Holonomic Poisson manifolds

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Based on joint works with

Travis Schedler [arXiv] [DOI]
Mykola Matviichuk & Travis Schedler [in prep.]

Holonomicity: nondegeneracy condition for Poisson structures, based on $\mathcal{D}$-modules

- Generalization of symplectic geometry (c.f. b-, log-, ...)
- Covers many natural examples
- Often makes deformation theory tractable
- Inspired by work of Etingof–Schedler on Poisson traces, Kaledin and Namikawa on symplectic singularities

X complex manifold

$\wedge^\bullet \mathcal{T}_X$ = sheaf of holomorphic polyvector fields

Holomorphic Poisson structure:

$$\pi \in \Gamma(X, \wedge^2 \mathcal{T}_X) = H^0(\wedge^2 \mathcal{T}_X) \qquad [\pi, \pi] = 0 \in H^0(\wedge^3 \mathcal{T}_X)$$

$$\pi = \sum_{ij} \pi^{ij}(z) \partial_{z_i} \wedge \partial_{z_j} \qquad \text{(no } \bar{z} \text{ appearing)}$$

Classification problem (hopeless): what are the possible pairs (X, π)?

moduli space $\qquad \mathcal{M}_{\mathsf{Pois}} := \dfrac{\{(X, \pi) \text{ compact}\}}{\sim}$

irreducible decomposition $\qquad \mathcal{M}_{\mathsf{Pois}} = \bigcup_i \mathcal{M}_{\mathsf{Pois},i}$

More tractable: describe neighbourhood of $[(X, \pi)] \in \mathcal{M}_{\mathsf{Pois}}$

Deformations are governed by Poisson cohomology

$$\mathcal{O}_X \xrightarrow{\mathrm{d}_\pi} \mathcal{T}_X \xrightarrow{\mathrm{d}_\pi} \wedge^2\mathcal{T}_X \longrightarrow \cdots \qquad \mathrm{d}_\pi = [\pi, -]$$

Local picture: deform germ $(X, \pi)_p$

$$\pi \rightsquigarrow \pi_t = \pi + t\eta + \cdots \qquad \eta \in \wedge^2\mathcal{T}_{X,p}$$

$$\begin{aligned}[\pi_t, \pi_t] &= [\pi, \pi] + 2t[\pi, \eta] + \cdots \\ &= 2t\mathrm{d}_\pi\eta + \cdots\end{aligned}$$

Trivial deformations:

$$\pi_t = \phi_t^*\pi \implies \eta = \mathscr{L}_\xi\pi = \mathrm{d}_\pi\xi$$

$$\frac{\{\text{1st-order defs}\}}{\sim} \cong \underbrace{\mathcal{H}^2_p(\wedge^\bullet\mathcal{T}_X, \mathrm{d}_\pi)}_{\text{stalk cohomology}}$$

Global picture: deform (X, π), including (generalized) complex structure of X

Local defs on opens U_i, glue by isomorphisms on $U_i \cap U_j$

$$\frac{\{\text{1st-order defs}\}}{\sim} \cong \underbrace{H^2(\wedge^\bullet\mathcal{T}_X, \mathrm{d}_\pi)}_{\text{hypercohomology}}$$

Use any model (Čech, Dolbeault, ...)

Local or global: cohomology only depends on $(\wedge^\bullet\mathcal{T}_X, \mathrm{d}_\pi) \in D(X)$

Higher order defs: Schouten bracket on $\wedge^\bullet\mathcal{T}_X \implies L_\infty$ structure on cohomology $\implies$ nonlinear obstruction map $H^2 \to H^3$

Case (X,π) nondegenerate (symplectic): Darboux coordinates

$$\pi = \sum_i \partial_{q^i} \wedge \partial_{p_i} \qquad \omega := \pi^{-1} = \sum_i dp_i \wedge dq^i$$

Nondegeneracy is an open condition $\implies$ defs. are locally trivial

$$(\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi) \cong (\Omega^\bullet_X, \mathrm{d}) \cong \underbrace{\mathbb{C}_X}_{\text{abelian!}} \qquad \text{in } D(X)$$

$$\mathcal{H}^2(\wedge \mathcal{T}_X, \mathrm{d}_\pi) = 0 \qquad H^2(\wedge \mathcal{T}_X, \mathrm{d}_\pi) \cong H^2(X;\mathbb{C})$$

Higher-order deformations: unobstructed (cf. Bogomolov, Tian–Todorov)

Case (X,π) trivial ($\pi = 0$): all Poisson germs give local deformations

$$\mathrm{d}_\pi = 0 \qquad \implies \qquad \mathcal{H}^\bullet = \wedge^\bullet \mathcal{T}_X \qquad \implies \qquad \dim_{\mathbb{C}} \mathcal{H}^\bullet = \infty$$

Higher-order deformations: highly obstructed

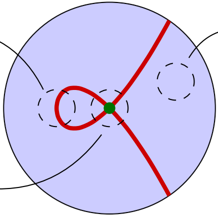
$\dim X = 2$: Poisson surfaces (Goto)

$\pi \in \wedge^2 \mathcal{T}_X = \det \mathcal{T}_X = K_X^{-1}$ anticanonical line bundle

$\partial X := Zeros(\pi) \subset X$ curve $\quad X^\circ = X \setminus \partial X$ symplectic

singular locus $\partial^2 X \subset \partial X$ (assume quasi-homogeneous for simplicity)

$$\pi = y\partial_y \wedge \partial_z$$
$$\mathcal{H}^0 \cong \mathbb{C}$$
$$\mathcal{H}^1 \cong \mathbb{C} \cdot \partial_z$$



$$\pi = \partial_q \wedge \partial_p$$
$$\mathcal{H}^0 = \mathbb{C}$$

$$\pi = f(u,v)\partial_u \wedge \partial_v$$
$$\mathcal{H}^0 \cong \mathbb{C}, \mathcal{H}^1 \cong H^1(B_\epsilon \cap X^\circ)$$
$$\mathcal{H}^2 = H^2(B_\epsilon \cap X^\circ) + \text{smoothings}$$

$$\begin{aligned} (\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi) &\cong Rj_*\mathbb{C}_{X^\circ} \oplus i_* K_X^{-1}|_{\partial^2 X}[-2] \\ H^\bullet(\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi) &\cong H^\bullet(X^\circ; \mathbb{C}) \oplus \underbrace{H^0(\partial^2 X, K_X^{-1}|_{\partial^2 X})}_{\text{degree 2}} \end{aligned}$$

Obstructions vanish!

$$\begin{array}{rcl} \text{Global:} & \dim H^2 < \infty & \text{automatic for } X \text{ compact} \\ & \Uparrow & \\ \text{Local:} & \dim \mathcal{H}^{\leq 2} < \infty & \text{finitely many gluing params.} \\ & \Uparrow & \\ \text{Local derived:} & \dim \mathcal{H}^{\bullet} < \infty & \end{array}$$

Question: when are stalks of $\mathcal{H}^{\bullet}$ finite-dimensional?

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Answer: use D-modules

$$\begin{aligned} \mathcal{D}_X &= \text{diff. ops. on } \mathcal{O}_X, \text{filtered by order} \\ \operatorname{gr} \mathcal{D}_X &\cong Sym(\mathcal{T}_X) \cong \text{poly. functions on } T^*X \end{aligned}$$

Complex of right $\mathcal{D}_X$-modules:

$$\wedge^\bullet \mathcal{T}_X \otimes \mathcal{D}_X = \left\{ \mathrm{d}_\pi \circlearrowright \wedge^\bullet \mathcal{T}_X \xleftarrow{\text{diff. ops.}} \mathcal{O}_X \circlearrowleft \mathcal{D}_X \right\}$$

$$\begin{array}{c} \operatorname{gr}(\wedge^\bullet \mathcal{T}_X \otimes \mathcal{D}_X) \in QCoh(T^*X) \\ \Big\downarrow \scriptstyle\text{support} \\ \underbrace{Char(X,\pi)}_{\text{characteristic variety}} \subset T^*X \end{array}$$

Definition

(X,π) is **holonomic** if $Char(X,\pi) \subset T^*X$ is Lagrangian.

Consequences of holonomicity

Definition

(X, π) is **holonomic** if $Char(X, \pi) \subset T^*X$ is Lagrangian.

Immediate consequences holonomicity:

- via Kashiwara constructibility: $\dim \mathcal{H}^\bullet < \infty$.
- via Kashiwara + Roos: $\wedge^\bullet \mathcal{T}_X[\dim X]$ is **a perverse sheaf**

$X = \sqcup_\alpha X_\alpha$ $\quad$ X_α locally closed $\quad$ $Char(X, \pi) = \bigcup \overline{N^* X_\alpha}$

$\mathcal{H}^\bullet$ locally constant on X_α $\quad$ $\mathrm{codim}(\mathrm{supp}(\mathcal{H}^k)) \geq k$

Proposition

Holonomicity is:

1. *A local condition: depends only on germs transverse to sympl. leaves*
2. *An open condition in proper families: $\mathcal{M}_{\mathrm{HolPois}} \subset \mathcal{M}_{\mathrm{Pois}}$ Zariski open.*

How to determine $Char(X, \pi)$?

Lemma

$$Char(X, \pi) \subset \bigcup_{L \text{ symp. leaf}} N^*L \subset T^*X$$

Sketch of proof.

1. Hamiltonian flows act homotopically trivially on $\wedge^\bullet \mathcal{T}_X$
2. Thus Hamiltonian vector fields (viewed as functions on T^*X) vanish on $Char(X, \pi)$
3. Zero set of all such functions $= \bigcup_L N^*L$ $\square$

Corollary

Every symplectic manifold is holonomic, with $Char(X, \pi) = \{0\} \subset T^*X$.

How to determine $Char(X,\pi)$? Redux

$$Char(X,\pi) \subset \bigcup_{L \text{ symp. leaf}} N^*L \subset T^*X$$

Definition

A symplectic leaf $L \subset X$ is **characteristic** if $N^*L \subset Char(X,\pi)$

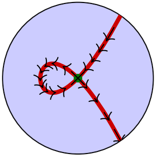
Theorem (P.–Schedler)

1. *symp. leaf L is characteristic $\iff$ modular vector field tangent to L*
2. *(X,π) is holonomic at $p \in X \implies \#\{\text{characteristic } L \mid p \in \overline{L}\} < \infty$ (Conjecturally, $\iff$)*
3. *finitely many modular orbits of sympl. leaves $\implies$ holonomic*
4. *holonomic away from codimension two $\iff$ log symplectic*

Definition (Goto)

(X,π) is **log symplectic** if $\exists$ open dense symplectic $X^\circ = X$, and symplectic form has first-order poles on anticanonical $\partial X = X \setminus X^\circ$

Surfaces: $\dim X = 2$.



$$\partial^2 X \subset \partial X \subset X \supset X^\circ$$

Characteristic leaves: X°, points of $\partial^2 X$

Holonomic $\iff \dim \partial^2 X = 0 \iff \partial X$ reduced $\iff$ log symplectic

Hilbert schemes of surface X: in progress w/ Matviichuk & Schedler

$$\underbrace{Sym^n(X) := X^n/S_n}_{\text{singular Poisson variety}} \quad \longleftarrow \quad \underbrace{Hilb^n(X)}_{\text{Hilbert scheme, smooth Poisson}}$$

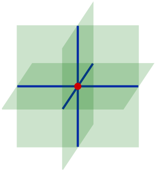
[Bottacin, Mukai], defs: [Hitchin, Ran]

$$Hilb^n(X) \text{ holonomic} \quad \Longleftarrow \quad \partial X \text{ smooth}$$
$$Hilb^2(X) \text{ holonomic} \quad \iff \quad \partial^2 X \text{ always of type } A_k$$

Feigin–Odesskii: "elliptic" Poisson structures $q_{d,r}(E,\zeta)$ on $\mathbb{P}^{d-1}$

P.–Schedler: d odd, $r = 1 \implies$ holonomic $\rightsquigarrow$ irred. cpts. of $\mathcal{M}_{\mathrm{Pois}}$

Assume (X, π, ω) log symplectic, ∂X simple normal crossings:



- all irreducible components $\partial_i X \subset X$ are smooth
- all intersections $\partial_{i_1} X \cap \cdots \cap \partial_{i_j} X$ are transverse (components called **strata**, automatically Poisson)

Stable local normal form along strata $S^\circ \subset S \subset X$:

$$\omega \sim \sum_i \mathrm{d}p_i \wedge \frac{\mathrm{d}q_i}{q_i} + \sum_{i<j} B_{ij} \frac{\mathrm{d}q_i}{q_i} \wedge \frac{\mathrm{d}q_j}{q_j}$$

$$B_{ij} := \tfrac{1}{(2\pi\sqrt{-1})^2} \int_{\Sigma_{ij}} \omega \in \mathbb{C} \qquad \Sigma_{ij} := \text{torus wrapping } \partial_i X \cap \partial_j X \text{ near } S^\circ$$

From which follows:

- $\operatorname{corank}(\pi|_{S^\circ}) = \operatorname{corank}(B_{ij})$
- leaves in S° are characteristic $\iff (1, \ldots, 1) \in \operatorname{img}(B_{ij})$

Vector fields tangent to ∂X:

$$\mathcal{T}_X(-\log \partial X) \subset \mathcal{T}_X$$

Exterior powers $\implies$ filtration by "weight" compatible with d_π:

$$\wedge^\bullet \mathcal{T}_X(-\log \partial X) = W_0 \subset W_1 \subset \cdots \subset W_{\dim X} = \wedge^\bullet \mathcal{T}_X$$

$$W_j / W_{j-1} \cong \bigoplus_{\operatorname{codim} S = j} i_{S*} \underbrace{\wedge^\bullet \mathcal{T}_S(-\log \partial S) \otimes \det \mathcal{N}_S}_{=:\mathcal{C}_S^\bullet}[-j]$$

Theorem (Matviichuk–P.–Schedler)

If (X, π) log symplectic with ∂X normal crossings

1. (X, π) *holonomic* $\iff$ *# char leaves locally finite*
2. *when holonomic, each $\mathcal{L}_{S^\circ} := \mathcal{H}^0(\mathcal{C}_S|_{S^\circ})$ is either trivial, or a rank one local system on S°, and we have an isomorphism in $D(X)$:*

$$\mathrm{gr}^W(\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi) \cong \bigoplus_{S^\circ \text{ characteristic}} Rj_*^{nr} \mathcal{L}_{S^\circ}[-\operatorname{codim} S]$$

Consequences for moduli space

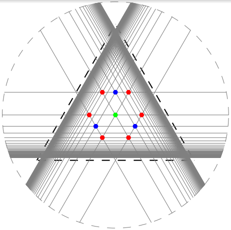
Corollary

If (X,π) holonomic normal crossings (away from codim 4), then

$$H^2(\wedge^\bullet \mathcal{T}_X, \mathrm{d}_\pi) \cong H^2(X^\circ;\mathbb{C}) \oplus \bigoplus_{S^\circ \text{ codim 2 } char} H^0(Rj_*^{nr}\mathcal{L}_{S^\circ})$$

$$\dim H^2 = b_2(X^\circ) + \#\ \text{“smoothable” codim-two strata}$$

$$[\omega] \in \mathbb{P}(H^2(X^\circ;\mathbb{C})) =$$



Theorem (Matviichuk–P.–Schedler)

In nice cases (e.g. toric), deformations governed by explicit formal dgLa

$$\textit{nbhd of } (X,\pi) \in \mathcal{M}_{\mathrm{Pois}} \cong \frac{\textit{union of linear subspaces in } H^2(\wedge^\bullet \mathcal{T}_X)}{\textit{linear action of } (\mathbb{C}^*)^k \rtimes \textit{finite}}$$

Example: projective space

$$X = \mathbb{P}^{2n} = \{[x_0 : \cdots : x_{2n}]\} \qquad \partial_i X = \{x_i = 0\} \qquad \omega = \sum B_{ij} \frac{\mathrm{d}x_i}{x_i} \wedge \frac{\mathrm{d}x_j}{x_j}$$

When is $S_{ij} := \partial_i X \cap \partial_j X$ smoothable?

generically symplectic:	$B_{ij} \neq 0$
$H^0(Rj_*^{nr}\mathcal{L}_{S_{ij}^\circ}) \neq 0$:	$\dfrac{B_{jk}+B_{ki}}{B_{ij}} \in \mathbb{Z}_{\geq 0} \quad \forall k \neq i,j$

Theorem (Matviichuk–P.–Schedler + computer)

$\mathcal{M}_{\mathrm{Pois}}$ has $\sim$40 irreducible components corresponding to holonomic Poisson structures on $\mathbb{P}^4$ admitting normal crossings degenerations.

$q_{5,1}$ $\quad$ $q_{5,2}$ $\quad$ new $\quad$ new $\quad$ new $\quad \cdots$

THANK YOU!