the set of all elements of G which are of the form gag^{-1} for some $g \in G$. Show that a normal subgroup of G is a disjoint union of conjugacy classes. List the conjugacy classes in S_4 and use this to give a complete list of all the normal subgroups of S_4. Same question for S_5.