THE ANALYTIC CONTINUATION OF DIRICHLET L-SERIES

CRISTINA TOROPU

Definition 1. *Let G be a finite abelian group.A character of $G$is a homomorphism of G in C^**

$$\chi : G \to C^* \textit{such that} \chi(ab) = \chi(a)\chi(b) \textit{and} \chi(1) = 1$$

$\overline{\chi}(a) = \overline{\chi(a)}$ *for all* $a \in G$

Definition 2. *Let* $G = \left(\frac{Z}{mZ}\right)^*$ *for all* $m \geq 1$

A function $\chi^* : Z \to C^*$

$$\chi^*(a) = \begin{cases} \chi(\overline{a}) & \textit{if } (a, m) = 1 \\ 0 & \textit{if } (a, m) > 1 \end{cases}$$

is called Dirichlet character modulo m.

Let $d \mid m$ and let χ be a character modulo d.Let's define

$$\chi'(a) = \begin{cases} \chi(a) & \text{if } (a, m) = 1 \\ 0 & \text{if } (a, m) > 1 \end{cases}$$

We say that χ induces χ'.

Definition 3. *Let χ be a character modulo m.If there exists $d \mid m$ and a character χ' modulo d such that χ' induces χ,then χ is called nonprimitive.Otherwise is called primitive.*

Proposition 1. *A χ modulo m is a primitive character if $\forall d \mid m \quad \exists x \equiv 1$ (mod m) such that $\chi(x) \neq 1$*

Proof. It is equivalent to prove that χ is nonprimitive if $\exists d \mid m$ for all $x \equiv 1$ mod m such that $\chi(x) = 1$.
Since χ is nonprimitive there exists a character χ' mod d such that $\chi(x) = \chi'(x) = 1$ □

Dirichlet L-functions

If χ is a Dirichlet character,primitive or not,we define a L-function

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

By comparison with Riemann zeta-function it is absolutely convergent for $Re(s) > 1$. It has an Euler product.
For $re(s) > 1$ we prove that

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \tag{1}$$

$\left(1-\frac{\chi(p)}{p^s}\right)^{-1}=\sum_{k=o}^{\infty}\frac{\chi(p)^k}{p^{ks}}$

For any positive integer n there exist exactly one way to write

$$n^{-s}=\left(\prod p_i^{k_i}\right)^{-s}$$

On expanding

$$\prod_p\sum_{k=0}^{\infty}\frac{\chi(p)^k}{p^{ks}}$$

the coefficient of n^{-s} is
$\prod_i \chi(p_i^{k_i})=\chi(n)$
Let χ be a primitive character modulo N

The Gauss sum $\tau(\chi)$ is defined by

$$\tau(\chi)=\sum_{n \bmod N}\chi(n)e^{\frac{2\pi in}{N}} \qquad n=1,\ldots,N-1 \tag{2}$$

We prove that :

$$\sum_{n \bmod N}\chi(n)e^{\frac{2\pi inm}{N}}=\overline{\chi(m)}\tau(\chi) \tag{3}$$

We have 2 cases: If $(m,N)=1 \quad |\chi(m)|=1$
Making the change of variables $n\to nm$

$$\tau(\chi)=\sum_{n \bmod N}\chi(nm)e^{\frac{2\pi inm}{N}}=\chi(m)\sum_{n \bmod N}\chi(n)e^{\frac{2\pi inm}{N}} \tag{4}$$

Multiplying this by $\chi^{-1}(m)=\overline{\chi(m)}$ we obtain (3)
If $(m,n)>1$ let $d=(m,n)\Rightarrow m=dM$ and $N=dN_1$
So

$$\sum_{n \bmod N}\chi(n)e^{\frac{2\pi inm}{N_1}}=\sum_{r \bmod N_1}\left\{\sum_{\substack{n \bmod N\\ n\equiv r \bmod N_1}}\chi(n)\right\}e^{\frac{2\pi irM}{N_1}}$$

The substitution

$$n\to cn$$

permutes the residue classes n mod N such that

$$n\equiv rmodN_1$$

among themselves,so

$$\sum_{\substack{n \bmod N \\ n\equiv rmodN_1}} \chi(n) = \sum_{\substack{n \bmod N \\ n\equiv rmodN_1}} \chi(cn) = \chi(c) \sum_{\substack{n \bmod N \\ n\equiv rmodN_1}} \chi(n)$$

Since χ is a primitive character $\Rightarrow \forall d|N$ there exist $c \equiv 1 \bmod d$ such that $\chi(c) \neq 1$
So

$$\sum_{\substack{n \bmod N \\ n\equiv rmodN_1}} \chi(n) = 0$$

,so (3) is proved
We prove $\tau(\chi) \neq 0$

(5) $$|\tau(\chi)| = \sqrt{N}$$

Because:

$$\overline{\chi(n)e^{\frac{2\pi inm}{N}}} = \overline{\chi(n)}e^{\frac{-2\pi inm}{N}}$$

$$|\sum_{n \bmod N} \chi(\tau)e^{\frac{2\pi inm}{N}}|^2 = \sum_{\substack{n_1,n_2 \bmod N \\ (n_1n_2,N)=1}} \chi(n_1)\overline{\chi(n_2}e^{\frac{2\pi i(n_1-n_2)m}{N}}$$

This becomes by(3)

$$\begin{cases} |\tau(\chi)|^2 & \text{if } (m,N)=1 \\ 0 & \text{if } (m,N)>1 \end{cases}$$

Summing over all m modulo N

$$\Phi(N)|\tau(\chi)|^2 = \sum_{m \bmod N} \sum_{\substack{n_1,n_2bmodN \\ (n_1n_2,N)=1}} \chi(n_1)\overline{\chi(n_2)}e^{\frac{2\pi i(n_1-n_2)m}{N}}$$

where$\Phi(N)$ is the number of residue classes mod N prime to N
Because

$$\sum_{m \bmod N} e^{\frac{2\pi iam}{N}} = \begin{cases} N & \text{if } N|a \\ 0 & \text{if N doesn't divide } a \end{cases}$$

Only terms with $n_1 \equiv n_2 \bmod N$ contribute

$$\chi(n_1)\overline{\chi(n_2)} = 1$$

So

$$\Phi(N)|\chi(\tau)|^2 = \sum_{\substack{n_1,n_2 \bmod N \\ (n_1n_2,N)=1 \\ n_1\equiv n_2modN}} N = \Phi(N)N$$

$$\Rightarrow |\chi(\tau)|^2 = \sqrt{N}$$

Since

$$\tau(\chi) \neq 0$$

we interpolate a primitive Dirichlet character between integers.
We interchange

$$\chi \to \overline{\chi}$$

in(3) and change n with m

$$\Rightarrow \chi(n) = \frac{1}{\tau(\overline{\chi})} \sum_{m \bmod N} \overline{\chi(m)} e^{\frac{2\pi i m n}{N}} \tag{6}$$

Since

$$\tau(\overline{\chi}) = \overline{\tau(\chi)} \cdot \chi(-1)$$

$$|\tau(\chi)|^2 = N \Rightarrow \tau(\chi)\overline{\tau(\chi)} = N$$

$$\to \chi(n) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} e^{\frac{2\pi i m n}{N}} \tag{7}$$

This is defined when n is an arbitrary real number.
We prove more twisted Poisson summation:

$$\sum_{n=-\infty}^{\infty} \chi(n) f(n) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{-\infty}^{\infty} \overline{\chi(n)} \hat{f}\left(\frac{n}{N}\right) \tag{8}$$

But

$$\sum_{-\infty}^{\infty} \chi(n) f(n) = \sum f_1(n)$$

where

$$f_1(x) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} e^{\frac{2\pi i x m}{N}} f(x)$$

Applying Poisson summation formula we get:

$$\hat{f}_1(x) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \overline{\chi(m)} \hat{f}\left(\frac{Nx+m}{N}\right)$$

$$\sum_{n=-\infty}^{\infty} \chi(n) f(n) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \sum_{n=-\infty}^{\infty} \overline{\chi(m)} \hat{f}\left(\frac{Nn+m}{N}\right) = \frac{\chi(-1)\tau(\chi)}{N} \sum_{m \bmod N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \hat{f}\left(\frac{n}{N}\right)$$

This is happend because

$$\chi(m) = \chi\left(Nm + m\right)$$

and $Nn + m$ runs uniquely through Z when m runs through a set of residue classes mod N and n runs through Z
We compute Fourier transform of the Gaussian distribution.Let t have positive real part and let

$$f_t = e^{-\pi t x^2}$$

$f_t(x)$ is of faster –than–polynomial decay as $x \to \pm\infty$

$$\hat{f}_t(x) = \frac{1}{\sqrt{t}} f_{\frac{1}{t}}$$

Both sides are analytical functions of t defined when$re(t) > 0$.By completing square:

$$\hat{f}_t(x) = \int_{-\infty}^{\infty} e^{-\pi(ty^2-2\pi ixy)}dy = e^{-\frac{\pi x^2}{t}} \int_{-\infty}^{\infty} e^{-\pi(\sqrt{t}y-\frac{ix}{\sqrt{t}})^2}dy$$

By using Cauchy's theorem to shift the path of integration with respect to y vertically by a constant distance depending on x.

$$y \to y + \frac{ix}{t}$$

$$\Rightarrow \hat{f}_t(x) = e^{\frac{-\pi x^2}{t}} \int_{-\infty}^{\infty} e^{-\pi ty^2}dy = \frac{c}{\sqrt{t}}e^{\frac{-\pi x^2}{t}}$$

where

$$c = \int_{-\infty}^{\infty} e^{-\pi y^2}dy$$

$$\Rightarrow \hat{\hat{t}}_t = \frac{c}{\sqrt{t}}\hat{f}_{\frac{1}{t}} = c^2 f_t$$

We use Fourier inversion formula

$$\hat{\hat{f}}_t(x) = f(-x)$$

if f is any continuous function such that f and $\hat{f}$ are in $L^1(R)$
Applying this to even function $\Rightarrow \hat{\hat{f}}_t = f_t$ and $\Rightarrow c^2 = 1 \Rightarrow c = 1$ and $c > 0$ Let's define some theta functions.

Let χ be a primitive character mod N.
First assume $\chi(-1) = 1$
$\theta_\chi = \frac{1}{2}\sum_{n=-\infty}^{\infty} \chi(n)e^{-\pi n^2 t} = \frac{1}{2}\chi(0) + \sum_{n=1}^{\infty} \chi(n)e^{-\pi n^2 t}$ when $re(s) > 0$
($\chi(0) = 0$ unless $N = 1$ because $\chi(n)$ is primitive)
Using $\hat{f}_t = \frac{1}{\sqrt{t}}f_{\frac{1}{t}}$ and twisted Poisson formula we obtain the functional equation:
$\sum_{n=-\infty}^{\infty} \chi(n)f(n) = \frac{\chi(-1)}{N}\tau(\chi)\sum_{n=-\infty}^{\infty} \overline{\chi(n)}\hat{f}(\frac{n}{N})$
$\theta_\chi(t) = \frac{\tau(\chi)}{N\sqrt{t}}\theta_{\overline{\chi}}\left(\frac{1}{N^2 t}\right)$ if $\chi(-1) = 1$
Now assume $\chi(-1) = -1$
Let $g_t(x) = xe^{-\pi tx^2} = -\frac{1}{2\pi t}\left(\frac{df_t}{dx}\right)(x)$
$\hat{g}_t(x) = \frac{i}{t^{\frac{3}{2}}}g_{\frac{1}{t}}$
$\hat{g}_t(x) = \int_{-\infty}^{\infty} g_t(y)e^{2\pi ixy}dy = -\frac{1}{2\pi t}\int_{-\infty}^{\infty}\left(\frac{df_t}{dy}\right)(y)e^{2\pi ixy}dy$
Integrating by parts this becomes equal to $\frac{ix}{t}\int_{-\infty}^{\infty} f_t(y)e^{2\pi ixy}dy = \frac{ix}{t}\hat{f}_t = \frac{ix}{t}\frac{1}{\sqrt{t}}f_1 t = \frac{i}{t^3 2}g_{\frac{1}{t}}$
If $\chi(-1) = -1$ we define
$\theta_\chi(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} n\chi(n)e^{-\pi n^2 t} = \sum_{n=1}^{\infty} n\chi(n)e^{-\pi n^2 t}$
Applying twisted Poisson summation to g_t
$\sum_{n=-\infty}^{\infty} \chi(n)f(n) = \frac{\chi(-1)\tau(\chi)}{N}\sum_{n=-\infty}^{\infty} \overline{\chi(n)}\hat{f}(\frac{n}{N})$
$\theta_\chi(t) = \frac{-i\tau(\chi)}{N^2 t^{\frac{3}{2}}}\theta_{\overline{\chi}}\left(\frac{1}{N^2 t}\right)$ if $\chi(-1) = -1$

Theorem 1. *Let χ be a primitive Dirichlet character with conductor N and let $\epsilon = 0 or 1$ be chosen so that $\chi(-1) = (-1)^{\epsilon}$.Let*

$$\Lambda(s,\chi) = \pi^{-\frac{(s+\epsilon)}{2}} \Gamma\left(\frac{s+\epsilon}{2}\right) L(s,\chi)$$

Then$\Lambda(s,\chi)$ has a meromorphic continuation to all s:indeed if$\chi \neq 1$ it is entire,while if $\chi = 1$ it is analytic for all s except $s = 1$ or $s = 0$ where it has simple poles.We have the functional equation

$$\Lambda(s,\chi) = (-1)^{\epsilon}\tau(\chi)N^{-s}\Lambda(1-s,\overline{\chi})$$

Proof. If χ is the trivial character,χ primitive $\Rightarrow N = 1 \Rightarrow L(s,\chi)$ is just Riemann zeta function.
Let$\chi \neq 1$.Because $\chi(0) = 0$,the series $\theta_{\chi}(t)$ is a sum of terms of $n^{\epsilon}\chi(n)e^{-\pi n^2 t}$ with $|n| \geq 1$,each of which is of very decay as $t \to \infty$
Combining θ_{χ} for $\chi(-1) = -1$

$$\theta_{\chi}(t) = \frac{(-1)^{\epsilon}\tau(\chi)}{N^{1+\epsilon}t^{\epsilon+\frac{1}{2}}}\theta_{\overline{\chi}}\left(\frac{1}{N^2 t}\right)$$

$\to \theta_{\chi(t)}$ is also of very decay as $t \to 0$
$\to$Mellin transform

$$\int_0^{\infty} \theta_{\chi}(t)t^{\frac{s+\epsilon}{2}}\frac{dt}{t}$$

is convergent for all s.If $re(s) > 1$ we have the identity:

$$\int_0^{\infty} e^{-\pi t n^2} t^{\frac{s+\epsilon}{2}}\frac{dt}{t} = \pi^{-\frac{s+\epsilon}{2}}\Gamma\left(\frac{s+\epsilon}{2}\right) n^{-s-\epsilon}$$

So$\int_0^{\infty} \theta_{\chi}(t)t^{\frac{s+\epsilon}{2}}\frac{dt}{t} = \Lambda(s,\chi)$

Because Mellin transform is convergent for all s,and clearly defines an analytic function,this gives the analytic continuation of $\Lambda(s,\chi)$
We are in the case $\chi(-1) = -1$ because $\chi \neq 1$
So substituting $\theta_{\chi}(t) = \frac{-i\tau(\chi)}{N^2 t^{\frac{3}{2}}}\theta_{\overline{\chi}}(\frac{1}{N^2 t})$ in Mellin transform and making the change of variables $t \to \frac{1}{N^2 t}$

$$\int_0^{\infty} \theta_{\chi}(t)t^{\frac{s+\epsilon}{2}}\frac{dt}{t} = (-1)^{\epsilon}\tau(\chi)N^{-s}\int_0^{\infty} \theta_{\overline{\chi}}(t)t^{\frac{1-s+\epsilon}{2}}\frac{dt}{t} = (-1)^{\epsilon}\tau(\chi)N^{-s}\Lambda(1-s,\overline{\chi})$$

So

$$\Lambda(s,\chi) = (-i)^{\epsilon}\tau(\chi)N^{-s}\Lambda(1-s,\overline{\chi})$$

$\square$