Online Supplement to the paper by C. Genest, M. Mesfioui and J. Schulz entitled
“A new bivariate Poisson common shock model covering all possible degrees of dependence”

This Online Supplement provides details about the form of the asymptotic variance of the moment-based estimator $\hat{\theta}$ appearing in Proposition 3 of the paper. As stated there, this asymptotic variance is given by

$$\{g'(\theta)\}^2 \times \text{var}\{(X_1 - \lambda_1)(X_2 - \lambda_2)\},$$

where $(X_1, X_2) \sim \mathcal{BP}(\Lambda, \theta)$ and $g : \theta \mapsto m_{\lambda_1, \lambda_2}(\theta)$ with $m_{\lambda_1, \lambda_2}(\theta) = \text{cov}(X_1, X_2)$. First note that by definition,

$$\text{var}\{(X_1 - \lambda_1)(X_2 - \lambda_2)\} = E\{(X_1 - \lambda_1)^2(X_2 - \lambda_2)^2\} - \{m_{\lambda_1, \lambda_2}(\theta)\}^2.$$

Now recall from Eq. (2) in the paper that for each $k \in \{1, 2\}$, $X_k = Y_k + Z_k$, where $Y_k \sim \mathcal{P}\{(1 - \theta)\lambda_k\}$ is independent of $Z_k \sim \mathcal{P}(\theta\lambda_k)$. One can then write, for $k \in \{1, 2\}$, $X_k - \lambda_k = \hat{Y}_k + \hat{Z}_k$, where $\hat{Y}_k = Y_k - (1 - \theta)\lambda_k$ and $\hat{Z}_k = Z_k - \theta\lambda_k$. Upon substitution and expansion, one gets

$$E\{(X_1 - \lambda_1)^2(X_2 - \lambda_2)^2\} = E\{(\hat{Y}_1^2\hat{Y}_2^2 + 2\hat{Y}_1^2\hat{Z}_2\hat{Z}_2 + \hat{Y}_1^2\hat{Z}_1^2 + 2\hat{Y}_1\hat{Z}_1\hat{Z}_2^2 + 4\hat{Y}_1\hat{Z}_1\hat{Z}_2 + 2\hat{Z}_1\hat{Z}_2^2 + 2\hat{Z}_1^2\hat{Z}_2 + \hat{Z}_1^2\hat{Z}_2^2)\}.$$

Exploiting the fact that the pairs $(\hat{Y}_1, \hat{Y}_2)$ and $(\hat{Z}_1, \hat{Z}_2)$ are independent, the right-hand side reduces to

$$E(\hat{Y}_1^2)E(\hat{Y}_2^2) + 2E(\hat{Y}_1^2)E(\hat{Z}_2) + E(\hat{Y}_1^2)E(\hat{Z}_1)E(\hat{Y}_2^2) + 4E(\hat{Y}_1)E(\hat{Y}_2)E(\hat{Z}_1)E(\hat{Z}_2) + 2E(\hat{Y}_1)E(\hat{Z}_1)E(\hat{Y}_2^2) + E(\hat{Z}_1)E(\hat{Z}_2)E(\hat{Y}_2^2) + 2E(\hat{Y}_2)E(\hat{Z}_1^2)E(\hat{Y}_2) + E(\hat{Z}_1^2)E(\hat{Y}_2^2).$$

Now by construction, the components of $(\hat{Y}_1, \hat{Y}_2)$ are themselves independent and $E(\hat{Y}_k) = E(\hat{Z}_k) = 0$ for $k \in \{1, 2\}$. The above expression thus becomes

$$E(\hat{Y}_1^2)E(\hat{Y}_2^2) + E(\hat{Y}_1^2)E(\hat{Z}_2) + E(\hat{Y}_1^2)E(\hat{Z}_1^2) + E(\hat{Z}_1)E(\hat{Z}_2)E(\hat{Y}_2^2) + 2E(\hat{Y}_1)E(\hat{Z}_1)E(\hat{Y}_2^2) + 2E(\hat{Y}_2)E(\hat{Z}_1^2)E(\hat{Y}_2) + E(\hat{Z}_1^2)E(\hat{Y}_2^2)$$

in terms of the original variables $Y_1, Y_2, Z_1, Z_2$. Moreover, the last summand satisfies

$$E\{((Z_1 - \theta\lambda_1)(Z_2 - \theta\lambda_2))^2\} = \text{var}\{((Z_1 - \theta\lambda_1)(Z_2 - \theta\lambda_2)) + [E\{(Z_1 - \theta\lambda_1)(Z_2 - \theta\lambda_2)\}]^2 = \text{var}\{((Z_1 - \theta\lambda_1)(Z_2 - \theta\lambda_2)) + \{m_{\lambda_1, \lambda_2}(\theta)\}^2.$$

Putting this all together, we deduce that

$$\text{var}\{(X_1 - \lambda_1)(X_2 - \lambda_2)\} = \text{var}(Y_1)\text{var}(Y_2) + \text{var}(Y_1)\text{var}(Z_2) + \text{var}(Z_1)\text{var}(Y_2) + \text{var}\{(Z_1 - \theta\lambda_1)(Z_2 - \theta\lambda_2)\},$$

which, upon substitution, further reduces to

$$\text{var}\{(X_1 - \lambda_1)(X_2 - \lambda_2)\} = (1 - \theta)^2\lambda_1\lambda_2 + 2(1 - \theta)\theta\lambda_1\lambda_2 + \text{var}\{(Z_1 - \theta\lambda_1)(Z_2 - \theta\lambda_2)\}. \quad (\dagger)$$

As mentioned below Proposition 3 in the paper, an additional simplification occurs when $\lambda_1 = \lambda_2 = \lambda$. In that case, $Z_1 = Z_2$ almost surely. Letting $Z$ be this common variable, one finds

$$\text{var}\{(Z_1 - \theta\lambda_1)(Z_2 - \theta\lambda_2)\} = \text{var}\{(Z - \theta\lambda)^2\} = \theta\lambda(1 + 3\theta\lambda) - \theta^2\lambda^2 = \theta\lambda(1 + 2\theta\lambda).$$

It then follows from Eq. (\dagger) that

$$\text{var}\{(X_1 - \lambda_1)(X_2 - \lambda_2)\} = (1 - \theta)^2\lambda^2 + 2(1 - \theta)\theta\lambda^2 + \theta\lambda(1 + 2\theta\lambda) = \lambda(\lambda + \theta + \theta^2\lambda).$$

Moreover, one has $m_{\lambda, \lambda}(\theta) = \theta\lambda$ and hence $g'(\theta) = 1/\lambda$. The asymptotic variance of the moment-based estimator $\hat{\theta}$ thus reduces to $(\lambda + \theta + \theta^2\lambda)/\lambda$, as claimed. \qed