189-235A: Basic Algebra I

Assignment 1

Due: Wednesday, September 15

Let S and T be the sets $\{a, b, c\}$and $\{x, y\}$ respectively.
1. How many functions are there from S to T?
2. How many injective functions?
3. How many surjective functions?

Let X be a set, and let $\mathcal{F}(X)$ be the set of all functions from X to itself. This set is equipped with a natural binary operation $(f, g) \mapsto fg$, given by the composition of functions.
4. Show that $f(gh) = (fg)h$ for all f, g, h in $\mathcal{F}(X)$. (In other words, the operation of composition of funcions is *associative*.)
5. Show, by providing an example, that fg need not be equal to gf.
6. Prove that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, and that $\sum_{k=0}^{n}(-1)^k \binom{n}{k} = 0$, using the binomial theorem.

7. Using the Euclidean algorithm compute the gcd of 910091 and 3619. Show the steps in your calculation.

8. Using induction, show that

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

9. Using induction, show that the addition law in $\mathbf{N}$ is associative directly from the axioms defining addition in $\mathbf{N}$.

10. Show that the expression

$$1^k + 2^k + 3^k + \cdots + n^k$$

can be written as a polynomial in n of degree at most $k + 1$.