Introduction to resurgence

Brent Pym

McGill

Common situation in physics:

$$\text{physical quantity} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots$$

where x is a small parameter, e.g. $x = \hbar$, but **series often diverges** and moreover this **ignores exponentially small corrections**

Stokes phenomenon:

- divergent power series can still be useful as asymptotic approximations
- exponentially small ambiguities cause discontinuities

Borel summation and resurgence (É. Borel, Écalle, . . .)

- "re-sums" the divergent series to get a function
- produces sums that are real-valued and non-perturbatively correct

This lecture: sketch basic mathematical ideas; reference list and (old, sketchy, unedited) lecture notes:

https://www.math.mcgill.ca/bpym/courses/resurgence/

Euler's equation

$$\frac{df}{dx} = \frac{1}{x} - \frac{f}{x^2} \qquad \text{Ansatz: } f = \sum_{n=0}^{\infty} a_n x^{n+1}$$

$$a_1 = 1, \qquad a_n = -n \cdot a_{n-1} \qquad \Longrightarrow \quad f = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}$$

Problems:

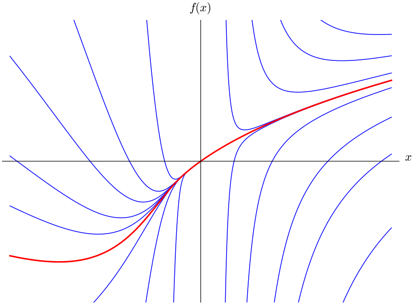
- divergent, i.e. radius of convergence is zero
- constant of integration is absent

Cause: ODE singular at $x = 0$.

NB: Homogenized equation

$$\frac{d\tilde{f}}{dx} = -\frac{\tilde{f}}{x^2} \qquad \rightsquigarrow \qquad \tilde{f} = Ce^{1/x}$$

$$f = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} + Ce^{1/x}$$



Borel summation of $f = \sum_{n=0}^{\infty}(-1)^n n! x^{n+1}$

Recall standard Laplace transform: $n! x^{n+1} = \int_0^\infty t^n e^{-t/x}\, \mathrm{d}t$

Therefore:

$$
\begin{aligned}
f &= \sum_{n=0}^{\infty}(-1)^n n! x^{n+1} \\
&= \sum_{n=0}^{\infty}\int_0^\infty (-1)^n t^n e^{-t/x}\,\mathrm{d}t \\
\text{“}=\text{”} &\quad \int_0^\infty \left(\sum_{n=0}^{\infty}(-1)^n t^n\right) e^{-t/x}\,\mathrm{d}t \\
\text{“}=\text{”} &\quad \int_0^\infty \frac{1}{1+t} e^{-t/x}\,\mathrm{d}t \\
&= e^{-1/x}\mathrm{Ei}(1/x) \qquad \text{for } x > 0 \qquad \text{solves our ODE!} \\
&= \sum_{n=0}^{N}(-1)^n n! x^{n+1} + O(x^{n+1}) \qquad \text{as } x \to 0^+
\end{aligned}
$$

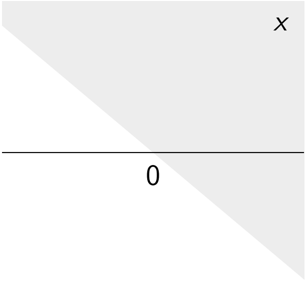
Idea: pass from $x > 0$ to $x < 0$ through the complex plane $\mathbb{C}$

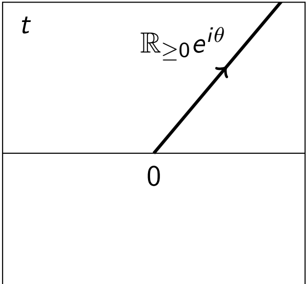
Integral for converges for $x \in \mathbb{C}$ as long as $|e^{-t/x}| \to 0$ as $t \to \infty$, i.e. for $\Re(x) > 0$.

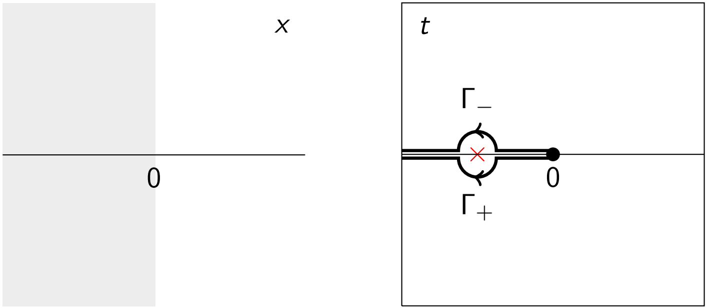
Analytically continue by varying integration contour:

$$f_\theta := \mathscr{L}_\theta(F(t)) := \int_{\mathbb{R}_{\geq 0} e^{i\theta}} F(t) e^{-t/x}\, \mathrm{d}t \qquad F(t) = \frac{1}{1+t}$$

Domain of f_θ:







$$f_{\pi-} - f_{\pi+} = \int_{\Gamma_- - \Gamma_+} \frac{e^{-t/x}}{1+t}\,\mathrm{d}t = 2\pi i \mathrm{Res}\left(\frac{e^{-t/x}}{1+t}; t = -1\right) = 2\pi i \cdot e^{1/x}$$

Solution of the homogeneous equation!

NB: ODE was not used; $e^{1/x}$ is encoded in the divergent series.

$$\begin{tikzcd} x\mathbb{C}[[x]] \arrow[r, bend left, "\text{formal Borel transform } \mathscr{B}"] & \mathbb{C}[[t]] \arrow[l, bend left, "\text{formal Laplace transform } \mathscr{L}"] \end{tikzcd}$$

$$n!x^{n+1} \longleftrightarrow t^n$$

$$(\mathscr{L}_\theta F)(x) = \int_{\mathbb{R}_{\geq 0} e^{i\theta}} F(t) e^{-t/x}\, \mathrm{d}t$$

Definition

The **Borel sum of** $f = \sum_{n=0}^{\infty} a_n x^{n+1} \in x\mathbb{C}[[x]]$ **in the direction** θ is

$$f_\theta(x) := (\mathscr{L}_\theta \mathscr{B} f)(x) = \int_{\mathbb{R}_{\geq 0} e^{i\theta}} \underbrace{\left(\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \right)}_{\text{analytically continued to all } t \in \mathbb{R}_{\geq 0} e^{i\theta}} e^{-t/x}\, \mathrm{d}t,$$

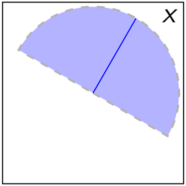
provided the expression converges (i.e. $\mathscr{B} f(t)$ grows at most exponentially)

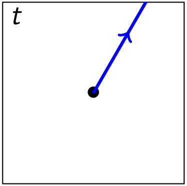
Check: f converges in a disk $\Longrightarrow$ $\mathscr{B} f$ is entire and $f_\theta =$ ordinary sum

Properties of Borel summation

When it exists, the Borel sum f_θ of $f = \sum_{n=0}^{\infty} a_n x^{n+1}$ is:

- defined in a sector centred at $x = 0$ with opening angle π





- asymptotic to f:

$$f_\theta(x) = \sum_{n=0}^{N} a_n x^{n+1} + O(x^{N+1}) \qquad \text{as } x \to 0 \text{ in domain of } f_\theta$$

- compatible with algebraic operations:

$$(f+g)_\theta = f_\theta + g_\theta \qquad (fg)_\theta = f_\theta g_\theta \qquad \frac{df_\theta}{dx} = \left(\frac{df}{dx}\right)_\theta$$

$\therefore$ takes "formal" solutions of polynomial ODEs to "actual" solutions

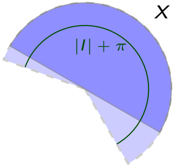
Definition

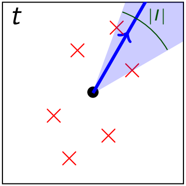
An angle θ is **singular for** $f \in x\mathbb{C}[[x]]$ if f is not Borel summable along the corresponding ray $e^{i\theta}\mathbb{R}_{\geq 0}$.

For I an open interval containing no singular angles:

$$f_\theta = f_{\theta'} \qquad \forall \theta, \theta' \in I$$

$\therefore$ can glue to Borel sum f_I defined on domain of opening angle $|I| + \pi$



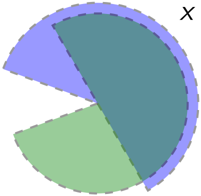


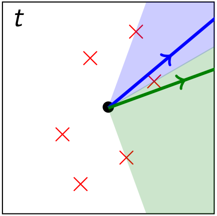
Theorem (Watson)

I nonsingular $\implies$ f_I is the unique function on its domain that is

- *analytic away from* $x = 0$
- *asymptotic to* f *as* $x \to 0$
- *of "Gevrey class":* $\sup |f_I^{(n)}| \leq M^n \cdot (n!)^2$ *on compact subsectors*

Stokes phenomenon: sum jumps as we cross a singular $\theta \in S^1$





Example: If $\mathscr{B}(f)$ has a pole at $t_0 \in \mathbb{R}_{\geq 0} e^{i\theta}$, then

$$f_{\theta-} - f_{\theta_+} = 2\pi i \mathrm{Res}(\mathscr{B}(f); t_0) e^{-t_0/x}$$

Example: If $\mathscr{B}(f)$ has logarithmic branching at $t_0 \in \mathbb{R}_{\geq 0} e^{i\theta}$, then

$$f_{\theta-} - f_{\theta_+} = \underbrace{\delta_\theta f}_{\substack{\mathscr{L} \text{ of} \\ \text{difference} \\ \text{of branches}}} e^{-t_0/x}$$

Resurgence (Écalle 1980s)

Morally: a formal power series is "resurgent" if it is Borel summable in most directions.

Definition (∃ other conventions)

A series $f \in x\mathbb{C}[[x]]$ is **resurgent** if $\mathscr{B}(f) \in \mathbb{C}[[t]]$ extends to a (possibly multi-valued) analytic function on $\mathbb{C}$ with only isolated singularities whose branches grow at most exponentially as $t \to \infty$.

NB: Stokes phenomenon $\implies$ need to track exponentially small corrections

Definition

A **resurgent trans-series** is a (possibly infinite) formal sum of the form

$$W = \sum_i f_i e^{-t_i/x}$$

where $f_i \in x\mathbb{C}[[x]]$ are resurgent series and $t_i \in \mathbb{C}$.

On the space of resurgent series:

- addition, multiplication, $\frac{d}{dx}$
- Borel sum: integrate over lifts of rays to Riemann surfaces $M(f_i)$
- Stokes phenomenon: unique transformation $W \mapsto S_\theta W$ such that

$$(\text{Borel sum of } W \text{ for } \theta^-) = (\text{Borel sum of } S_\theta W \text{ for } \theta^+),$$

Rmk: $S_\theta W = W +$ exponentially small corrections

Problem: even if f_i, t_i are real, the Borel sum with $\theta = 0$ need not be, due to singularities when $t \in \mathbb{R}_{\geq 0}$. This is an issue since physical quantities like energy are real numbers!

Solution: average over all contours $t \to +\infty$, dodging singularities

Theorem (Écalle)

If $f_i \in x\mathbb{R}[[x]]$ and $t_i \in \mathbb{R}$ then the Borel sum of $\sqrt{S_{\theta=0}} \cdot W$ in the direction $\theta = 0^+$ is real-valued for $x > 0$, whenever it converges.