Poisson modules and degeneracy loci

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1. Overview

We examine the degeneracy loci of holomorphic Poisson structures—the subvarieties where the rank of the Poisson tensor drops. We discuss:

- Bondal's conjecture regarding degeneracy loci on Fano manifolds
- The singularities of degeneracy loci
- The "residues" of Poisson structures

Throughout, X is a connected complex manifold, $\omega_{\mathsf{X}} = \Omega^{top}_{\mathsf{X}}$ is the canonical sheaf (the holomorphic volume forms), $\mathscr{X}^k_{\mathsf{X}}$ is the sheaf of holomorphic k-derivations on X and

$$\sigma \in \Gamma(\mathsf{X}, \mathscr{X}^2_{\mathsf{X}})$$

is a holomorphic Poisson structure.

2. Degeneracy loci

For $k \in \mathbb{Z}_{\geq 0}$, the $2k^{th}$ ***degeneracy locus of*** σ is the analytic subvariety

$$\begin{aligned} \mathsf{D}_{2k}(\sigma) &= \{x \in \mathsf{X} \mid \operatorname{rank}(\sigma|_x) \leq 2k\} \\ &= \mathsf{Zeros}\left(\sigma^{\wedge(k+1)}\right), \end{aligned}$$

which is the union of all symplectic leaves of (complex) dimension $\leq 2k$.

3. Bondal's conjecture

A ***Fano manifold*** is a compact complex manifold X with $c_1(\mathsf{X}) > 0$ (eg., projective space $\mathbb{P}^d$).

Conjecture (Bondal [2]). *If X is Fano, then $\mathsf{D}_{2k}(\sigma)$ has a component of dimension $\geq 2k+1$ for all $k < \frac{1}{2}\dim \mathsf{X}$.*

Theorem (Beauville [1], Polishchuk [6]). *If X is Fano and $M = \max \operatorname{rank}(\sigma)$, then $\mathsf{D}_{M-2}(\sigma)$ has a component of dimension $\geq M-1$. Hence, Bondal's conjecture is true for Fano threefolds.*

References

[1] A. Beauville, *Holomorphic symplectic geometry: a problem list*, 1002.4321.

[2] A. I. Bondal, *Non-commutative deformations and Poisson brackets on projective spaces*, Max-Planck-Institute Preprint (1993), no. 93-67.

[3] B. L. Feĭgin and A. V. Odesskiĭ, *Sklyanin's elliptic algebras*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 3, 45–54, 96.

[4] M. Gualtieri and B. Pym, *Poisson modules and degeneracy loci*, 1203.4293.

[5] S. Holzer and O. Labs, surfex 0.90, Tech. report, University of Mainz, University of Saarbrücken, 2008, www.surfex.AlgebraicSurface.net.

[6] A. Polishchuk, *Algebraic geometry of Poisson brackets*, J. Math. Sci. (N. Y.) **84** (1997), no. 5, 1413–1444. Algebraic geometry, 7.

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4. Main result: evidence for Bondal's conjecture

Theorem. *Let X be a connected Fano manifold of dimension $2n$. Then every component of $\mathsf{D}_{2n-2}(\sigma)$ has dimension $\geq 2n-1$, and $\mathsf{D}_{2n-4}(\sigma)$ has at least one component of dimension $\geq 2n-3$.*

Corollary. *Bondal's conjecture is true for Fano manifolds of dimension four.*

Right: An outline of the proof of the theorem for Poisson structures that are generically symplectic. In this case, $\mathsf{D}_{2n-2}(\sigma)$ is a hypersurface—the zero locus of the anti-canonical section σ^n. This poster discusses steps **4A** and **4B**.

Use Lefschetz' hyperplane theorem to show that $\mathsf{D}_{2n-2}(\sigma)$ is singular.

↓

4A Control the ***singular locus*** $\mathsf{W} = \mathsf{D}_{2n-2}(\sigma)_{sing}$.

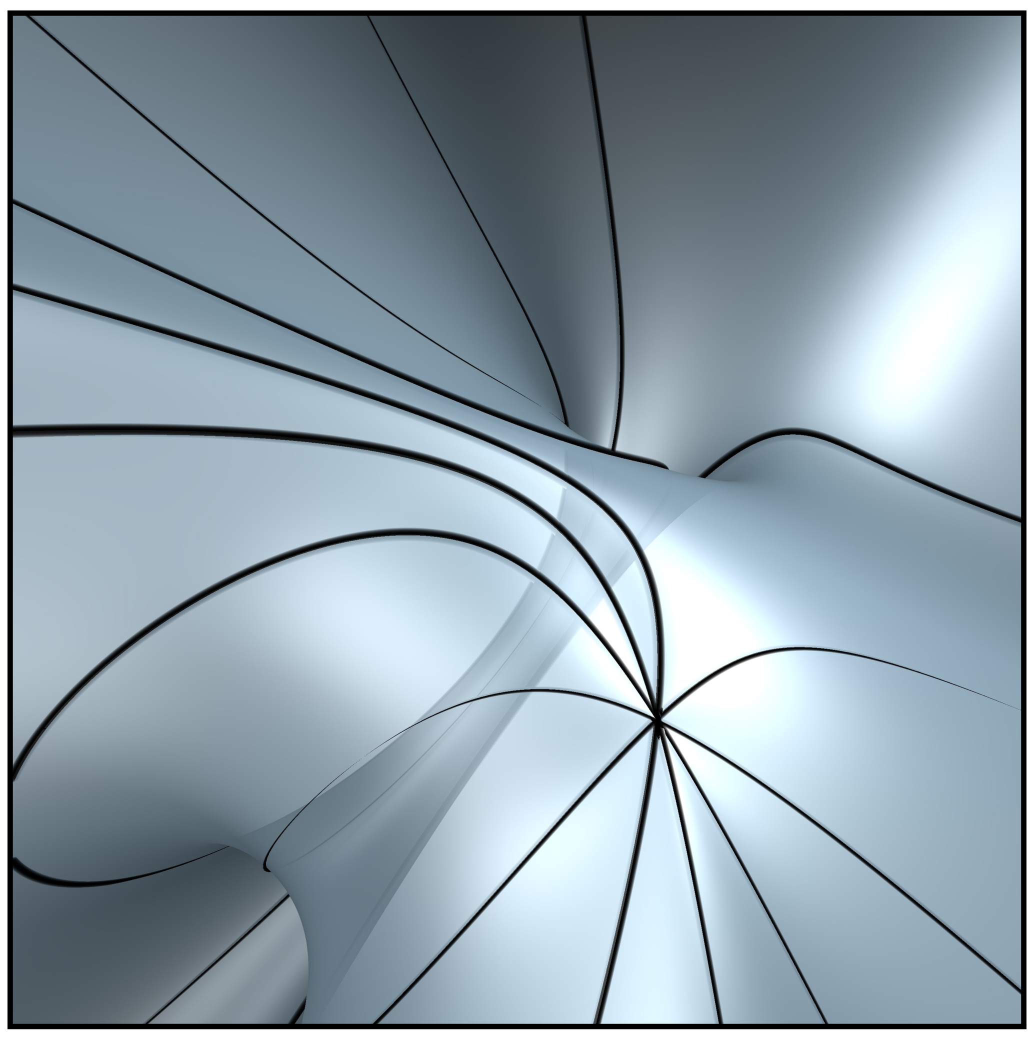
↓

4B Use ***modular residues*** to study the Poisson module $\omega_{\mathsf{X}}|_{\mathsf{W}}$.

↓

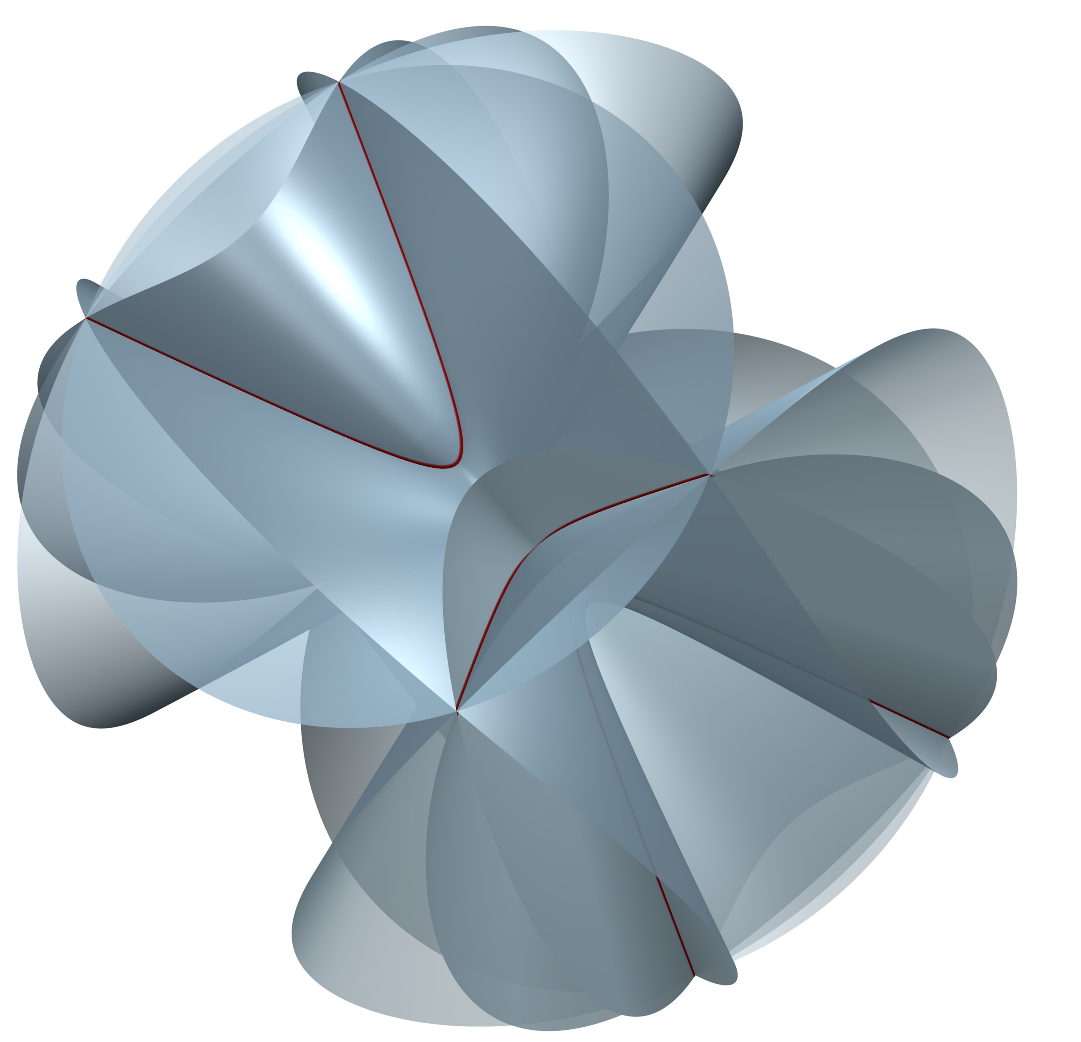
Use Bott's vanishing theorem on $\omega_{\mathsf{X}}|_{\mathsf{W}}$ and $c_1(\mathsf{X}) > 0$ to bound $\dim \mathsf{D}_{2n-4}(\sigma)$.

Example: Feigin-Odesskii $\mathbb{P}^4$

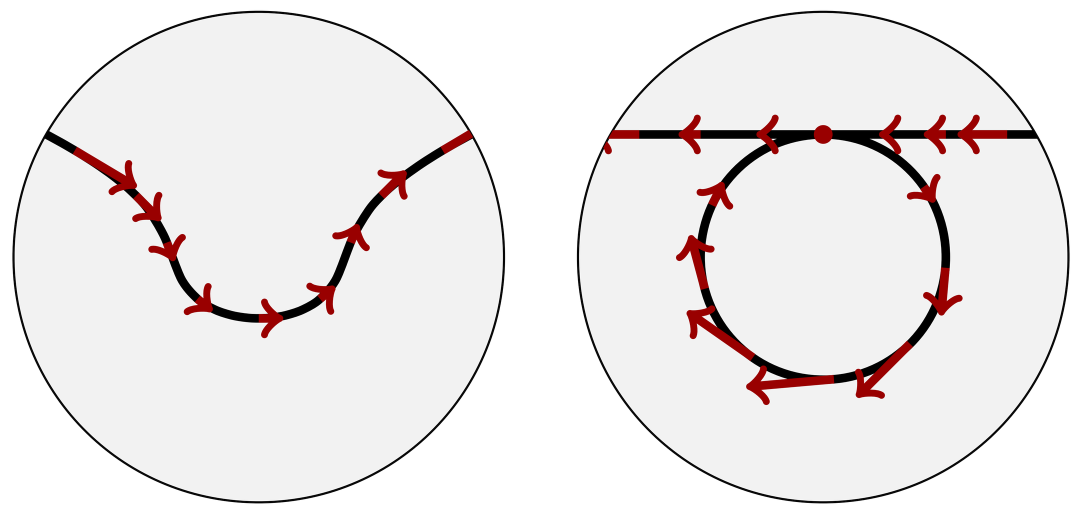


Above: A cross-section of the generically symplectic Poisson structure $q_{5,1}$ on $\mathbb{P}^4$ defined by Feigin and Odesskii [3]. The black curves represent the 2D symplectic leaves. Their closures intersect at the singular locus of the degeneracy hypersurface, which is an elliptic curve in $\mathbb{P}^4$.

Below: Projection of the 2D symplectic leaves of $q_{5,1}$ to $\mathbb{P}^3$ gives the Poisson structure $q_{4,1}$, which vanishes on an elliptic curve (red).



Example: residues on $\mathbb{P}^2$



Above: The zero locus of a Poisson structure on $\mathbb{P}^2$ is a cubic curve. The red arrows represent the modular residue $\operatorname{Res}^0_{mod}(\sigma)$.

4A. The singular locus

Degeneracy loci are highly singular:

$$\mathsf{D}_{2k-2}(\sigma) \subset \mathsf{D}_{2k}(\sigma)_{sing}$$

whenever $\mathsf{D}_{2k}(\sigma) \neq \mathsf{X}$. To prove the main theorem, we use the following facts about the singular locus of a degeneracy hypersurface:

Theorem. *Suppose $\dim \mathsf{X} = 2n$ and σ is generically symplectic. Let $\mathsf{W} = \mathsf{D}_{2n-2}(\sigma)_{sing}$. If $\mathsf{W} \neq \varnothing$, then*

$$\dim \mathsf{W} \geq 2n-3.$$

If $\dim \mathsf{W} = 2n-3$, then $\mathcal{O}_{\mathsf{W}}$ has a locally-free resolution obtained from the natural Poisson structure

$$\sigma_\nabla \in \Gamma\left(\mathsf{X}, \wedge^2 \mathcal{A}_{\omega_{\mathsf{X}}}\right)$$

on the Atiyah algebroid of the canonical sheaf (see right).

$$\begin{array}{c} 0 \\ \downarrow \\ \omega^2_{\mathsf{X}} \\ \downarrow{\scriptstyle \sigma^n_\nabla} \\ \mathcal{A}^\vee_{\omega_{\mathsf{X}}} \otimes \omega_{\mathsf{X}} \\ \downarrow{\scriptstyle \sigma^\sharp_\nabla \otimes 1} \\ \mathcal{A}_{\omega_{\mathsf{X}}} \otimes \omega_{\mathsf{X}} \\ \downarrow{\scriptstyle \sigma^n_\nabla} \\ \mathcal{O}_{\mathsf{X}} \\ \downarrow \\ \mathcal{O}_{\mathsf{W}} \\ \downarrow \\ 0 \end{array}$$

Corollary. *If $\dim \mathsf{W} = 2n-3$, then W is a Gorenstein scheme with dualizing sheaf $\omega^{-1}_{\mathsf{X}}|_{\mathsf{W}}$ and fundamental class $[\mathsf{W}] = c_1(\mathsf{X})c_2(\mathsf{X}) - c_3(\mathsf{X})$.*

4B. Modular residues

Since the modular vector field Z of σ is defined modulo Hamiltonian vector fields, $Z \wedge \sigma^k$ is well-defined wherever $\operatorname{rank}(\sigma) \leq 2k$. Since Z is tangent to every degeneracy locus, we may define the ***modular residue***

$$\begin{aligned} \operatorname{Res}^k_{mod}(\sigma) &= Z \wedge \sigma^k|_{\mathsf{D}_{2k}(\sigma)} \\ &\in \Gamma\left(\mathsf{D}_{2k}(\sigma), \mathscr{X}^{2k+1}_{\mathsf{D}_{2k}(\sigma)}\right). \end{aligned}$$

This residue is often nonzero, suggesting that there may be a local reason for the prediction $\dim \mathsf{D}_{2k}(\sigma) \geq 2k+1$ of Bondal's conjecture.

Theorem. *The modular residue is given by*

$$\operatorname{Res}^k_{mod}(\sigma) = \frac{-1}{k+1}\operatorname{Tr}(D\sigma^{k+1})$$

where $D\sigma^{k+1}$ is the derivative of σ^{k+1} along $\mathsf{D}_{2k}(\sigma)$, and $\operatorname{Tr} : \Omega^1_{\mathsf{X}} \otimes \mathscr{X}^{2k+2}_{\mathsf{X}} \to \mathscr{X}^{2k+1}_{\mathsf{X}}$ is the contraction.

Notice that if $\dim \mathsf{X} = 2n$ and σ is generically symplectic, then $\mathsf{W} = \mathsf{D}_{2n-2}(\sigma)_{sing}$ is the zero locus of $D\sigma^n$. Therefore, $Z \wedge \sigma^{n-1}|_{\mathsf{W}} = 0$ and Z is tangent to the $(2n-2)$-dimensional leaves of W. Hence, $\omega_{\mathsf{X}}|_{\mathsf{W}}$ has a flat connection along the leaves, and Bott's vanishing theorem applies.