Poisson Structures and Lie Algebroids in Complex Geometry

by

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Abstract

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This thesis is devoted to the study of holomorphic Poisson structures and Lie algebroids, and their relationship with differential equations, singularity theory and noncommutative algebra.

After reviewing and developing the basic theory of Lie algebroids in the framework of complex analytic and algebraic geometry, we focus on Lie algebroids over complex curves and their application to the study of meromorphic connections. We give concrete constructions of the corresponding Lie groupoids, using blowups and the uniformization theorem. These groupoids are complex surfaces that serve as the natural domains of definition for the fundamental solutions of ordinary differential equations with singularities. We explore the relationship between the convergent Taylor expansions of these fundamental solutions and the divergent asymptotic series that arise when one attempts to solve an ordinary differential equation at an irregular singular point.

We then turn our attention to Poisson geometry. After discussing the basic structure of Poisson brackets and Poisson modules on analytic spaces, we study the geometry of the degeneracy loci—where the dimension of the symplectic leaves drops. We explain that Poisson structures have natural residues along their degeneracy loci, analogous to the Poincaré residue of a meromorphic volume form. We discuss the local structure of degeneracy loci that have small codimensions, and place strong constraints on the singularities of the degeneracy hypersurfaces of log symplectic manifolds. We use these results to give new evidence for a conjecture of Bondal.

Finally, we discuss the problem of quantization in noncommutative projective geometry. Using Cerveau and Lins Neto's classification of degree-two foliations of projective space,

we give normal forms for unimodular quadratic Poisson structures in four dimensions, and describe the quantizations of these Poisson structures to noncommutative graded algebras. As a result, we obtain a (conjecturally complete) list of families of quantum deformations of projective three-space. Among these algebras is an “exceptional” one, associated with a twisted cubic curve. This algebra has a number of remarkable properties: for example, it supports a family of bimodules that serve as quantum analogues of the classical Schwarzenberger bundles.

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Chapter 1

Introduction

1.1 Motivation

This thesis is motivated by three specific problems that, at first glance, seem to have little to do with one another:

Question 1.1.1. *Do divergent asymptotic series expansions, such as*

$$\frac{1}{z}\int_0^\infty \frac{e^{-t/z}}{1+t}\,dt \sim \sum_{n=0}^{\infty}(-1)^n n! z^n$$

have some intrinsic geometric meaning?

Conjecture 1.1.2 (Bondal [19])**.** *If σ is a holomorphic Poisson structure on a Fano manifold X, and if $2k < \dim \mathsf{X}$, then the locus in X defined by the symplectic leaves of dimension no greater than $2k$ is a subvariety of dimension at least $2k+1$.*

Problem 1.1.3. *Classify the Artin–Schelter regular algebras [4, 6] of global dimension four.*

Our viewpoint in this thesis is that these seemingly disparate issues are linked by a common theme: the presence of a geometric structure known as a holomorphic Lie algebroid. Thus, by developing an understanding of Lie algebroids, we can shed some light on all of these questions. Therefore, this thesis is devoted to the study of Poisson structures and Lie algebroids in the complex analytic setting, and in particular their connection with differential equations, algebraic geometry, singularity theory, and noncommutative algebra.

To address Question 1.1.1 regarding asymptotic expansions, we discuss Lie algebroids on complex curves and their global counterparts, Lie groupoids. The latter spaces serve as the natural domains for the parallel transport of meromorphic connections (i.e., ordinary differential equations with singularities). We show that they give a canonical, geometric way to obtain holomorphic functions from certain divergent series, such as $\sum_{n=0}^{\infty}(-1)^n n! z^n$, that arise when one attempts to solve a differential equation at an irregular singular point.

For Conjecture 6.1.1, we develop the complex analytic geometry of Poisson structures and their modules, in which various Lie algebroids play an important role. By combining these methods with more classical techniques from algebraic geometry, we are able to study the local and global structure of the degeneracy loci—where the dimension of the symplectic leaves drops. This approach allows us to prove, for example, that Bondal's conjecture is true for Fano manifolds of dimension four.

We also explore in some detail the geometry of Poisson structures on projective space. We recall the classification [34, 101, 117] of Poisson structures on $\mathbb{P}^3$ and use our understanding of the geometry to describe the quantizations. As a result, we give a conjectural classification of the noncommutative deformations of $\mathbb{P}^3$—an important subset of the larger classification sought in Problem 1.1.3.

Throughout the thesis, we find that the language of algebraic geometry—particularly coherent sheaves—can be extremely useful in describing the geometry of Poisson structures and Lie algebroids. While we focus in this thesis on the complex analytic situation, wherein we can take the greatest advantage of these methods, the author believes that the ideas can also be useful in the C^∞ world.

To make this philosophy more concrete, we will now consider a simple and well-known example from Poisson geometry that already displays many of these interesting complexities. We shall return to this example at many points in the thesis. On the one hand, we shall take advantage of its familiarity to illustrate the definitions and methods we develop. On the other, we shall explore some aspects of its geometry that may be less routine. Owing to its central role in the thesis, we shall refer to it throughout the text as The Example.

1.2 The Example

For the moment, we assume that the reader has some basic familiarity with Poisson structures and Lie algebroids. The formal definitions will be reviewed in later chapters.

Let x, y and z be linear coordinates on $\mathbb{C}^3$ (or $\mathbb{R}^3$), and consider the Poisson structure with elementary brackets

$$\begin{aligned} \{x, y\} &= 2y \\ \{x, z\} &= -2z \\ \{y, z\} &= x, \end{aligned}$$

i.e., the standard Lie–Poisson structure on the dual of the $\mathfrak{sl}(2)$ Lie algebra. Let

$$\sigma = x\partial_y \wedge \partial_z + 2y\partial_x \wedge \partial_y - 2z\partial_x \wedge \partial_z$$

be the corresponding bivector field.

The two-dimensional symplectic leaves of this Poisson structure (the surface to which σ is tangent) are illustrated in Figure 1.1). They are the level sets of the Casimir function

$$f = x^2 + 4yz.$$

For $c \in \mathbb{C} \setminus \{0\}$, the level set $f^{-1}(c)$ is smooth, giving a holomorphic symplectic manifold. However, 0 is not a regular value. Instead, the preimage $\mathsf{Y} = f^{-1}(0)$ is the famous nilpotent cone, which plays an important role in representation theory. Thus, it is a very interesting space, but it is not a manifold; rather, it has a singularity at the origin in $\mathbb{C}^3$. This singular point is special from the point of view of Poisson geometry: it is the only zero-dimensional symplectic leaf. Notice that, although Y is not a manifold, it is a perfectly good algebraic variety and its ring of functions inherits a Poisson structure that vanishes at the singular point. In this sense, it is a Poisson subspace of $\mathbb{C}^3$. Thus, although we started our discussion with a holomorphic Poisson structure on a smooth manifold, we were very quickly led to a singular space that forms an important and interesting feature of the geometry.

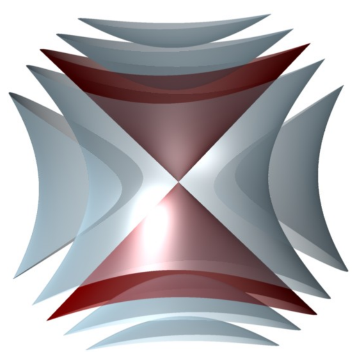


Figure 1.1: Some symplectic leaves of the Poisson structure σ on the dual of the $\mathfrak{sl}(2)$ Lie algebra. The special singular variety Y—the nilpotent cone—is shown in red.

.

Suppose now that we wish to describe the holomorphic vector fields that are tangent to all of the two-dimensional symplectic leaves. Any such vector field Z must satisfy the equation $\sigma \wedge Z = 0$. In other words, the kernel of the vector bundle map

$$\begin{array}{rcccc} \phi_\sigma & : & \mathsf{T}\mathbb{C}^3 & \to & \Lambda^3\mathsf{T}\mathbb{C}^3 \\ & & Z & \mapsto & \sigma \wedge Z \end{array}$$

should be thought of as the "tangent bundle" to the two-dimensional symplectic leaves. But there is a problem: the Poisson structure vanishes at the origin, and so the dimension of the kernel of ϕ_σ jumps at this point. Hence, this "tangent bundle" is not actually a bundle at all.

In contrast, if we consider the corresponding map $\phi_\sigma : \mathcal{T}_{\mathbb{C}^3} \to \Lambda^3\mathcal{T}_{\mathbb{C}^3}$ on the sheaves of sections, we may form the sheaf-theoretic kernel

$$\mathcal{F} = \mathcal{K}er(\phi_\sigma) \subset \mathcal{T}_{\mathbb{C}^3}.$$

Thus, $\mathcal{F}$ is the sheaf whose sections are exactly those vector fields that are tangent to all of the two-dimensional leaves. These sections remain tangent to the leaves when added together or multiplied by arbitrary holomorphic functions and so $\mathcal{F}$ is naturally a module over the sheaf of holomorphic functions. One may verify that this module is generated by the Hamiltonian vector fields $X_x = 2y\partial_y - 2z\partial_z$, $X_y = x\partial_z - 2y\partial_x$ and $X_z = 2z\partial_x - x\partial_y$. In other words, every vector field on $\mathbb{C}^3$ that is tangent to all of the two-dimensional leaves can be written as a linear combination

$$Z = f_1 X_x + f_2 X_y + f_3 X_z$$

where f_1, f_2 and f_3 are holomorphic functions.

Away from the origin, the vector fields X_x, X_y and X_z span a rank-two integrable subbundle of the tangent bundle, and thus they are linearly dependent. Correspondingly, the sheaf $\mathcal{F}|_{\mathbb{C}^3\setminus\{0\}}$ is a locally free module of rank two over the sheaf of holomorphic functions on $\mathbb{C}^3 \setminus \{0\}$. However, at the origin, this sheaf is no longer locally free; rather, the stalk $\mathcal{F}_0$ is a rank-three module, meaning that in a neighbourhood of 0, it is not possible to express a general section of $\mathcal{F}$ as a linear combination of only two of the three vector fields X_x, X_y and X_z. To describe all of the vector fields tangent to the two-dimensional leaves, all three of these generators are truly required. Furthermore, we see that the Lie bracket of vector fields gives a bracket $[\cdot,\cdot] : \mathcal{F} \times \mathcal{F} \to \mathcal{F}$ and the natural inclusion $\mathcal{F} \hookrightarrow \mathcal{T}_{\mathsf{X}}$ of sheaves is obviously compatible with this bracket. In other words, $\mathcal{F}$ is an example of a "Lie algebroid" that is not a vector bundle and its structure encodes some interesting information about the geometry of the symplectic leaves in a neighbourhood of the origin.

1.3 Guiding principles

The inevitable conclusion of our discussion of The Example so far is that, even if we set out to understand Lie algebroids that are vector bundles on manifolds, we soon encounter new objects that look like Lie algebroids, except that they are *not* vector bundles and they live on spaces that are *not* smooth. Moreover, these objects tend to display some of the most interesting features of the geometry. We emphasize that this discussion is still valid if we replace $\mathbb{C}^3$ by $\mathbb{R}^3$ and holomorphic functions with C^∞ ones, and thus sheaves and singularities are equally present and interesting in the smooth category.

The author's basic contention in this thesis is that these singular objects should be admired for their beauty, embraced for their utility and—as much as possible—treated on equal footing with their smooth counterparts. Thus, we shall expand our definitions to include objects like $\mathcal{F}$ that behave like Lie algebroids, but are not vector bundles, and allow them to live on singular spaces, like Y. By adopting this viewpoint, we are able to take advantage of a host of useful tools from algebraic geometry and singularity theory. As a result, we can obtain much stronger conclusions about holomorphic Lie algebroids than one might traditionally expect to find in the smooth category.

However, most of the basic geometric ideas, such as the modular residues of Poisson structures (defined in Section 5.5) and many of the examples we present (including the "free divisors" in Section 2.3 as well as some constructions of Poisson structures) make equally good sense in the C^∞ or algebraic categories. Indeed, the reader interested in those cases will find a number of proposed problems and conjectures throughout the text that are intended to appeal to his or her tastes. Among them are Problem 2.8.1, which asks for a natural generalization of the Crainic–Fernandes integrability theorem [42]; Problem 2.9.4 and Conjecture 2.9.6, related to the existence of certain special Lie groupoids on complex algebraic varieties; Question 6.6.3 regarding a possible skew-symmetric version of free divisors; Conjecture 7.6.8 regarding the Poisson geometry of the secant varieties to elliptic normal curves; and Problem 8.1.6 which puts forward an inherently geometric programme for the classification of quadratic Poisson structures on $\mathbb{R}^4$. The author hopes that these questions will be of interest and perhaps provide the reader with some inspiration for future work.

As an antidote to the somewhat more abstract algebro-geometric language of sheaves that is required in order to deal efficiently with singularities, the author has attempted to include plenty of concrete examples, as well as several diagrams; he hopes that these additions will help to clarify the geometric intuition.

1.4 Summary of the thesis

The thesis is laid out as follows: Chapter 2 gives an overview of the basic definitions and properties of holomorphic Lie algebroids, holomorphic Lie groupoids and their modules. While most of this material is well known, the approach and emphasis are, perhaps, somewhat unorthodox. We formulate the theory in a more algebro-geometric manner, and include a brief review of analytic spaces and coherent sheaves with the hope that it will make the thesis more accessible to readers from other fields. We recall the notions of logarithmic vector fields and free divisors, which recur at various points in the thesis as both a useful tool and a source of examples. We also mention some differences between the smooth, analytic and algebraic settings.

Chapter 3 consists of new results obtained in joint work with Marco Gualtieri and Songhao Li [72]. We explore in detail the geometry of some very simple Lie algebroids on complex curves (i.e., Riemann surfaces) and the corresponding Lie groupoids. We give several concrete descriptions of the groupoids using blowups and the uniformization theorem. The main theme in this chapter is the relationship between these Lie-theoretic objects and a classical topic in analysis: the study of ordinary differential equations with irregular singularities. The chapter culminates in a proof that the groupoids can be used to extract analytic functions from the divergent series that arise in this context.

In Chapter 4, we begin our discussion of Poisson geometry, reviewing the basic definitions of multiderivations, Poisson structures and Poisson subspaces familiar from the smooth setting. We recall the basic properties of Poisson hypersurfaces and log symplectic manifolds; and the relationship between the symplectic leaves and certain coherent Lie algebroids, which explains why the Lie algebroid $\mathcal{F}$ from The Example fails to be a vector bundle. Most of the material in this chapter is review, but we also introduce the useful notion of a "strong Poisson subspace"—one that is preserved by all of the infinitesimal symmetries of the Poisson structure.

Chapter 5 is devoted to the study of Poisson modules, the analogues in Poisson geometry of vector bundles with flat connections. After recalling the definition, we develop their geometry in detail. We introduce a number of new concepts, including natural Higgs fields that are associated with Poisson modules; "adapted" modules that are flat along the symplectic leaves; Lie bialgebroids that are associated with Poisson line bundles; and residues for Poisson line bundles, which are natural tensors supported on Poisson subspaces.

Chapter 6 discusses the properties of the degeneracy loci of Poisson structures and Lie algebroids, and contrasts them with the classical theory of degeneracy loci of vector bundle maps. This chapter contains several new results—most importantly, a description of the singular locus of the degeneracy divisor on a log symplectic manifold, and a proof of Bondal's conjecture for Fano fourfolds. A number of the results and definitions described in Chapter 4 through Chapter 6 appeared in the joint work [73] with Gualtieri, but they have been substantially reorganized for this thesis, with the goal of a more leisurely and comprehensive presentation.

For the rest of the thesis, we focus our attention on projective spaces. In Chapter 7 we recall the connection between quadratic Poisson structures and Poisson structures on projective space. We then prove a comparison theorem, relating the cohomology of Poisson line bundles on projective space with the cohomology of the corresponding quadratic Poisson structures. We ask when a Poisson line bundle can be used to embed a projective Poisson variety as a Poisson subspace in projective space and show that a Poisson structure on projective space is completely determined by its linearization along any reduced Poisson divisor of degree at least four. We construct an example of a generically symplectic Poisson structure on $\mathbb{P}^4$ that is associated with a linear free divisor in $\mathbb{C}^5$, and show that it is equipped with a natural Lagrangian fibration. We close the chapter with a study of the Poisson structures of Feigin and Odesskii, where our results on degeneracy loci have implications for the secant varieties elliptic normal curves.

Finally, in Chapter 8, we undertake a detailed study of Poisson structures on $\mathbb{P}^3$. Using the remarkable classification results of [34, 101, 117], we give normal forms for the generic unimodular quadratic Poisson structures on $\mathbb{C}^4$. After a brief review of quantization in the context of graded algebras, we describe the deformation quantizations of these Poisson structures. Among them is an algebra associated with a twisted cubic curve, which we construct using a formula of Coll, Gerstenhaber and Giaquinto. As a result, we obtain a conjecturally complete list of (suitably generic) deformations of $\mathbb{P}^3$ as a noncommutative projective scheme.

Chapter 2

Lie algebroids in complex geometry

This chapter contains an overview of Lie algebroids in the complex analytic setting, emphasizing the role is played by singular spaces and sheaves. We therefore begin with a brief review of analytic spaces and coherent sheaves for the benefit of those readers who have less experience with these concepts. Readers who are familiar with these notions are invited to skip to Section 2.2

2.1 Preliminaries

2.1.1 Analytic spaces

We now recall the basic definitions and properties of a complex analytic spaces that we shall need in this thesis, and illustrate them with several examples. Our aim is to be as concrete as possible without going into technicalities, in order to make the reader comfortable with the basic language that we will employ. We refer the reader to [67] for a thorough treatment.

The basic point is that when dealing with geometric structures on complex manifolds, we are often interested in loci that are described as the zero sets of some collection of holomorphic functions; such objects are known as analytic subspaces. The formal definition is as follows. Suppose that X is a complex manifold, and denote by $\mathcal{O}_{\mathsf{X}}$ its sheaf of holomorphic functions. Thus $\mathcal{O}_{\mathsf{X}}$ assigns to every open set $\mathsf{U} \subset \mathsf{X}$ the ring of holomorphic functions on U. Similarly, a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ assigns to every open set U an ideal in $\mathcal{O}_{\mathsf{X}}(\mathsf{U})$ in a way that is compatible with the restriction to smaller open sets. A sheaf of ideals $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ is a ***coherent sheaf of ideals*** if it is locally finitely generated—i.e., at every point $x \in \mathsf{X}$ there exists a neighbourhood U of x and a finite number of holomorphic functions

$f_1, \ldots, f_n \in \mathcal{O}_{\mathsf{X}}(\mathsf{U})$ such that $\mathcal{I}(\mathcal{U})$ is the ideal generated by $f_1, \ldots, f_n$ in $\mathcal{O}_{\mathsf{X}}(\mathsf{U})$.

Associated with such a sheaf of ideals is its vanishing set $\mathsf{Y} = \mathsf{V}(\mathcal{I}) \subset \mathsf{X}$ defined as the closed subset of X on which every function in $\mathcal{I}$ vanishes. This subset comes equipped with its own sheaf of rings, namely the quotient $\mathcal{O}_{\mathsf{Y}} = \mathcal{O}_{\mathsf{X}}/\mathcal{I}$. Since the quotient kills the equations used to define Y, this sheaf of rings serves as a model for the sheaf of holomorphic functions on Y. We therefore say that the pair $(\mathsf{Y}, \mathcal{O}_{\mathsf{Y}})$ is an ***analytic subspace of*** X.

Definition 2.1.1. An ***analytic space*** is a pair $(\mathsf{Y}, \mathcal{O}_{\mathsf{Y}})$ of a topological space Y and a sheaf of rings $\mathcal{O}_{\mathsf{Y}}$ that arises as the vanishing set of a coherent ideal sheaf on some complex manifold X. A complex analytic space is ***smooth*** if $\mathsf{Y} \subset \mathsf{X}$ is a complex submanifold and the ideal $\mathcal{I}$ consists of all of the functions on X that vanish on this submanifold—in other words, Y is smooth if it is a complex manifold in its own right. Otherwise, it is ***singular***.

We will regularly abuse notation and refer to the pair $(\mathsf{Y}, \mathcal{O}_{\mathsf{Y}})$ simply as Y. Notice that, given a complex analytic space Y, we may also define a complex analytic subspace of Y as the vanishing set of a coherent ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathsf{Y}}$; this definition makes sense since the ideal defines the ideal $\mathcal{I} + \mathcal{I}_{\mathsf{Y}}$ in $\mathcal{O}_{\mathsf{X}}$, where $\mathcal{I}_{\mathsf{Y}}$ is the ideal of Y.

Example 2.1.2. The cone $\mathsf{Y} \subset \mathbb{C}^3$ defined as the zero set of the function $f = x^2 + 4yz$, and considered in The Example, is an analytic subspace of $\mathbb{C}^3$. Having a singularity at the origin, it is not a submanifold.

A function on Y is defined to be holomorphic if and only if it extends a holomorphic function in a neighbourhood of Y. Any two such extensions must differ by a multiple of f and hence we have an identification of the holomorphic functions on Y as the quotient $\mathcal{O}_{\mathsf{Y}} = \mathcal{O}_{\mathbb{C}^3}/f\mathcal{O}_{\mathbb{C}^3}$ by the ideal generated by f in the sheaf of holomorphic functions on $\mathbb{C}^3$.

Moreover, Y has a privileged subspace: its singular locus. This subspace is the set of critical points of f and is therefore defined by the vanishing of the components of the derivative $df = 2x\,dx + 4z\,dy + 4y\,dz$. The corresponding ideal is the one generated by x, y and z. We can view this as an ideal in $\mathcal{O}_{\mathbb{C}^3}$ or restrict these functions to Y to obtain generators for the ideal $(x|_{\mathsf{Y}}, y|_{\mathsf{Y}}, z|_{\mathsf{Y}}) \subset \mathcal{O}_{\mathsf{Y}}$. Either way, the resulting subspace is the point $\mathsf{Y}_{sing} = \{0\} \in \mathbb{C}^3$ with the ring

$$\mathcal{O}_{\mathbb{C}^3}/(x, y, z) \cong \mathcal{O}_{\mathsf{Y}}/(x|_{\mathsf{Y}}, y|_{\mathsf{Y}}, z|_{\mathsf{Y}}) \cong \mathbb{C}$$

of constant functions on the point. □

One important remark is in order: different ideals $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ will produce different rings $\mathcal{O}_{\mathsf{Y}} = \mathcal{O}_{\mathsf{X}}/\mathcal{I}$, but can nevertheless produce the same underlying topological space Y. The quintessential example of this phenomenon is the case where $\mathsf{X} = \mathbb{C}$ is the complex line and $\mathcal{I}_k \subset \mathcal{O}_{\mathbb{C}}$ is the ideal generated by the function z^k, where z is a coordinate. The resulting topological space Y_k is simply the origin, but the ring $\mathcal{O}_{\mathsf{Y}_k} = \mathcal{O}_{\mathbb{C}}/\mathcal{I}_k$ depends on k. Its elements are expressions of the form

$$p = a_0 + a_1 z + \cdots + a_{k-1} z^{k-1}$$

with $a_0, \ldots, a_{k-1} \in \mathbb{C}$. They are multiplied like polynomials, except that we force $z^k = 0$. Thus, for $k \geq 2$ the ring has nilpotent elements. Clearly, the resulting rings for different values of k are not isomorphic. Thus, the analytic space captures not only the set of points where the ideal vanishes, but also some information about transversal derivatives. For this reason Y_k is called the k^{th}-order neighbourhood of the origin. Notice that there are natural quotient maps $\mathcal{O}_{\mathsf{Y}_k} \to \mathcal{O}_{\mathsf{Y}_{k-1}}$ corresponding to a chain of inclusions

$$(\mathsf{Y}_1, \mathcal{O}_{\mathsf{Y}_1}) \subset (\mathsf{Y}_2, \mathcal{O}_{\mathsf{Y}_2}) \subset (\mathsf{Y}_3, \mathcal{O}_{\mathsf{Y}_3}) \subset \cdots$$

of analytic spaces supported on the same topological space.

Definition 2.1.3. The analytic space Y is ***reduced*** if $\mathcal{O}_{\mathsf{Y}}$ contains no nilpotent elements.

Notice that complex manifolds have no nilpotent elements in their sheaves of holomorphic functions. Thus, smooth analytic spaces are necessarily reduced. However, the converse does not hold. For example, consider $\mathbb{C}^2$ with coordinates x and y. The analytic subspace $\mathsf{Y} \subset \mathbb{C}^2$ defined by the function $xy \in \mathcal{O}_{\mathbb{C}^2}$ is reduced, but it is the union of the two coordinate axes and is therefore singular.

Every analytic space Y has a ***reduced subspace***, which is the unique reduced analytic subspace $\mathsf{Y}_{red} \subset \mathsf{Y}$ having the same underlying topological space. It is defined by the ideal $\mathcal{I}_{nil} \subset \mathcal{O}_{\mathsf{Y}}$ consisting of nilpotent elements—the so-called nilradical. Every analytic Y space has a subspace Y_{sing}, called its ***singular locus***, such that $\mathsf{Y} \setminus \mathsf{Y}_{sing}$ (which may be empty) is smooth. If Y is reduced, then Y_{sing} will be a subspace of strictly smaller dimension.

At first glance, one is therefore tempted to always get rid of the nilpotence and work with the reduced space Y_{red}. However, there are several reasons why we should allow for non-reduced spaces:

1. Firstly, we shall encounter a number of ideals defined in an intrinsic way by the geometry; there is no reason to expect them to define reduced spaces in general (although often they will). As such, nonreduced spaces come up naturally. We would rather not bother with reducing them every time arise.

2. We want to deal with intersections or fibre products that are not transverse in the same way as we would deal with transverse ones; sometimes these intersections will not be reduced so we should include these objects if we want to treat all intersections in a uniform way.

3. Relatedly, we need non-reduced spaces if we want to properly count intersection points:

for example, a generic line L in the plane will intersect a parabola C in exactly two points, but if L is tangent to C there is only one point of intersection. To correctly account for this situation, Bézout's Theorem tells us that we must assign an intersection multiplicity of two to this point. This multiplicity is exactly detected by the presence of nilpotent elements in $\mathcal{O}_{\mathsf{L}\cap\mathsf{C}}$: this ring is a two-dimensional algebra over $\mathbb{C}$, with a one-dimensional space of nilpotent elements.

We would also like to say when a given analytic space can be broken into a union of smaller pieces:

Definition 2.1.4. An analytic space Y is ***irreducible*** if it cannot be written as the union $\mathsf{Y} = \mathsf{Y}_1 \cup \mathsf{Y}_2$ of two closed analytic subspaces.

Every analytic space can be written uniquely as the union of a collection of irreducible subspaces, called its ***irreducible components***. In the previous example of the coordinate axes $\mathsf{Y} \subset \mathbb{C}^2$, there are two irreducible components: the x- and y-axes.

Notice that there is a confusing point with regard to terminology: it is possible for an analytic space to be reduced (i.e., have no nilpotence) but reducible (i.e., have multiple irreducible components). The union of the coordinate axes $\mathsf{Y} \subset \mathbb{C}^2$ gives just such an example.

2.1.2 Holomorphic vector bundles and sheaves

Let X be a complex manifold. If $\mathsf{E} \to \mathsf{X}$ is a holomorphic vector bundle, we may consider its sheaf $\mathcal{E}$ of holomorphic sections. Elements of $\mathcal{E}$ may be added together and multiplied by holomorphic functions and therefore $\mathcal{E}$ forms a module over the sheaf of holomorphic functions. We say that $\mathcal{E}$ is an $\mathcal{O}_{\mathsf{X}}$-module.

In general, we have the

Definition 2.1.5. Let X be a complex manifold or analytic space. An $\mathcal{O}_{\mathsf{X}}$***-module*** is a sheaf that assigns to every open set $\mathsf{U} \subset \mathsf{X}$ a module over the ring $\mathcal{O}_{\mathsf{X}}(\mathsf{U})$, in a way that is compatible with the restriction maps. A morphism between two $\mathcal{O}_{\mathsf{X}}$-modules is a morphism of sheaves that is compatible with the module structures.

If E is a rank r vector bundle, a local trivialization of E over an open set $\mathsf{U} \subset \mathsf{X}$ gives rise to an isomorphism $\mathcal{E}|_{\mathsf{U}} \cong \mathcal{O}_{\mathsf{X}}|_{\mathsf{U}}^{\oplus r}$ with the module of sections of the trivial bundle, which is a free module over $\mathcal{O}_{\mathsf{X}}|_{\mathsf{U}}$. We therefore say that the $\mathcal{O}_{\mathsf{X}}$-module $\mathcal{E}$ is ***locally free***. There is a natural one-to-one correspondence between holomorphic vector bundles and locally free $\mathcal{O}_{\mathsf{X}}$-modules. (In fact, this correspondence is an equivalence of categories.) For this reason, we shall often abuse notation and say that the locally free sheaf $\mathcal{E}$ itself is "a vector bundle".

One of the main problems in dealing only with vector bundles is that they do not form an abelian category. If $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ is a morphism of vector bundles that has constant rank, then we obtain a new vector bundle whose fibres are the quotients of the fibres of $\mathcal{E}_2$ be the images of the fibres of $\mathcal{E}_2$. However, if ϕ drops rank at some point $x \in \mathsf{X}$, then these fibres will jump in dimension and so we cannot assemble them into a vector bundle.

However, we can still form the quotient sheaf $\mathcal{E}_1/\phi(\mathcal{E}_2)$ and it will be an $\mathcal{O}_{\mathsf{X}}$-module; it will simply fail to be locally free at x. In so doing, we obtain new objects that are more general than vector bundles:

Definition 2.1.6. An $\mathcal{O}_{\mathsf{X}}$-module $\mathcal{E}$ is ***coherent*** if for every $x \in \mathsf{X}$ there is a neighbourhood U of X together with a map of finite-rank vector bundles $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ on U such that $\mathcal{E}|_{\mathsf{U}} \cong \mathcal{E}_1/\phi(\mathcal{E}_2)$.

The usefulness of coherent sheaves is that they do form an abelian category: we can form the kernels and cokernels of an arbitrary morphism of coherent sheaves, and these kernels and cokernels are themselves coherent.

Example 2.1.7. Let $\mathsf{X} = \mathbb{C}$ be the complex line with coordinate z. Consider the map $\phi : \mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}$ on the trivial line bundle, given by multiplication by z. The image of this map is the ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ of all functions vanishing at the origin $\mathsf{Y} = \{0\} \subset \mathsf{X}$, and the quotient is $\mathcal{O}_{\mathsf{Y}}$, which is a coherent sheaf supported on Y. Notice that all of the elements in $\mathcal{I}$ kill $\mathcal{O}_{\mathsf{Y}}$ when we think of $\mathcal{O}_{\mathsf{Y}}$ as an $\mathcal{O}_{\mathsf{X}}$-module. □

Given a coherent sheaf $\mathcal{E}$, we obtain an ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ by declaring that $f \in \mathcal{I}$ if and only if $fs = 0$ for all $f \in \mathcal{E}$. This ideal is called the ***annihilator of*** $\mathcal{E}$, and the corresponding subspace $\mathsf{Y} \subset \mathsf{X}$ is the ***support of*** $\mathcal{E}$.

In the previous example, if we take $\mathcal{E} = \mathcal{O}_{\mathsf{Y}}$ the annihilator is the ideal of functions vanishing at the origin in $\mathbb{C}$ and the support is Y. On the other hand, if X is reduced (e.g., a complex manifold) than the support of any vector bundle on X is X itself.

Definition 2.1.8. Suppose that X is a connected analytic space. A ***torsion sheaf on*** X is a coherent sheaf $\mathcal{E}$ whose support is a proper closed subspace of X, i.e., $\mathcal{E}(\mathsf{U})$ is a torsion module for all open sets $\mathsf{U} \subset \mathsf{X}$.

Thus, if $\mathsf{Y} \subset \mathsf{X}$ is a closed analytic subspace, then $\mathcal{O}_{\mathsf{Y}}$ is a torsion sheaf on X. If $\mathcal{E}$ is a coherent sheaf on X, there is a maximal subsheaf of $\mathcal{E}$ that is torsion. This subsheaf is called the ***torsion subsheaf of*** $\mathcal{E}$. If the torsion subsheaf is trvial, $\mathcal{E}$ is called ***torsion-free***. Clearly any subsheaf of a torsion-free sheaf is torsion-free.

If $\mathcal{E}_1$ and $\mathcal{E}_2$ are coherent sheaves, then so are the direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$, the tensor products $\mathcal{E}_1 \otimes_{\mathcal{O}_{\mathsf{X}}} \mathcal{E}_2$ and the sheaf $\mathcal{H}om_{\mathcal{O}_{\mathsf{X}}}(\mathcal{E}_1, \mathcal{E}_2)$ of $\mathcal{O}_{\mathsf{X}}$-linear maps from $\mathcal{E}_1$ to $\mathcal{E}_2$. In particular, every coherent sheaf $\mathcal{E}$ has a ***dual*** $\mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_{\mathsf{X}})$. These constructions coincide with the usual operations on vector bundles.

Any coherent sheaf $\mathcal{E}$ has a natural morphism $\phi : \mathcal{E} \to (\mathcal{E}^\vee)^\vee$ to its double dual that sends $s \in \mathcal{E}$ to the map $\phi(s) : \mathcal{E}^\vee \to \mathcal{O}_{\mathsf{X}}$ defined by evaluation: $\phi(s) \cdot \alpha = \alpha(s)$. In general, this map is neither injective nor surjective; in fact, the kernel of this map is exactly the torsion subsheaf of $\mathcal{E}$. If the map ϕ is an isomorphism, then $\mathcal{E}$ is said to be ***reflexive***. Clearly any vector bundle is reflexive.

We recall some useful facts from [78], which show that reflexive sheaves behave in many ways like vector bundles:

1. If X is reduced and irreducible (for example, a connected manifold), then the kernel of any vector bundle map $\mathcal{E}_1 \to \mathcal{E}_2$ is reflexive. In fact, every reflexive sheaf is locally of this form.

2. If X is reduced and irreducible, then the dual of any coherent sheaf is automatically reflexive.

3. If X is a manifold, then the locus where a reflexive sheaf fails to be locally free has codimension ≥ 3 in X. Thus, every reflexive sheaf has a well-defined rank, defined as its rank on the open dense set where it is a vector bundle.

4. Any reflexive sheaf of rank one is automatically a line bundle, i.e., locally free.

5. If X is a manifold, then reflexive sheaves exhibit the Hartogs phenomenon: if $\mathsf{Y} \subset \mathsf{X}$ is a subspace of (complex) codimension at least two, then any section $s \in \mathcal{E}|_{\mathsf{X}\setminus\mathsf{Y}}$ extends uniquely to a section of $\mathcal{E}$.

6. If X is a manifold, and $\mathcal{E}$ is a reflexive sheaf of rank r, then $\mathcal{E}$ has a well-defined determinant line bundle $\det \mathcal{E}$. By the previous three points, it is enough to define $\mathcal{E}$ on the open set $\mathsf{U} \subset \mathsf{X}$ where $\mathcal{E}$ is locally free, and there we simply declare that $\det \mathcal{E} = \wedge^r \mathcal{E}$.

7. If X is a manifold and $\mathcal{E}$ is a rank-two reflexive sheaf, then $\mathcal{E} \cong \mathcal{E}^\vee \otimes \det \mathcal{E}$.

Example 2.1.9. Recall from the The Example, the sheaf $\mathcal{F} \subset \mathcal{T}_{\mathbb{C}^3}$ of vector fields tangent to the two-dimensional symplectic leaves of the Poisson structure σ on $\mathbb{C}^3$. We saw that $\mathcal{F}$ is the kernel of the vector bundle map $\sigma\wedge : \mathcal{T}_{\mathbb{C}^3} \to \det \mathcal{T}_{\mathbb{C}^3}$ and so we conclude that it is a reflexive sheaf of rank two. Notice that the Poisson structure σ defines a section of $\wedge^2\mathcal{F}$ away from the origin, since it is tangent to all of the leaves. Therefore, by the Hartogs phenomenon, it extends to a section of $\det \mathcal{F}$.

We claim that this section is non-vanishing. Indeed, if it were to vanish, it would have to do so on a hypersurface because it is a section of a line bundle. But σ is nonvanishing away from the origin, so it cannot possibly vanish on a hypersurface. Thus, σ is a nonvanishing section of $\det \mathcal{F}$, even though it vanishes as a section of $\wedge^2\mathcal{T}_{\mathsf{X}}$. In particular, by Item 7 above, σ defines an isomorphism $\sigma^\sharp : \mathcal{F}^\vee \to \mathcal{F}$. If we think of $\mathcal{F}^\vee$ as the cotangent sheaf of the symplectic leaves, then σ behaves like a symplectic form: it gives an isomorphism between the cotangent and tangent sheaves even though these sheaves are not bundles at the origin. We shall return to this theme later in the thesis. □

2.1.3 Calculus on analytic spaces

Since an analytic space X need not be smooth, it has no tangent bundle in general, but we may still speak of vector fields as defining derivations. Thus, we may define the ***tangent sheaf*** $\mathcal{T}_{\mathsf{X}}$ as the sheaf of all $\mathbb{C}$-linear maps $Z : \mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}$ that obey the Leibniz rule $Z(fg) = Z(f)g + fZ(g)$. As usual, the commutator gives this sheaf a Lie bracket with the usual properties familiar from the smooth case. Since derivations can be multiplied by functions $\mathcal{T}_{\mathsf{X}}$ is naturally an $\mathcal{O}_{\mathsf{X}}$-module. In fact, it is a coherent sheaf.

We shall often be interested in understanding when a collection of vector fields $\mathcal{F} \subset \mathcal{T}_{\mathsf{X}}$ are tangent to an analytic subspace $\mathsf{Y} \subset \mathsf{X}$. This property can be characterized as follows: $\mathcal{F}$ is ***tangent to*** Y if it preserves the ideal defining Y. In other words, if $f \in \mathcal{O}_{\mathsf{X}}$ is a function vanishing on Y we require that $\mathscr{L}_Z f$ also vanishes on Y for all $Z \in \mathcal{F}$. This definition is very useful because it is compatible with various natural operations on subspaces. Indeed, we will make repeated use of the following

Theorem 2.1.10. *Let $\mathsf{Y} \subset \mathsf{X}$ be a closed analytic subspace and let $\mathcal{F} \subset \mathcal{T}_{\mathsf{X}}$ be tangent to Y. Then $\mathcal{F}$ is also tangent to the following subspaces:*

1. *the reduced subspace $\mathsf{Y}_{red} \subset \mathsf{Y}$,*

2. *the singular locus $\mathsf{Y}_{sing} \subset \mathsf{Y}$, and*

3. *each of the irreducible components of Y.*

Moreover, if $\mathsf{Z} \subset \mathsf{X}$ is another analytic subspace to which $\mathcal{F}$ is tangent, then $\mathcal{F}$ is also tangent to

4. *the union $\mathsf{Y} \cup \mathsf{Z}$, and*

5. *the intersection $\mathsf{Y} \cap \mathsf{Z}$.*

Proof. Statements 1 and 3 follow from [128, Theorem 1], using the correspondence between primary ideals and irreducible components. Statement 2 follows from [77, Corollary 2], using the fact that the singular locus is defined by the first Fitting ideal of Ω^1_{Y}, which describes the locus where Ω^1_{Y} is not locally free; see [51, §16.6].

If $\mathcal{I}$ and $\mathcal{J}$ are the ideals defining Y and Z, then $\mathsf{Y} \cup \mathsf{Z}$ and $\mathsf{Y} \cap \mathsf{Z}$ are defined by $\mathcal{I} \cap \mathcal{J}$ and $\mathcal{I} + \mathcal{J}$, respectively. Statements 4 and 5 follow immediately. □

We can also define a complex of differential forms on X as follows: consider the diagonal embedding $\Delta_{\mathsf{X}} \subset \mathsf{X} \times \mathsf{X}$ of X, and let $\mathcal{I} \subset \mathcal{O}_{\mathsf{X} \times \mathsf{X}}$ be the ideal defining this closed subspace. Then the conormal sheaf of Δ_{X} is given by $\mathcal{I}/\mathcal{I}^2$. If X were a complex manifold, this conormal sheaf would be identified with the cotangent bundle of X by the projections $p_1, p_2 : \mathsf{X} \times \mathsf{X} \to \mathsf{X}$, but if X is a general analytic space, we *define* the cotangent sheaf to be $\Omega^1_{\mathsf{X}} = \mathcal{I}/\mathcal{I}^2$. The key difference from the case of manifolds is that Ω^1_{X} will fail to be locally free at the singular points.

We can now define the k-forms for $k > 1$ by setting $\Omega^k_{\mathsf{X}} = \wedge^k \Omega^1_{\mathsf{X}}$, the kth exterior power as an $\mathcal{O}_{\mathsf{X}}$-module. A function $f \in \mathcal{O}_{\mathsf{X}}$ has an exterior derivative $df = [p_1^* f - p_2^* f] \in \mathcal{I}/\mathcal{I}^2$, and this derivative $d : \mathcal{O}_{\mathsf{X}} \to \Omega^1_{\mathsf{X}}$ extends to the exterior algebra in the usual way, giving the de Rham complex of the analytic space X. The sheaf $\mathcal{T}_{\mathsf{X}}$ of vector fields is identified with the dual of Ω^1_{X} as an $\mathcal{O}_{\mathsf{X}}$-module, and so we can define contractions with vector fields, Lie derivatives, etc., obeying the usual identities familiar from manifolds. Note, though, that some care must be taken since Ω^k_{X} is not, in general reflexive. Hence the double dual $(\Omega^k_{\mathsf{X}})^{\vee\vee}$ is a different object when X is singular, sometimes called the sheaf of reflexive k-forms.

2.2 Lie algebroids

Let X be a complex manifold. We denote by $\mathcal{T}_{\mathsf{X}}$ the tangent sheaf of X—that is, the sheaf of holomorphic vector fields. A ***Lie algebroid on*** X is a triple $(\mathcal{A}, [\cdot,\cdot], a)$, where $\mathcal{A}$ is the sheaf of sections of a holomorphic vector bundle, $[\cdot,\cdot]$ is a $\mathbb{C}$-linear Lie bracket on $\mathcal{A}$, and $a : \mathcal{A} \to \mathcal{T}_{\mathsf{X}}$ is a map of holomorphic vector bundles. These data must satisfy the following compatibility conditions:

1. a is a homomorphism of Lie algebras, where the bracket on $\mathcal{T}_{\mathsf{X}}$ is the usual Lie bracket of vector fields

2. We have the Leibniz rule

$$[\xi, f\eta] = (\mathscr{L}_{a(\xi)} f)\eta + f[\xi, \eta]$$

   for all $\xi, \eta \in \mathcal{A}$ and $f \in \mathcal{O}_{\mathsf{X}}$.

The map $a : \mathcal{A} \to \mathcal{T}_{\mathsf{X}}$ is called the ***anchor map***. We will regularly abuse notation and denote the whole triple $(\mathcal{A}, [\cdot,\cdot], a)$ simply by $\mathcal{A}$.

More generally, we can relax our assumptions and allow for more singular objects: we can let X be an analytic space or replace $\mathcal{A}$ with an arbitrary coherent sheaf, or both. The same definition applies. Thus, when we write something like "let $\mathcal{A}$ be a Lie algberoid on X", we typically mean that X is a complex analytic space and $\mathcal{A}$ is a Lie algebroid that is a coherent sheaf, but not necessarily a vector bundle. If, at some point, we need to assume that $\mathcal{A}$ comes from a vector bundle (i.e., is locally free), we will be careful to indicate this assumption.

Let us now discuss several examples of Lie algebroids:

Example 2.2.1. If X is a complex manifold then the tangent bundle $\mathcal{T}_{\mathsf{X}}$ is obviously a Lie algebroid. □

Example 2.2.2. The action of a complex Lie algebra $\mathfrak{g}$ on a complex manifold X (or, more generally, an analytic space) is defined by a Lie algebra homomorphism $\mathfrak{g} \to \mathsf{H}^0(\mathsf{X}, \mathcal{T}_{\mathsf{X}})$ to the space of global holomorphic vector fields. This map gives the trivial bundle $\mathfrak{g} \times \mathsf{X} \to \mathsf{X}$ the structure of a Lie algebroid, called the ***action algebroid*** $\mathfrak{g} \ltimes \mathsf{X}$. □

Example 2.2.3. If $\mathcal{F} \subset \mathcal{T}_{\mathsf{X}}$ is an involutive subbundle of the tangent bundle (meaning that $\mathcal{F}$ is closed under Lie brackets), the inclusion $\mathcal{F} \to \mathcal{T}_{\mathsf{X}}$ makes $\mathcal{F}$ into a locally free Lie algebroid. □

Example 2.2.4. The previous example can be generalized considerably. Suppose that X is an analytic space and $\mathcal{F} \subset \mathcal{T}_{\mathsf{X}}$ is a coherent subsheaf that is involutive, i.e., $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$. Then the inclusion $a : \mathcal{F} \hookrightarrow \mathcal{T}_{\mathsf{X}}$ gives $\mathcal{F}$ the structure of a Lie algebroid. In general, even if $\mathcal{F}$ is locally free, it will not arise from a subbundle of $\mathcal{T}_{\mathsf{X}}$ since the rank of the vector bundle map a can drop along a subspace of X. A locally free Lie algebroid for which the anchor map $a : \mathcal{A} \to \mathcal{T}_{\mathsf{X}}$ is an embedding of sheaves is called ***almost injective*** in [41, 43]. In Section 2.3, we shall discuss an interesting class of almost injective Lie algebroids that are associated with hypersurfaces. □

Example 2.2.5. If $\mathcal{E}$ is a vector bundle or a torsion-free coherent sheaf on X, we may consider its ***Atiyah algebroid*** $\mathfrak{gl}(\mathcal{E})$, which is the sheaf of first order differential operators $D : \mathcal{E} \to \mathcal{E}$ with scalar symbol. The commutator of differential operators gives $\mathfrak{gl}(\mathcal{E})$ a bracket, and the inclusion $\mathcal{E}nd(\mathcal{E}) \to \mathfrak{gl}(\mathcal{E})$ of the $\mathcal{O}_{\mathsf{X}}$-linear endomorphisms as the zeroth order operators gives an exact sequence

$$0 \longrightarrow \mathcal{E}nd(\mathcal{E}) \longrightarrow \mathfrak{gl}(\mathcal{E}) \longrightarrow \mathcal{T}_{\mathsf{X}} \longrightarrow 0,$$

where the anchor map $\mathfrak{gl}(\mathcal{E}) \to \mathcal{T}_{\mathsf{X}}$ is given by taking symbols. □

Notice that if $\mathcal{A}$ is any Lie algebroid, then the image $\mathcal{F} = a(\mathcal{A}) \subset \mathcal{T}_{\mathsf{X}}$ of the anchor map is necessarily an involutive subsheaf, that is

$$[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}.$$

Hence, by Nagano's theorem on the integrability of singular analytic distributions, X is partitioned into immersed complex analytic submanifolds that are maximal integral submanifolds of $\mathcal{F}$. We call these submanifolds the ***orbits of*** $\mathcal{A}$.

In general, it is useful to consider subspaces $\mathsf{Y} \subset \mathsf{X}$ that are unions of orbits; in other words, we ask that all of the vector fields coming from $\mathcal{A}$ are tangent to Y. The precise definition is as follows:

Definition 2.2.6. A closed complex analytic subspace $\mathsf{Y} \subset \mathsf{X}$ is ***$\mathcal{A}$-invariant*** if the corresponding sheaf of ideals $\mathcal{I}_{\mathsf{Y}} \subset \mathcal{O}_{\mathsf{X}}$ is preserved by the action of $\mathcal{A}$. In other words, we require that $\mathscr{L}_{a(\xi)} f \in \mathcal{I}_{\mathsf{Y}}$ for all $\xi \in \mathcal{A}$ and $f \in \mathcal{I}_{\mathsf{Y}}$.

One of the main reasons for the usefulness of $\mathcal{A}$-invariant subspaces is the following lemma, which follows directly from the definition:

Lemma 2.2.7. *If $\mathsf{Y} \subset \mathsf{X}$ is $\mathcal{A}$-invariant, then the bracket and anchor restrict to give $\mathcal{A}|_{\mathsf{Y}}$ the structure of a Lie algebroid on Y.*

Notice that Theorem 2.1.10 applies directly with $\mathcal{F} = \mathcal{I}mg(a) \subset \mathcal{T}_{\mathsf{X}}$ to show that the singular locus, reduced subspace and all of the irreducible components of an $\mathcal{A}$-invariant subspace are themselves $\mathcal{A}$-invariant. Moreover, intersections and unions of $\mathcal{A}$-invariant subspaces are also $\mathcal{A}$-invariant.

In particular, a given Lie algebroid has a number of important $\mathcal{A}$-invariant subspaces—for example, the singular locus and irreducible components of X itself.

Example 2.2.8. Let us return to The Example. Recall that the vector fields $X_x = 2y\partial_y - 2z\partial_z$, $X_y = x\partial_z - 2y\partial_x$ and $X_z = 2z\partial_x - x\partial_y$ span an involutive subsheaf $\mathcal{F} \subset \mathcal{T}_{\mathbb{C}^3}$ that is not a vector bundle. Nevertheless, according to our definition, it is still a Lie algebroid—just not a locally free one.

Since the function $f = x^2 + 4yz$ is annihilated by X_x, X_y and X_z, its zero locus—the cone Y—defines an $\mathcal{F}$-invariant subspace. This fact is visible in the geometry: Y is the union of the two $\mathcal{F}$-orbits $\{0\}$ and $\mathsf{Y} \setminus \{0\}$. Thus $\mathcal{F}|_{\mathsf{Y}}$ defines a Lie algebroid on Y, giving an example of a Lie algebroid that is not locally free on a space that is not smooth.

To explain the Lie algebroid structure, we describe how a section $\xi \in \mathcal{F}|_{\mathsf{Y}}$ acts on a function $g \in \mathcal{O}_{\mathsf{Y}}$: first extend g and ξ to corresponding objects $g' \in \mathcal{O}_{\mathbb{C}^3}$ and $\xi' \in \mathcal{F}$ in a neighbourhood of Y. Then $\mathscr{L}_{\xi} g = (\mathscr{L}_{\xi'} g')|_{\mathsf{Y}}$ is obtained by restricting the action on $\mathbb{C}^3$. The condition that Y be $\mathcal{F}$-invariant is exactly what is required for this process is well-defined (independent of the choices of extensions).

Recall that the singular locus of Y is the origin. The corresponding ideal is the ideal generated by x, y and z. This ideal is preserved by the action of the Hamiltonian vector fields X_x, X_y and X_z since they all vanish at the origin. Thus the singular locus is an invariant subspace for the Lie algebroid $\mathcal{F}|_{\mathsf{Y}}$. □

Perhaps the most important invariant subspaces for a Lie algebroid are the degeneracy loci, which are the loci where the rank of the anchor map a drops. The structure of these subspaces will be the subject of Chapter 6, but we define them now since they will appear often.

Notice that for $k \geq 0$, the locus where the rank is k or less is exactly the locus where the $(k+1)$st exterior power $\wedge^{k+1} a$ vanishes. Since the coefficients of this tensor are holomorphic, the zero locus is an analytic subspace.

We give a more precise definition, valid for an any analytic space X and any Lie algebroid $\mathcal{A}$ (not necessarily locally free) as follows: the $(k+1)$st exterior power of a defines a natural map

$$\wedge^{k+1}a : \wedge^{k+1}\mathcal{A} \to \wedge^{k+1}\mathcal{T}_{\mathsf{X}}.$$

Dually, this map defines a morphism

$$\wedge^{k+1}a : \wedge^{k+1}\mathcal{A} \otimes \Omega^{k+1}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}.$$

The image of $\wedge^{k+1}a$ is an ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ and the ***kth degeneracy locus of*** $\mathcal{A}$ is the analytic subspace $\mathsf{Dgn}_k(\mathcal{A}) \subset \mathsf{X}$ defined by $\mathcal{I}$.

Proposition 2.2.9. *For every $k \geq 0$, the degeneracy locus $\mathsf{Dgn}_k(\mathcal{A})$ is $\mathcal{A}$-invariant.*

Proof. The proof follows the related result for Poisson structures [117, Corollary 2.3]; see also [73]. Let $\phi = \wedge^{k+1}a : \wedge^{k+1}\mathcal{A} \to \wedge^{k+1}\mathcal{T}_{\mathsf{X}}$. By definition, the ideal $\mathcal{I}$ defining $\mathsf{Dgn}_k(\mathcal{A})$ is generated by functions of the form

$$f = \langle \phi(\xi), \omega \rangle$$

where $\xi \in \wedge^{k+1}\mathcal{A}$, $\omega \in \Omega^{k+1}_{\mathsf{X}}$ and $\langle \cdot, \cdot \rangle : \wedge^{k+1}\mathcal{T}_{\mathsf{X}} \otimes \Omega^{k+1}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}$ is the natural pairing. We must verify that if we take a section $\eta \in \mathcal{A}$ then the derivative $\mathscr{L}_{a(\eta)}f$ is also in the ideal. But we easily compute

$$\begin{aligned} \mathscr{L}_{a(\eta)}f &= \left\langle \mathscr{L}_{a(\eta)}\phi(\xi), \omega \right\rangle + \left\langle \phi(\xi), \mathscr{L}_{a(\eta)}\omega \right\rangle \\ &= \langle \phi(\mathscr{L}_{\eta}\xi), \omega \rangle + \left\langle \phi(\xi), \mathscr{L}_{a(\eta)}\omega \right\rangle \\ &\in \mathcal{I}, \end{aligned}$$

where we have used the compatibility of the anchor and bracket of $\mathcal{A}$ to pass the Lie derivative through ϕ. □

2.3 Lie algebroids associated with hypersurfaces

Suppose that X is a complex manifold, and suppose that $\mathsf{D} \subset \mathsf{X}$ is a hypersurface in X (possibly singular, reducible and non-reduced). We follow the standard convention and denote by $\mathcal{O}_{\mathsf{X}}(-\mathsf{D})$ the ideal defining D. Thus $\mathcal{O}_{\mathsf{X}}(-\mathsf{D})$ is an invertible sheaf—a holomorphic line bundle. Its dual is denoted by $\mathcal{O}_{\mathsf{X}}(\mathsf{D})$ and consists of functions having poles bounded by D; if f is a local equation for D then f^{-1} is a generator for $\mathcal{O}_{\mathsf{X}}(\mathsf{D})$. More generally, if $\mathcal{E}$ is a coherent sheaf, we denote by $\mathcal{E}(-\mathsf{D}) = \mathcal{E} \otimes \mathcal{O}_{\mathsf{X}}(-\mathsf{D}) \subset \mathcal{E}$ the sheaf of sections of $\mathcal{E}$ that vanish on $\mathcal{E}$, and by $\mathcal{E}(\mathsf{D}) = \mathcal{E} \otimes \mathcal{O}_{\mathsf{X}}(\mathsf{D})$ the sections with appropriate poles.

There are two natural Lie algebroids associated to D that arise as involutive subsheaves of $\mathcal{T}_\mathsf{X}$. The first is the subsheaf $\mathcal{T}_\mathsf{X}(-\mathsf{D}) \subset \mathcal{T}_\mathsf{X}$ consisting of vector fields that vanish on D. If $Z_1, \ldots, Z_n$ is a local basis for $\mathcal{T}_\mathsf{X}$ and $f \in \mathcal{O}_\mathsf{X}(-\mathsf{D})$ is a defining equation for D, then the vector fields $fZ_1, \ldots, fZ_n$ give a local basis for $\mathcal{T}_\mathsf{X}(-\mathsf{D})$, and hence $\mathcal{T}_\mathsf{X}(-\mathsf{D})$ is locally free. Notice that even though this Lie algebroid is locally free, it is not given by a subbundle of $\mathcal{T}_\mathsf{X}$: locally, the anchor map is just multiplication by the function f. Hence it is generically an isomorphism and drops rank to 0 along D.

The second Lie algebroid related to D is its sheaf of ***logarithmic vector fields*** [125] $\mathcal{T}_\mathsf{X}(-\log \mathsf{D}) \subset \mathcal{T}_\mathsf{X}$, which consists of all of the vector fields that are tangent to D. In algebraic terms, $Z \in \mathcal{T}_\mathsf{X}$ lies in $\mathcal{T}_\mathsf{X}(-\log \mathsf{D})$ if it preserves the ideal $\mathcal{O}_\mathsf{X}(-\mathsf{D})$ defining D. In other words if f is a local equation for D, we require that $\mathscr{L}_Z f = gf$ for some $g \in \mathcal{O}_\mathsf{X}$. From the latter description, it is straightforward to verify that $\mathcal{T}_\mathsf{X}(-\log \mathsf{D})$ is involutive, and that $\mathcal{T}_\mathsf{X}(-\log \mathsf{D}) = \mathcal{T}_\mathsf{X}(-\log \mathsf{D}_{red})$, where D_{red} is the reduced space underlying D. In contrast with $\mathcal{T}_\mathsf{X}(-\mathsf{D})$, the Lie algebroid $\mathcal{T}_\mathsf{X}(-\log \mathsf{D})$ will not, in general, be locally free.

Definition 2.3.1. Let X be a complex manifold. A reduced hypersurface $\mathsf{D} \subset \mathsf{X}$ is a ***free divisor*** if $\mathcal{T}_\mathsf{X}(-\log \mathsf{D})$ is locally free.

Remark 2.3.2. The author thanks the external thesis examiner for observing that, that freeness in the above sense should not be confused with base point freeness of linear systems. Both terms are standard in the literature, but the latter notion will not be used in this thesis.

The term "free divisor" was introduced by Kyoji Saito, who provided a useful way to check that a given hypersurface is free, now known as ***Saito's criterion***:

Theorem 2.3.3 (Saito [125])**.** *Suppose that X is a complex manifold of dimension n and that $\mathsf{D} \subset \mathsf{X}$ is a reduced hypersurface. Then D is a free divisor if and only if in a neighbourhood of every point $p \in \mathsf{D}$ we can find n vector fields $Z_1, \ldots, Z_n \in \mathcal{T}_\mathsf{X}(-\log \mathsf{D})$ such that the corresponding covolume form $\mu = Z_1 \wedge \cdots \wedge Z_n$ vanishes transversally on the smooth locus of D, i.e., μ is a generator for $\omega_\mathsf{X}^{-1}(-\mathsf{D})$. In this case, $Z_1, \ldots, Z_n$ give a local basis for $\mathcal{T}_\mathsf{X}(-\log \mathsf{D})$.*

The basis $Z_1, \ldots, Z_n$ is sometimes called a ***Saito basis***. The proof of the theorem is essentially an appeal to Cramer's rule, which allows one to express every vector field tangent to D uniquely as a linear combination of the given ones by computing some determinants. In so doing, we must divide by the determinant $Z_1 \wedge \cdots \wedge Z_n$. The transversality assumption assures this apparent singularity is cancelled by a similar factor in the numerator, resulting in a holomorphic expression.

Example 2.3.4. Every smooth hypersurface is free, because we can pick coordinates $(x_1, \ldots, x_n)$ in such a way that D is given by the zero set of x_1. Then the vector fields $Z_1 = x_1\partial_{x_1}, Z_2 = \partial_{x_2}, \ldots, Z_n = \partial_{x_n}$ give a Saito basis.

Example 2.3.5. A hypersurface $\mathsf{D} \subset \mathsf{X}$ is said to have ***normal crossings singularities*** if, in a neighbourhood of every $p \in \mathsf{D}$, we can find coordinates $x_1, \ldots, x_n$ so that D is the zero locus of the function $x_1 x_2 \cdots x_k$ for some $k \geq 1$. Then the vector fields $Z_1 = x_1 \partial_{x_1}, \ldots, Z_k = x_k \partial_{x_k}$ and $Z_{k+1} = \partial_{x_{k+1}}, \ldots, Z_n = \partial_{x_n}$ give a Saito basis and hence D is free. Notice that these vector fields commute; this is not an accident. In fact, a hypersurface D is normal crossings if and only if D is free and at every point of D we can find a basis for $\mathcal{T}_{\mathsf{X}}(-\log \mathsf{D})$ consisting of commuting vector fields [54, Theorem 1.52 and Proposition 1.54]. $\square$

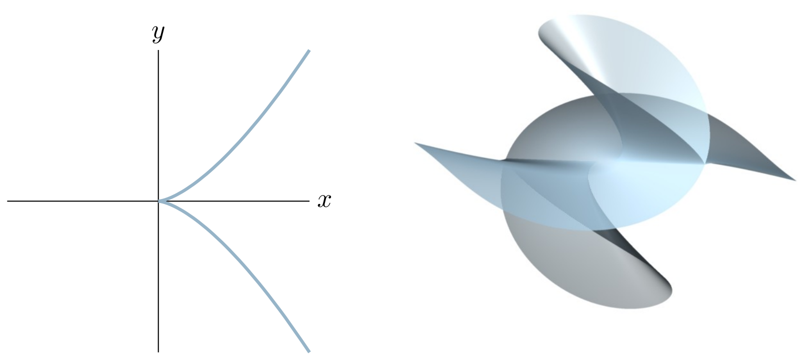
Example 2.3.6. The cusp singularity $\mathsf{D} \subset \mathbb{C}^2$, illustrated in Figure 2.1a, is defined as the zero set of the function $f = x^3 - y^2$. This hypersurface is not normal crossings, but nevertheless it is a free divisor. We claim that the vector fields

$$
\begin{aligned}
Z_1 &= 2x\partial_x + 3y\partial_y \\
Z_2 &= 2y\partial_x + 3x^2\partial_y
\end{aligned}
$$

form a Saito basis. Indeed, one readily verifies that $Z_1(f) = 6f$ and $Z_2(f) = 0$ so that $Z_1, Z_2 \in \mathcal{T}_{\mathbb{C}^2}(-\log \mathsf{D})$, while

$$Z_1 \wedge Z_2 = 6f\partial_x \wedge \partial_y$$

gives a reduced equation for D. The claim now follows from Saito's criterion. $\square$



(a) The cusp $x^3 - y^2 = 0$ (b) The surface $xy^3z + y^5 + z^3 = 0$

Figure 2.1: Some free divisors.

Example 2.3.7. Sekiguchi [129] has classified free divisors in $\mathbb{C}^3$ that are defined by quasi-homogeneous polynomials. For example the surface $\mathsf{D} \subset \mathbb{C}^3$ defined by the equation $xy^3z +$

$y^5 + z^3 = 0$ is a free divisor that Sekiguchi calls $F_{H,7}$. In this case, the vector fields

$$
\begin{aligned}
Z_1 &= x\partial_x + 3y\partial_y + 5z\partial_z \\
Z_2 &= 3y\partial_x - \tfrac{3}{5}(2x^2y + z)\partial_y + \tfrac{10}{3}xy^2\partial_z \\
Z_3 &= 5z\partial_x + \tfrac{3}{5}x(y^2 - 3xz)\partial_y + \tfrac{5}{3}y(3xz - y^2)\partial_z
\end{aligned}
$$

give a basis for $\mathcal{T}_{\mathbb{C}^3}(-\log \mathsf{D})$. This surface is illustrated in Figure 2.1b; we refer the reader to the web site of E. Faber for pictures of the rest of the free divisors from Sekiguchi's classification. □

In general, the freeness of a hypersurface is intimately connected with the structure of its singular locus. By examining the syzygies of the Jacobian ideal, one obtains the following remarkable characterization:

Theorem 2.3.8 ([3, 138])**.** *A singular reduced hypersurface* D *is a free divisor if and only if its singular locus has codimension two in* X *and is Cohen–Macaulay.*

Remark 2.3.9. The Cohen–Macaulay property for an analytic space Y is a weakening of smoothness that nevertheless shares many of the same good properties; see, for example, [50, Chapter 18].

Corollary 2.3.10. *If* X *is a two-dimensional complex manifold (i.e., a complex surface) then every complex curve* $\mathsf{D} \subset \mathsf{X}$*—no matter how singular—is a free divisor.*

2.4 Lie algebroid modules

Suppose that $\mathcal{A}$ is a Lie algebroid on X and that E is a holomorphic vector bundle on X, with $\mathcal{E}$ its sheaf of holomorphic sections. More generally, we could replace $\mathcal{E}$ with an arbitrary sheaf of $\mathcal{O}_\mathsf{X}$-modules. An ***$\mathcal{A}$-connection on $\mathcal{E}$*** is a $\mathbb{C}$-bilinear operator

$$
\begin{array}{rccc}
\nabla & : & \mathcal{A} \times \mathcal{E} & \to & \mathcal{E} \\
 & & (\xi, s) & \mapsto & \nabla_\xi s
\end{array}
$$

with the following properties:

1. ∇ is $\mathcal{O}_\mathsf{X}$-linear with respect to $\mathcal{A}$: that is, the identity
$$\nabla_{f\xi} s = f \nabla_\xi s$$
holds for all $f \in \mathcal{O}_\mathsf{X}$, $\xi \in \mathcal{A}$ and $s \in \mathcal{E}$; and

2. the Leibniz rule
$$\nabla_\xi (fs) = (\mathscr{L}_{a(\xi)} f)s + f\nabla_\xi s$$

holds for all $f \in \mathcal{O}_{\mathsf{X}}$, $\xi \in \mathcal{A}$ and $s \in \mathcal{E}$.

Remark 2.4.1. Equivalently, we may view ∇ as a map $\mathcal{E} \to \mathcal{H}om_{\mathcal{O}_{\mathsf{X}}}(\mathcal{A}, \mathcal{E})$. Provided that either $\mathcal{A}$ or $\mathcal{E}$ is locally free, we have the more familiar form $\mathcal{H}om_{\mathcal{O}_{\mathsf{X}}}(\mathcal{A}, \mathcal{E}) = \mathcal{A}^\vee \otimes \mathcal{E}$ where $\mathcal{A}^\vee$ is the dual of $\mathcal{A}$.

The connection ∇ is ***flat*** if the identity

$$\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi = \nabla_{[\xi,\eta]}$$

holds for all $\eta, \xi \in \mathcal{A}$. In this case, we say that the pair $(\mathcal{E}, \nabla)$—or simply $\mathcal{E}$ itself—is an ***$\mathcal{A}$-module***. A section $s \in \mathcal{E}$ is a ***flat section*** if $\nabla_\xi s = 0$ for all $\xi \in \mathcal{A}$.

Notice that if $\mathcal{E}$ and $\mathcal{E}'$ are $\mathcal{A}$-modules, then there are natural $\mathcal{A}$-module structures on $\mathcal{E} \oplus \mathcal{E}'$, $\mathcal{H}om(\mathcal{E}, \mathcal{E}')$, $\mathcal{E} \otimes \mathcal{E}'$, etc. A ***morphism from $\mathcal{E}$ to $\mathcal{E}'$*** is a morphism of vector bundles (or sheaves of modules) that respects the action of $\mathcal{A}$. A section of $\mathcal{H}om(\mathcal{E}, \mathcal{E}')$ defines a morphism of $\mathcal{A}$-modules if and only if it is a flat section.

Definition 2.4.2. An ***invertible $\mathcal{A}$-module*** is an $\mathcal{A}$-module $(\mathcal{L}, \nabla)$ such that $\mathcal{L}$ a holomorphic line bundle.

A locally free Lie algebroid $\mathcal{A}$ on a complex manifold X comes equipped with two natural invertible modules. The first is the trivial line bundle $\mathcal{O}_{\mathsf{X}}$, on which $\mathcal{A}$ acts by the formula

$$\nabla_\xi f = \mathscr{L}_{a(\xi)} f$$

for $\xi \in \mathcal{A}$ and $f \in \mathcal{O}_{\mathsf{X}}$.

The second natural, defined in [53], is the ***canonical module*** or ***modular representation***, $\omega_{\mathcal{A}} = \det \mathcal{A} \otimes \omega_{\mathsf{X}}$, on which $\mathcal{A}$ acts by the formula

$$\nabla_\xi (u \otimes \omega) = \mathscr{L}_\xi u \otimes \omega + u \otimes \mathscr{L}_{a(\xi)} \omega$$

for $\xi \in \mathcal{A}$, $u \in \det \mathcal{A}$ and $\omega \in \omega_{\mathsf{X}}$. Here $\mathscr{L}_\xi u$ denotes the action of $\mathcal{A}$ on $\det \mathcal{A}$ obtained by mimicking the formula for the Lie derivative of a top-degree multivector field along a vector field. This module plays an important role in the theory of Poincaré duality for Lie algebroid cohomology that we recall in Section 2.5.

Example 2.4.3. Let X be a complex manifold and let $\mathsf{D} \subset \mathsf{X}$ be a reduced hypersurface that is a free divisor in the sense of Definition 2.3.1. Let $\mathcal{A} = \mathcal{T}_{\mathsf{X}}(-\log \mathsf{D})$ be the corresponding Lie algebroid. Then Saito's criterion (Theorem 2.3.3) guarantees that $\det \mathcal{A} \cong \omega_{\mathsf{X}}^{-1}(-\mathsf{D})$ where $\omega_{\mathsf{X}}^{-1} = \det \mathcal{T}_{\mathsf{X}}$ is the anticanonical line bundle. We conclude that the canonical module for $\mathcal{A}$ is

$$\omega_{\mathcal{A}} = \omega_{\mathsf{X}}^{-1}(-\mathsf{D}) \otimes \omega_{\mathsf{X}} \cong \mathcal{O}_{\mathsf{X}}(-\mathsf{D}),$$

the sheaf of functions that vanish on D. One readily verifies that the action of $\mathcal{T}_{\mathsf{X}}(-\log \mathsf{D})$ is simply the usual action by differentiation of functions along vector fields. □

A well-known fact in the theory of $\mathcal{D}$-modules is that a module over the tangent sheaf $\mathcal{T}_{\mathsf{X}}$ that is coherent as an $\mathcal{O}_{\mathsf{X}}$-module is necessarily locally free over $\mathcal{O}_{\mathsf{X}}$—i.e., a vector bundle. However, other Lie algebroids may admit $\mathcal{O}_{\mathsf{X}}$-coherent modules that are not locally-free. For example, if $\mathsf{Y} \subset \mathsf{X}$ is an $\mathcal{A}$-invariant closed subspace with ideal sheaf $\mathcal{I}_{\mathsf{Y}}$, then $\mathcal{I}_{\mathsf{Y}}$ is preserved by $\mathcal{A}$ and is therefore an $\mathcal{A}$-submodule of $\mathcal{O}_{\mathsf{X}}$. Hence the quotient $\mathcal{O}_{\mathsf{Y}} = \mathcal{O}_{\mathsf{X}}/\mathcal{I}_{\mathsf{Y}}$ inherits the structure of an $\mathcal{A}$-module in a natural way, even though it is a torsion coherent sheaf.

Nevertheless, the structure of coherent $\mathcal{A}$-modules is heavily constrained:

Proposition 2.4.4. *Let $\mathcal{A}$ be a Lie algebroid on the analytic space X and let $(\mathcal{E}, \nabla)$ be a sheaf equipped with an $\mathcal{A}$-connection. Then the support of $\mathcal{E}$ is an $\mathcal{A}$-invariant subspace of X.*

Proof. The proof of the first statement follows [117, Lemma 2.1]: suppose that $f\mathcal{E} = 0$ for some $f \in \mathcal{O}_{\mathsf{X}}$, and that $\xi \in \mathcal{A}$. Then

$$0 = \nabla_\xi(f\mathcal{E}) = \mathscr{L}_{a(\xi)}(f)\mathcal{E} + f\nabla_\xi\mathcal{E} = \mathscr{L}_{a(\xi)}(f)\mathcal{E}$$

Hence $\mathscr{L}_{a(\xi)}(f)$ also annihilates $\mathcal{E}$. Thus the annihilator of $\mathcal{E}$ is $\mathcal{A}$-invariant. But the annihilator is exactly the ideal that defines the support of $\mathcal{E}$. □

Proposition 2.4.5. *Suppose that X is a complex manifold and that $\mathcal{A}$ is a locally free Lie algebroid on X. Let $(\mathcal{E}, \nabla)$ be a coherent sheaf equipped with an $\mathcal{A}$-connection. If the anchor map $a : \mathcal{A} \to \mathcal{T}_{\mathsf{X}}$ is surjective, then $\mathcal{E}$ is locally free (a vector bundle).*

Proof. Since the question is local in nature, and $\mathcal{A}$ and $\mathcal{T}_{\mathsf{X}}$ are vector bundles, we work locally and choose a splitting $b : \mathcal{T}_{\mathsf{X}} \to \mathcal{A}$ with $ab = \mathrm{id}_{\mathcal{T}_{\mathsf{X}}}$. Then $b^\vee \circ \nabla : \mathcal{E} \to \Omega^1_{\mathsf{X}} \otimes \mathcal{E}$ is a connection on $\mathcal{E}$ in the usual sense. This connection need not be flat, and hence it does not, in general, give $\mathcal{E}$ the structure of a $\mathcal{D}_{\mathsf{X}}$-module. Nevertheless, the proof in [20, VI, Proposition 1.7] applies word-for-word using this connection to show that $\mathcal{E}$ is locally free. □

The following theorem is an immediate corollary:

Theorem 2.4.6. *Suppose that $\mathcal{A}$ is a locally free Lie algebroid on the complex manifold X and that $\mathcal{E}$ is a coherent sheaf that is an $\mathcal{A}$-module. Then the restriction of $\mathcal{E}$ to any orbit of $\mathcal{A}$ is locally free. Hence the rank of $\mathcal{E}$ is constant on all of the orbits of $\mathcal{A}$.*

Remark 2.4.7. Intuitively, this theorem is expected since we should be able to use the connection to perform a parallel transport between fibres of $\mathcal{E}$ that lie over a given orbit, and hence these fibres must all be isomorphic. □

2.5 Lie algebroid cohomology

Let X be a complex manifold and let $\mathcal{A}$ be a holomorphic vector bundle that is a Lie algebroid on X. The ***k-forms for $\mathcal{A}$*** are given by

$$\Omega^k_{\mathcal{A}} = \wedge^k \mathcal{A}^\vee,$$

where $\mathcal{A}^\vee$ is the dual vector bundle to $\mathcal{A}$. There is a natural differential $d_{\mathcal{A}} : \mathcal{O}_{\mathsf{X}} \to \Omega^1_{\mathcal{A}}$ that takes the function $f \in \mathcal{O}_{\mathsf{X}}$ to the 1-form $a^\vee(df)$, where $a^\vee : \Omega^1_{\mathsf{X}} \to \mathcal{A}^\vee$ is the dual of the anchor. Using the Lie bracket on $\mathcal{A}$, this differential can be extended to a complex of sheaves

$$d\mathcal{R}(\mathcal{A}) = \left(0 \longrightarrow \mathcal{O}_{\mathsf{X}} \xrightarrow{d_{\mathcal{A}}} \Omega^1_{\mathcal{A}} \xrightarrow{d_{\mathcal{A}}} \Omega^2_{\mathcal{A}} \xrightarrow{d_{\mathcal{A}}} \cdots \right)$$

by mimicking the usual formula for the exterior derivative of differential forms. The ***cohomology of $\mathcal{A}$*** is the hypercohomology

$$\mathsf{H}^\bullet(\mathcal{A}) = \mathbb{H}^\bullet(d\mathcal{R}(\mathcal{A}))$$

of this complex of sheaves.

More generally, associated to any $\mathcal{A}$-module $(\mathcal{E}, \nabla)$ is its de Rham complex

$$d\mathcal{R}(\mathcal{A}, \mathcal{E}) = \left(0 \longrightarrow \mathcal{E} \xrightarrow{\nabla} \Omega^1_{\mathcal{A}} \otimes \mathcal{E} \xrightarrow{d^\nabla_{\mathcal{A}}} \Omega^2_{\mathcal{A}} \otimes \mathcal{E} \xrightarrow{d^\nabla_{\mathcal{A}}} \cdots \right)$$

obtained by combining the differential of $\mathcal{A}$ with the flat connection ∇ in the usual way. The ***de Rham cohomology of $\mathcal{A}$ with values in $\mathcal{E}$*** is the hypercohomology

$$\mathsf{H}^\bullet(\mathcal{A}, \mathcal{E}) = \mathbb{H}^\bullet(d\mathcal{R}(\mathcal{A}, \mathcal{E}))$$

of this complex of sheaves. When $\mathcal{A}$ is a vector bundle, we have

$$\mathsf{H}^\bullet(\mathcal{A}, \mathcal{E}) = \mathsf{Ext}^\bullet_{\mathcal{A}}(\mathcal{O}_{\mathsf{X}}, \mathcal{E})$$

where the right hand side is the usual Yoneda group of extensions in the abelian category of $\mathcal{A}$-modules [123]. When $\mathcal{A}$ is not a vector bundle, we prefer to *define* the cohomology of $\mathcal{A}$ by this formula.

We remark that if

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

is an exact sequence of $\mathcal{A}$-modules, we obtain a corresponding long exact sequence in cohomology. Notice that this approach becomes particularly useful when we allow sheaves that are not vector bundles:

Example 2.5.1. If $\mathsf{Y} \subset \mathsf{X}$ is an $\mathcal{A}$-invariant closed subspace corresponding to the ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ and $\mathcal{E}$ is an $\mathcal{A}$-module, then we have the exact sequence

$$0 \longrightarrow \mathcal{I}\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}|_{\mathsf{Y}} \longrightarrow 0$$

of $\mathcal{A}$-modules. We therefore have the long exact sequence

$$\cdots \longrightarrow \mathsf{H}^\bullet(\mathcal{A}, \mathcal{I}\mathcal{E}) \longrightarrow \mathsf{H}^\bullet(\mathcal{A}, \mathcal{E}) \longrightarrow \mathsf{H}^\bullet(\mathcal{A}|_{\mathsf{Y}}, \mathcal{E}|_{\mathsf{Y}}) \longrightarrow \mathsf{H}^{\bullet+1}(\mathcal{A}, \mathcal{I}\mathcal{E}) \longrightarrow \cdots$$

relating the cohomology of $\mathcal{E}$ to that of its restriction.

One reason for the importance of the canonical module $\omega_{\mathcal{A}}$ is its role as a dualizing object: in [53], it was shown that the contractions

$$\Omega^j_{\mathcal{A}} \otimes \Omega^{r-j}_{\mathcal{A}} \otimes \det \mathcal{A} \otimes \omega_{\mathsf{X}} \to \omega_{\mathsf{X}}$$

for $j \geq 0$ and $r = \mathrm{rank}(\mathcal{A})$, give rise to a natural pairing of complexes

$$d\mathcal{R}(\mathcal{A}, \mathcal{E}) \otimes d\mathcal{R}(\mathcal{A}, \mathcal{E})\,[-r] \to \omega_{\mathsf{X}}.$$

When X is a compact manifold, we obtain a Poincaré duality-type pairing on Lie algebroid cohomology.

The original definition of the pairing was made in the smooth category, where it is not, in general, perfect. In contrast, the complex analytic situation is much better behaved and one may show that the induced pairing is perfect in two different ways. In [135], Stiénon used a Dolbeault-type approach based on Block's duality theory [13] for elliptic Lie algebroids. In [35], Chemla followed the approach to duality used in the theory of $\mathcal{D}$-modules, obtaining a relative version of the duality theory, applicable in the more general situation of a proper morphisms of Lie algebroids. We shall recall only the global version here, which corresponds to the case of a morphism to a point:

Theorem 2.5.2 ([35, Corollary 4.3.6],[135, Proposition 6.3])**.** *Let X be a compact complex manifold of dimension n and let $\mathcal{A}$ a holomorphic Lie algebroid on X that is locally free of rank r. If $\mathcal{E}$ is a holomorphic vector bundle of finite rank that is an $\mathcal{A}$-module, then the natural pairing*

$$\mathsf{H}^i(\mathcal{A}, \mathcal{E}) \otimes_{\mathbb{C}} \mathsf{H}^{n+r-i}(\mathcal{A}, \mathcal{E}^\vee \otimes \omega_{\mathcal{A}}) \to \mathbb{C}$$

is perfect.

Example 2.5.3. When $\mathcal{A} = 0$ is the trivial Lie algebroid, the duality is just Serre duality. $\square$

Example 2.5.4. When $\mathcal{A} = \mathcal{T}_{\mathsf{X}}$, the cohomology of $\mathcal{A}$ is the de Rham cohomology of X and the duality is the usual Poincaré duality defined by integration of forms. □

2.5.1 The Picard group

Let $\mathcal{A}$ be a Lie algebroid on X. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are invertible $\mathcal{A}$-modules (i.e., line bundles with flat $\mathcal{A}$-connections), then the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$ is another invertible module, as is the dual $\mathcal{L}_1^\vee$. Moreover, the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_1^\vee$ is canonically isomorphic to the trivial line bundle $\mathcal{O}_{\mathsf{X}}$ as an $\mathcal{A}$-module. Thus, the isomorphism classes of invertible $\mathcal{A}$-modules form an abelian group, where the group operation is tensor product and the inversion is given by taking duals. This group is called the ***Picard group of*** $\mathcal{A}$. We denote by $\mathsf{Pic}_0(\mathcal{A}) \subset \mathsf{Pic}(\mathcal{A})$ the subgroup consisting of line bundles whose first Chern class is trivial.

The Picard group has a cohomological interpretation, which we now explain. This description is similar to the usual description of the usual Picard group of holomorphic line bundles on a complex manifold; see, e.g., [69]. An immediate corollary is the observation that the Picard group of a Lie algebroid on a compact complex manifold naturally has the structure of a finite dimensional abelian complex Lie group.

Let $\mathcal{O}_{\mathsf{X}}^\times$ be the sheaf of non-vanishing holomorphic functions on X, given the structure of a sheaf of abelian groups via multiplication of functions. There is a natural operator $d_\sigma \log : \mathcal{O}_{\mathsf{X}}^\times \to \mathcal{A}^\vee$ taking a non-vanishing function f to $f^{-1} d_{\mathcal{A}}(f)$, and we can extend this map to a complex

$$d\mathcal{R}(\mathcal{A}, \mathcal{O}_{\mathsf{X}}^\times) = \left(0 \longrightarrow \mathcal{O}_{\mathsf{X}} \xrightarrow{d_\sigma} \Omega^1_{\mathcal{A}} \xrightarrow{d_\sigma} \Omega^2_{\mathcal{A}} \xrightarrow{d_\sigma} \cdots \right)$$

using the usual Lie algebroid differential. We denotes its cohomology by $\mathsf{H}^\bullet(\mathcal{A}, \mathcal{O}_{\mathsf{X}}^\times) = \mathbb{H}^\bullet(d\mathcal{R}(\mathcal{A}, \mathcal{O}_{\mathsf{X}}^\times))$.

Proposition 2.5.5. *There is a canonical isomorphism* $\mathsf{Pic}(\mathcal{A}) \cong \mathsf{H}^1(\mathcal{A}, \mathcal{O}_{\mathsf{X}}^\times)$.

Proof. The proof is the same as for usual flat holomorphic connections (see, e.g., [52]). Computing the hypercohomology using a Čech resolution for a good open covering $\mathsf{X} = \bigcup_{i \in \mathsf{I}} \mathsf{U}_i$, one finds that a one-cocycle is represented by non-vanishing functions $g_{ij} \in \mathcal{O}_{\mathsf{X}}^\times(\mathsf{U}_{ij})$ on double overlaps and one-forms $\alpha_i \in \Omega^1_{\mathcal{A}}(\mathsf{U}_i)$ on open sets.

The cocycle condition forces $g_{ij} g_{jk} g_{ki} = 1$, so that the functions g_{ij} give the transition functions for a holomorphic line bundle over X. The condition also forces $\alpha_i - \alpha_j = d_{\mathcal{A}} \log g_{ij}$, so that the one-forms α_i glue together to give the resulting line bundle an $\mathcal{A}$-connection. Finally, the cocycle condition forces $d_{\mathcal{A}} \alpha_i = 0$ for all i so that this connection is flat.

Similarly, one checks that two cocycles are cohomologous if and only if they define isomorphic $\mathcal{A}$-modules, completing the proof. □

Notice that the morphism $\exp : \mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}^{\times}$ induces a surjective morphism of complexes $d\mathcal{R}(\mathcal{A}) \to d\mathcal{R}\big(\mathcal{A}, \mathcal{O}_{\mathsf{X}}^{\times}\big)$, where the morphism $\Omega^{\bullet}_{\mathcal{A}} \to \Omega^{\bullet}_{\mathcal{A}}$ in positive degree is simply the identity. We therefore have an exact sequence of complexes

$$0 \longrightarrow \mathbb{Z}_{\mathsf{X}} \xrightarrow{2\pi i} d\mathcal{R}(\mathcal{A}) \xrightarrow{\exp} d\mathcal{R}\big(\mathcal{A}, \mathcal{O}_{\mathsf{X}}^{\times}\big) \longrightarrow 0,$$

where $\mathbb{Z}_{\mathsf{X}}$ is the sheaf of locally-constant $\mathbb{Z}$-valued functions. The long exact sequence in hypercohomology then gives the exactness of the sequence

$$\mathsf{H}^0(\mathcal{A}) \xrightarrow{\exp} \mathsf{H}^0\big(\mathcal{A}, \mathcal{O}_{\mathsf{X}}^{\times}\big) \longrightarrow \mathsf{H}^1(\mathsf{X}, \mathbb{Z}) \longrightarrow \mathsf{H}^1(\mathcal{A}) \longrightarrow \mathsf{Pic}(\mathcal{A}) \xrightarrow{c_1} \mathsf{H}^2(\mathsf{X}, \mathbb{Z})$$

where $\mathsf{H}^{\bullet}(\mathsf{X}, \mathbb{Z})$ is the usual singular cohomology of X and c_1 sends an invertible module $(\mathcal{L}, \nabla)$ to the first Chern class of $\mathcal{L}$.

Notice that the zeroth cohomology groups in this sequence are spaces of global holomorphic functions. If X is compact, all global holomorphic functions are constant, and so the leftmost map becomes the exponential $\exp : \mathbb{C} \to \mathbb{C}^*$, which is surjective. Thus, the map $\mathsf{H}^1(\mathsf{X}, \mathbb{Z}) \to \mathsf{H}^1(\mathcal{A})$ is injective. Moreover, $\mathsf{H}^1(\mathsf{X}, \mathcal{A})$ is finite dimensional because it is built from the cohomology of coherent sheaves on X. Finally, the map $\mathsf{Pic}(\mathcal{A}) \to \mathsf{H}^2(\mathsf{X}, \mathbb{Z})$ is exactly the first Chern class. We therefore arrive at the following

Corollary 2.5.6. *If X is compact, then $\mathsf{Pic}(\mathcal{A})$ is a finite-dimensional complex Lie group. Moreover, the connected component containing the the identity is identified with*

$$\mathsf{Pic}_0(\mathcal{A}) \cong \mathsf{H}^1(\mathcal{A}) \,/ \mathsf{H}^1(\mathsf{X}, \mathbb{Z})\,.$$

In particular, the Lie algebra of $\mathsf{Pic}(\mathcal{A})$ is naturally isomorphic to $\mathsf{H}^1(\mathcal{A})$. Moreover, there is an exact sequence

$$0 \longrightarrow \mathsf{H}^0\Big(\mathsf{X}, \Omega^1_{\mathcal{A},cl}\Big) \longrightarrow \mathsf{Pic}_0(\mathcal{A}) \longrightarrow \mathsf{Pic}_0(\mathsf{X})$$

where $\mathsf{H}^0\Big(\mathsf{X}, \Omega^1_{\mathcal{A},cl}\Big)$ is the space of global, holomorphic, closed 1-forms for $\mathcal{A}$ and $\mathsf{Pic}_0(\mathsf{X})$ is the usual Picard group of holomorphic line bundles on X with trivial first Chern class.

Proof. It remains to prove the final statement. To do so, we simply note that if the holomorphic line bundles underlying two invertible $\mathcal{A}$-modules $(\mathcal{L}, \nabla)$ and $(\mathcal{L}, \nabla')$ are the same, then their difference $\nabla - \nabla'$ is a closed global section of $\Omega^1_{\mathcal{A}}$. □

Notice that when X is compact, all global holomorphic functions on X are constant and so there are no nonzero, globally-defined exact 1-forms for $\mathcal{A}$. Hence there is a natural inclusion $\mathsf{H}^0\Big(\mathsf{X}, \Omega^1_{\mathcal{A},cl}\Big) \to \mathsf{H}^1(\mathcal{A})$, which is the derivative of the map $\mathsf{H}^0\Big(\mathsf{X}, \Omega^1_{\mathcal{A},cl}\Big) \to \mathsf{Pic}_0(\mathcal{A})$ at the identity.

Corollary 2.5.7. *If* X *is a compact complex manifold and its fundamental group is finite, then the first Lie algebroid cohomology of* X *coincides with the space of* $d_{\mathcal{A}}$*-closed global sections of* $\Omega^1_{\mathcal{A}}$.

Proof. In this case, $\mathsf{H}^1(\mathsf{X}, \mathcal{O}_\mathsf{X}) = 0$ so that $\mathsf{Pic}(\mathsf{X})$ is discrete. □

2.6 The universal envelope, jets and higher order connections

We now recall the construction of the ring of differential operators for a Lie algebroid. We shall make relatively little use of this notion, but it will be useful in describing jets and higher order operators in the next section. This construction is described in many references and is particularly useful if one wants to do homological algebra; see, for example [11, 35, 36, 117, 123]

Let $\mathcal{A}$ be a Lie algebroid on X. The ***universal enveloping algebroid of*** $\mathcal{A}$, also known as the ***ring of*** $\mathcal{A}$***-differential operators*** is the sheaf $\mathcal{D}_\mathcal{A}$ of $\mathbb{C}$-algebras generated by $\mathcal{O}_\mathsf{X}$ and $\mathcal{A}$, subject to the relations

$$\begin{aligned} f \otimes g &= fg \\ f \otimes \xi &= f\xi \\ \xi \otimes f - f \otimes \xi &= \mathscr{L}_{a(\xi)} f \\ \xi \otimes \eta - \eta \otimes \xi &= [\xi, \eta] \end{aligned} \tag{2.1}$$

for all $f, g \in \mathcal{O}_\mathsf{X}$ and $\xi, \eta \in \mathcal{A}$.

Example 2.6.1. When $\mathcal{A} = \mathcal{T}_\mathsf{X}$ is the tangent sheaf of a complex manifold X, this construction yields the usual sheaf $\mathcal{D}_\mathsf{X}$ of differential operators on X. □

Example 2.6.2. When X is a point so that $\mathcal{A} = \mathfrak{g}$ is just a Lie algebra, we recover the universal enveloping algebra of $\mathfrak{g}$.

The sheaf of rings $\mathcal{D}_\mathcal{A}$ has a natural filtration by the order of a differential operator: $\mathcal{D}^{\leq k}_\mathcal{A}$ is simply the $\mathcal{O}_\mathsf{X}$-submodule of $\mathcal{D}_\mathcal{A}$ generated by the words of degree $\leq k$ in $\mathcal{A}$. The associated graded sheaf of rings is the symmetric algebra $\mathcal{S}ym^\bullet_{\mathcal{O}_\mathsf{X}}\mathcal{A}$.

If $(\mathcal{E}, \nabla)$ is an $\mathcal{A}$-module, then the morphism

$$\nabla : \mathcal{A} \to \mathcal{E}nd_{\mathbb{C}_\mathsf{X}}(\mathcal{E})$$

defined by the action of $\mathcal{A}$ on $\mathcal{E}$ is compatible with the relations (2.1) above, and it therefore extends to a map

$$\widetilde{\nabla} : \mathcal{D}_\mathcal{A} \to \mathcal{E}nd_{\mathbb{C}_\mathsf{X}}(\mathcal{E}) \,,$$

making $\mathcal{E}$ into a left $\mathcal{D}_{\mathcal{A}}$-module. Conversely, any left $\mathcal{D}_{\mathcal{A}}$-module $\mathcal{E}$ inherits an $\mathcal{O}_{\mathsf{X}}$-module structure from the embedding $\mathcal{O}_{\mathsf{X}} \to \mathcal{D}_{\mathcal{A}}$ and a flat $\mathcal{A}$-connection from the embedding $\mathcal{A} \to \mathcal{D}_{\mathcal{A}}$.

The basic result is the following

Lemma 2.6.3. *There is a natural equivalence between the categories of $\mathcal{A}$-modules and left $\mathcal{D}_{\mathcal{A}}$-modules and this equivalence preserves the underlying $\mathcal{O}_{\mathsf{X}}$-module structure.*

This allows one to do homological algebra with $\mathcal{A}$-modules by treating them as modules over this sheaf of rings. For example, this lemma makes it clear that the category of $\mathcal{A}$-modules has enough injectives.

Dual to the ring of differential operators is the sheaf of jets, as we now briefly recall. See [29] for a more detailed discussion.

Definition 2.6.4. Let $\mathcal{A}$ be a Lie algebroid on X and $\mathcal{E}$ an $\mathcal{O}_{\mathsf{X}}$-module. The ***sheaf of k-jets of $\mathcal{E}$ along $\mathcal{A}$*** is the $\mathcal{O}_{\mathsf{X}}$-module

$$\mathcal{J}^k_{\mathcal{A}}\mathcal{E} = \mathcal{H}om_{\mathcal{O}_{\mathsf{X}}}\left(\mathcal{D}^{\leq k}_{\mathcal{A}}, \mathcal{E}\right)$$

When $\mathcal{A} = \mathcal{T}_{\mathsf{X}}$ is the tangent bundle of a complex manifold, this construction recovers the usual sheaf of jets of $\mathcal{E}$. The anchor map induces a natural morphism

$$j^k a^* : \mathcal{J}^k\mathcal{E} \to \mathcal{J}^k_{\mathcal{A}}\mathcal{E}$$

and so we may define the ***k-jet of $s \in \mathcal{E}$ along $\mathcal{A}$*** to be the section $j^k_{\mathcal{A}} s = j^k a^*(j^k s)$, where $j^k s \in \mathcal{J}^k\mathcal{E}$ is the usual k-jet of s.

Since $\mathcal{D}^{\leq k}_{\mathcal{A}}/\mathcal{D}^{\leq k-1}_{\mathcal{A}} \cong \mathcal{S}ym^k_{\mathcal{O}_{\mathsf{X}}}\mathcal{A}$, we have a natural exact sequence

$$0 \longrightarrow \mathcal{S}ym^k_{\mathcal{O}_{\mathsf{X}}}\mathcal{A}^\vee \otimes_{\mathcal{O}_{\mathsf{X}}} \mathcal{E} \longrightarrow \mathcal{J}^k_{\mathcal{A}}\mathcal{E} \longrightarrow \mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E} \longrightarrow 0 \tag{2.2}$$

In order to model higher-order ordinary differential equations in Section 3.2.2 it is useful to introduce the notion of a higher-order connection on a sheaf. Following [49], we have the

Definition 2.6.5. Let $\mathcal{A}$ be a Lie algebroid on X and let $\mathcal{E}$ be an $\mathcal{O}_{\mathsf{X}}$-module. A kth-order $\mathcal{A}$-connection on $\mathcal{E}$ is an $\mathcal{O}_{\mathsf{X}}$-linear morphism

$$\Delta : \mathcal{J}^k_{\mathcal{A}}\mathcal{E} \to \mathsf{Sym}^k_{\mathcal{O}_{\mathsf{X}}}\mathcal{A}^\vee \otimes_{\mathcal{O}_{\mathsf{X}}} \mathcal{E}$$

that splits the jet sequence (2.2). If $s \in \mathcal{E}$, we abuse notation and write

$$\Delta s = \Delta(j^k_{\mathcal{A}} s) \in \mathsf{Sym}^k_{\mathcal{O}_{\mathsf{X}}}\mathcal{A}^\vee \otimes_{\mathcal{O}_{\mathsf{X}}} \mathcal{E}.$$

If $\Delta s = 0$, we say that s is a ***solution of*** Δ.

In particular, a first-order $\mathcal{A}$-connection on $\mathcal{E}$ is nothing but an $\mathcal{A}$-connection on $\mathcal{E}$ in the usual sense. Notice that this connection is not required to be flat.

In studying higher order differential equations, it is often useful to convert them to a system of first order equations by introducing new variables for the derivatives. The invariant description of this procedure is a canonical method for turning a kth-order $\mathcal{A}$-connection on $\mathcal{E}$ into an $\mathcal{A}$-connection on the sheaf $\mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E}$, which we now describe.

As for usual jet bundles, there is an embedding $\mathcal{J}^k_{\mathcal{A}}\mathcal{E} \to \mathcal{J}^1_{\mathcal{A}}\mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E}$ defined by $j^k_{\mathcal{A}}s \mapsto j^1_{\mathcal{A}}j^{k-1}_{\mathcal{A}}s$, giving the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{S}ym^k_{\mathcal{O}_{\mathsf{X}}}\mathcal{A}^\vee \otimes \mathcal{E} \arrow[r] \arrow[d, hook] & \mathcal{J}^k_{\mathcal{A}}\mathcal{E} \arrow[r] \arrow[d, hook] & \mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathcal{A}^\vee \otimes \mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E} \arrow[r] & \mathcal{J}^1_{\mathcal{A}}\mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E} \arrow[r] & \mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E} \arrow[r] & 0 \end{tikzcd} \tag{2.3}$$

where the leftmost vertical map is given by the inclusions

$$\mathcal{S}ym^k_{\mathcal{O}_{\mathsf{X}}}\mathcal{A}^\vee \otimes \mathcal{E} \to \mathcal{A}^\vee \otimes \mathsf{Sym}^{k-1}_{\mathcal{O}_{\mathsf{X}}}\mathcal{A}^\vee \otimes \mathcal{E} \to \mathcal{A}^\vee \otimes \mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E}.$$

Since specifying a kth-order connection $\Delta : \mathcal{J}^k_{\mathcal{A}}\mathcal{E} \to \mathcal{S}ym^k_{\mathcal{O}_{\mathsf{X}}}\mathcal{A}^\vee \otimes \mathcal{E}$ is equivalent to specifying a right splitting $\mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E} \to \mathcal{J}^k_{\mathcal{A}}\mathcal{E}$ of (2.2) we see that Δ induces a splitting

$$\mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E} \to \mathcal{J}^k_{\mathcal{A}}\mathcal{E} \to \mathcal{J}^1_{\mathcal{A}}\mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E}$$

of the one-jet sequence for $\mathcal{J}^{k-1}_{\mathcal{A}}\mathcal{E}$. The latter splitting is, in turn, the same as an $\mathcal{A}$-connection on $\mathcal{J}^k_{\mathcal{A}}\mathcal{E}$. We see immediately that $s \in \mathcal{E}$ is a solution of Δ if and only if $j^{k-1}_{\mathcal{A}}s$ is a flat section for the resulting connection.

2.7 Holomorphic Lie groupoids

Lie algebroids are the infinitesimal counterparts of Lie groupoids, which originated in the work of Erhesmann and Pradines. In this section, we briefly review the definitions and basic constructions. We refer the reader to, e.g., [103, 107] for a thorough introduction.

Recall that a groupoid is simply a category in which every object is invertible. A Lie groupoid is a groupoid with a compatible manifold structure, defined as follows

Definition 2.7.1. A ***holomorphic Lie groupoid*** is a tuple $(\mathsf{G}, \mathsf{X}, s, t, m, \mathrm{id})$ defining a groupoid whose arrows are a complex manifold G and whose morphisms are a complex manifold X. The maps $s, t : \mathsf{G} \to \mathsf{X}$ which take an arrow to its source and target are required to be holomorphic submersions, and the composition of arrows is given by a holomorphic map

$$m : \mathsf{G}^{(2)} = \mathsf{G}_s \times_t \mathsf{G} \to \mathsf{G},$$

where $\mathsf{G}_s \times_t \mathsf{G} = \{(g,h) \in \mathsf{G} \times \mathsf{G} \mid s(g) = t(h)\}$ denotes the fibre product of G with itself over s and t. The map $\mathrm{id} : \mathsf{X} \to \mathsf{G}$ is a closed embedding which takes $x \in \mathsf{X}$ to the identity arrow over X. If $g, h \in \mathsf{G}$ satisfies $s(g) = t(h)$, we write $gh = m(g,h)$ for their composition.

We will often abuse notation and denote a groupoid simply by $\mathsf{G} \rightrightarrows \mathsf{X}$, or even G when X is clear from context. Since in this thesis we work only in the complex analytic category, we will omit the adjective "holomorphic" and simply refer to $\mathsf{G} \rightrightarrows \mathsf{X}$ as a ***Lie groupoid over*** X.

Definition 2.7.2. A ***morphism between the Lie groupoids*** $\mathsf{G} \rightrightarrows \mathsf{X}$ ***and*** $\mathsf{H} \rightrightarrows \mathsf{Y}$ is a holomorphic map $F : \mathsf{G} \to \mathsf{H}$ that induces a functor between the corresponding categories. $\square$

Example 2.7.3. If X is a complex manifold, the ***pair groupoid*** $\mathsf{Pair}(\mathsf{X}) \rightrightarrows \mathsf{X}$ has arrows $\mathsf{Pair}(\mathsf{X}) = \mathsf{X} \times \mathsf{X}$. The groupoid structure is completely fixed by requiring that the source and target maps are given by

$$s(x,y) = y$$
$$t(x,y) = x$$

The composition of $(x,y), (y,z) \in \mathsf{Pair}(\mathsf{X})$ given by

$$(x,y) \cdot (y,z) = (x,z)$$

and the identity map $\mathrm{id} : \mathsf{X} \to \mathsf{Pair}(\mathsf{X})$ is the diagonal embedding of X. The situation is illustrated in Figure 2.2. Since every Lie groupoid $\mathsf{G} \rightrightarrows \mathsf{X}$ maps to $\mathsf{Pair}(\mathsf{X})$ via its source and target maps, $\mathsf{Pair}(\mathsf{X})$ is the terminal object in the catgeory of Lie groupoids over X, and therefore this picture serves as a good intuitive guide for the structure of other groupoids. $\square$

Example 2.7.4. If X is a complex manifold and H is a complex Lie group acting holomorphically on X, we may form the ***action groupoid*** $\mathsf{H} \ltimes \mathsf{X} \rightrightarrows \mathsf{X}$. The space of arrows is $\mathsf{H} \ltimes \mathsf{X} = \mathsf{H} \times \mathsf{X}$, and the source and target maps are given by

$$s(h,x) = x$$
$$t(h,x) = h \cdot x$$

with composition

$$(h,x) \cdot (h',y) = (hh',y)$$

defined whenever $x = h'y$. The identity map is given by the identification $\mathsf{X} \cong \{1\} \times \mathsf{X} \subset$

$\mathsf{H} \times \mathsf{X}$ where $1 \in \mathsf{H}$ is the identity element in the group. $\square$

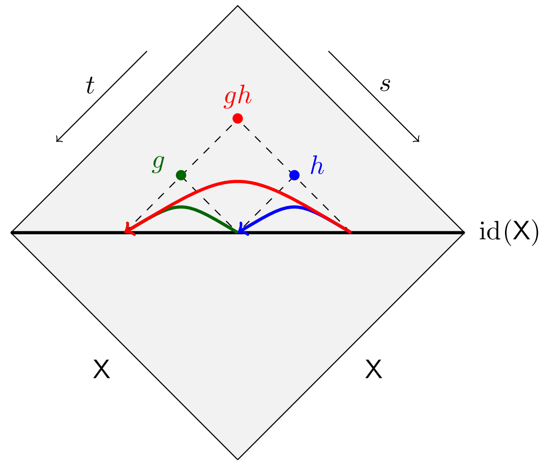


Figure 2.2: A schematic diagram of the pair groupoid of X, which is the product $\mathsf{X} \times \mathsf{X}$. This picture is rotated by $\pi/4$ from the usual illustration of $\mathsf{X} \times \mathsf{X}$ so that the manifold of identity arrows—the diagonal embedding of X—is a horizontal line.

Let $\mathsf{G} \rightrightarrows \mathsf{X}$ be a Lie groupoid, and let $\mathsf{Y} \subset \mathsf{X}$ be an open submanifold or a closed analytic subspace. The ***full subgroupoid of*** G ***over*** Y, is the subspace

$$\mathsf{G}|_{\mathsf{Y}} = s^{-1}(\mathsf{Y}) \cap t^{-1}(\mathsf{Y}) \subset \mathsf{G}$$

consisting of all arrows in G which start and end on Y. This subspace inherits the structure of a groupoid over Y by restricting the structure maps for G. If Y is smooth and an appropriate transversality condition is satisfied, then will $\mathsf{G}|_{\mathsf{Y}}$ will be a Lie groupoid. Of particular importance is the case when $\mathsf{Y} = \{p\}$ is a single point. In this case, the full subgroupoid $\mathsf{G}|_p$ forms a Lie group, called the ***isotropy group of*** G ***at*** p***.***

Example 2.7.5. If H is Lie group acting on X, we may form the action groupoid $\mathsf{H} \ltimes \mathsf{X}$ as in Example 2.7.4. For $p \in \mathsf{X}$, the isotropy group $(\mathsf{H} \ltimes \mathsf{X})|_p$ is $\mathsf{S} \times \{p\}$, where $\mathsf{S} \subset \mathsf{H}$ is the stabilizer of p. $\square$

Just as the infinitesimal version of a Lie group is a Lie algebra, the infinitesimal version of a Lie groupoid $\mathsf{G} \rightrightarrows \mathsf{X}$ is a Lie algebroid $\mathcal{L}ie(\mathsf{G})$ on X, constructed as follows. Consider the subsheaf $\mathcal{T}_t = \mathsf{ker}(Tt) \subset \mathcal{T}_{\mathsf{G}}$ consisting of vector fields that are tangent to the target fibres. Since t is a submersion, this sheaf is locally free—a vector bundle. Moreover if $x \in \mathsf{X}$, then any vector field $\xi \in \mathcal{T}_t$ restricts to a vector field on $t^{-1}(x)$.

If $g \in \mathsf{G}$, then left multiplication by G gives an isomorphism

$$L_g : t^{-1}(s(g)) \to t^{-1}(t(g)).$$

We say that $\xi \in \mathcal{T}_t$ is ***left-invariant*** if for every $g \in \mathsf{G}$, the isomorphism L_g sends the vector field $\xi|_{t^{-1}(s(g))}$ to the vector field $\xi|_{t^{-1}(t(g))}$. One readily checks that left-invariant vector fields are closed under the Lie bracket on $\mathcal{T}_t$, and that a left-invariant vector field is completely determined by its restriction to $\mathrm{id}(\mathsf{X}) \subset \mathsf{G}$. Hence the $\mathcal{O}_\mathsf{X}$-module

$$\mathcal{L}ie(\mathsf{G}) = \mathcal{T}_t|_{\mathrm{id}(\mathsf{X})},$$

which is isomorphic to the normal bundle of $\mathrm{id}(\mathsf{X})$ in G, inherits a Lie bracket. It also inherits a map to the tangent sheaf via the composition

$$\mathcal{T}_t|_{\mathrm{id}(\mathsf{X})} \longrightarrow \mathcal{T}_\mathsf{G}|_{\mathrm{id}(\mathsf{X})} \xrightarrow{Ts} \mathcal{T}_\mathsf{X}.$$

With this structure, $\mathcal{L}ie(\mathsf{G})$ becomes a locally free Lie algebroid on X. This construction is functorial: if $F : \mathsf{G} \to \mathsf{H}$ is a morphism between the Lie groupoids $\mathsf{G} \rightrightarrows \mathsf{X}$ and $\mathsf{H} \rightrightarrows \mathsf{Y}$, its derivative defines a morphism $\mathcal{L}ie(\mathsf{G}) \to \mathcal{L}ie(\mathsf{H})$.

Example 2.7.6. The Lie algebroid of $\mathsf{Pair}(\mathsf{X})$ is naturally identified with the tangent sheaf $\mathcal{T}_\mathsf{X}$. □

Example 2.7.7. If $\mathcal{E}$ is a locally-free sheaf on X, the ***gauge groupoid of*** $\mathcal{E}$ is the Lie groupoid $\mathsf{GL}(\mathcal{E})$ over X for which the morphisms between the points p and q in X are the $\mathbb{C}$-linear isomorphisms $\mathcal{E}|_p \to \mathcal{E}|_q$ between the fibres of $\mathcal{E}$. The isotropy group at p is therefore identified with $\mathsf{GL}(\mathcal{E}|_p)$, the group of linear automorphisms of the fibre over p.

The Lie algebroid of $\mathsf{GL}(\mathcal{E})$ is the sheaf $\mathcal{D}^1(\mathcal{E})$ of first order differential operators on $\mathcal{E}$—the Atiyah algebroid discussed in Example 2.2.5. The isotropy Lie algebra at p is $\mathsf{End}_\mathbb{C}(\mathcal{E}|_p)$, which is the Lie algebra of the isotropy group. □

This example is important to us because it is related to the notion of a representation of a Lie groupoid on a vector bundle:

Definition 2.7.8. Let $\mathsf{G} \rightrightarrows \mathsf{X}$ be a Lie groupoid, let $p_1, p_2 : \mathsf{G}^{(2)} = \mathsf{G}_s\times_t\mathsf{G} \to \mathsf{G}$ be the projections on the first and second factors, and let $m : \mathsf{G}^{(2)} \to \mathsf{G}$ be the composition in the groupoid. A G-***equivariant sheaf*** is a pair $(\mathcal{E}, \rho)$, where $\mathcal{E}$ is an $\mathcal{O}_\mathsf{X}$-module, and $\rho \in \mathsf{Hom}_\mathsf{G}(s^*\mathcal{E}, t^*\mathcal{E})$ is an isomorphism such that

$$p_1^*\rho \circ p_2^*\rho = m^*\rho,$$

i.e., ρ is multiplicative. If $\mathcal{E}$ is locally free, we say that the pair $(\mathcal{E}, \rho)$ is a ***representation of*** G.

A morphism between two equivariant sheaves $(\mathcal{E}, \rho)$ and $(\mathcal{E}', \rho')$ is an $\mathcal{O}_\mathsf{X}$-linear map

$\psi : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes:

$$\begin{tikzcd} s^*\mathcal{E} \arrow[r, "\rho"] \arrow[d, "s^*\phi"'] & t^*\mathcal{E} \arrow[d, "t^*\phi"] \\ s^*\mathcal{E}' \arrow[r, "\rho'"] & t^*\mathcal{E}' \end{tikzcd}$$

We denote by $\mathscr{R}ep(\mathsf{G})$ the category of representations of G with these morphisms. □

Remark 2.7.9. If $\mathcal{E}$ is a locally-free sheaf, then a representation of G on $\mathcal{E}$ is equivalent to the specification of a morphism $\rho : \mathsf{G} \to \mathsf{GL}(\mathcal{E})$ of Lie groupoids, i.e., a holomorphic assignment to every $g \in \mathsf{G}$ of a linear isomorphism $\rho(g) : \mathcal{E}|_{s(g)} \to \mathcal{E}|_{t(g)}$, such that

$$\rho(gh) = \rho(g)\rho(h).$$

for all composable elements $g, h \in \mathsf{G}$. □

Example 2.7.10. For the case of groupoid $\mathsf{H} \ltimes \mathsf{X} \rightrightarrows \mathsf{X}$ associated with a group action as in Example 2.7.4, a representation is the same thing as an equivariant vector bundle for the group action. □

2.7.1 Integration and source-simply connected groupoids

By analogy with the case of Lie algebras and Lie groups, it is natural to ask whether a given Lie algebroid $\mathcal{A}$ can be "integrated" to a Lie groupoid. If $\mathcal{A}$ is a Lie algebroid on X that is a vector bundle (locally free), then an ***integration of*** $\mathcal{A}$ is a pair (G, ϕ) of a Lie groupoid $\mathsf{G} \rightrightarrows \mathsf{X}$, together with an isomorphism of Lie algebroids $\phi : \mathcal{L}ie(\mathsf{G}) \to \mathcal{A}$. If an integration of $\mathcal{A}$ exists, we say that $\mathcal{A}$ is ***integrable***. Not all Lie algberoids are integrable, but a large class of them are. A celebrated theorem of Cranic and Fernandes [42] gives a set of necessary and sufficient conditions for such an integration to exist in the C^∞ setting. Subsequent work by Laurent-Gengoux, Stiénon and Xu [94] verified that these condition are also valid in the holomorphic context.

Since the general integrability conditions are somewhat technical, we shall not recall them here. Instead, we state a simple condition that guarantees integrability for the examples that concern us in this thesis:

Theorem 2.7.11 (Debord [43], Crainic–Fernandes [42])**.** *If $\mathcal{A}$ is a locally free Lie algebroid on X for which the anchor map $a : \mathcal{A} \to \mathcal{T}_\mathsf{X}$ is an embedding of sheaves, then $\mathcal{A}$ is integrable to a Lie groupoid $\mathsf{G} \rightrightarrows \mathsf{X}$ having the property that the only map $\mathsf{X} \to \mathsf{G}$ which is a section of both s and t is the identity map* $\mathrm{id} : \mathsf{X} \to \mathsf{G}$.

Corollary 2.7.12. *If X is a complex manifold and $\mathsf{D} \subset \mathsf{X}$ is any hypersurface—no matter how singular—then the Lie algebroid $\mathcal{T}_\mathsf{X}(-\mathsf{D})$ is integrable. Moreover, if D is a free divisor,* *then* $\mathcal{T}_{\mathsf{X}}(-\log \mathsf{D})$ *is also integrable.*

Just as a Lie algebra may integrate to many different Lie groups, an integrable Lie algebroid may have many non-isomorphic integrations. Among all Lie groupoids, there is an important class that play a role analogous to the simply connected group Lie groups:

Definition 2.7.13. A Lie groupoid $\mathsf{G} \rightrightarrows \mathsf{X}$ is ***source-simply connected*** if the fibres of the source map $s : \mathsf{G} \to \mathsf{X}$ are connected and simply connected.

If $\mathsf{G} \rightrightarrows \mathsf{X}$ and $\mathsf{H} \rightrightarrows \mathsf{Y}$ are Lie groupoids with $\mathsf{G} \rightrightarrows \mathsf{X}$ source-simply connected, and if $\phi : \mathcal{L}ie(\mathsf{G}) \to \mathcal{L}ie(\mathsf{H})$ is a morphism between the corresponding Lie algebroids, then there exists a unique morphism $\Phi : \mathsf{G} \to \mathsf{H}$ that induces ϕ (see, e.g., [107, Proposition 6.8]). Hence source-simply connected groupoids are initial objects in the category of integrations of a given Lie algebroid. In particular, they are unique up to isomorphism.

If a Lie algebroid is integrable, it has a canonical source-simply connected integration:

Theorem 2.7.14. *There is a natural equivalence $\mathcal{A} \mapsto \Pi_1(\mathcal{A})$ between the categories of integrable Lie algebroids and the category of source-simply connected Lie groupoids.*

Example 2.7.15. We saw in Example 2.7.3 that the tangent sheaf $\mathcal{T}_{\mathsf{X}}$ is always integrated by the pair groupoid $\mathsf{Pair}(\mathsf{X})$, but the source fibre over a point $x \in \mathsf{X}$ is $\mathsf{X} \times \{x\}$, which need not be simply connected. The canonical source-simply connected integration is given by the ***fundamental groupoid*** $\Pi_1(\mathsf{X})$ ***of*** X, which is the set of equivalence classes of paths $\gamma : [0,1] \to \mathsf{X}$, where $\gamma \sim \gamma'$ if there is an end-point-preserving homotopy between γ and γ'. The source and target maps are given by $s([\gamma]) = \gamma(0)$ and $t([\gamma]) = \gamma(1)$, and the composition is concatenation of paths. □

2.8 Groupoids in analytic spaces

Our discussion of groupoids so far has focused on Lie groupoids, in which the spaces involved are manifolds and the structure maps satisfy some transversality assumptions. However, in general, it is also useful to consider groupoids for which G and X are simply analytic spaces and the maps s and t are analytic but need not be submersions. Such objects are ***groupoids in analytic spaces***, and they are necessary if we wish to discuss groupoids whose Lie algebroids are not vector bundles. Notice that the existence of fibre products such as $\mathsf{G}{}_s\times_t\mathsf{G}$, which is one of the main reasons for assuming that the source and target maps of a Lie groupoid are submersions, is automatic when we work with analytic spaces; no transversality is required.

Given a Lie groupoid $\mathsf{G} \rightrightarrows \mathsf{X}$ in analytic spaces, we obtain its Lie algebroid as follows. Consider the embedding $\mathrm{id} : \mathsf{X} \to \mathsf{G}$, which we assume to be closed. If $\mathcal{I} \subset \mathcal{O}_{\mathsf{G}}$ is the ideal defining X, then the conormal sheaf is, by definition, the coherent module $\mathcal{O}_{\mathsf{X}}$-module

$\mathcal{I}/\mathcal{I}^2$. The Lie algebroid is the normal bundle of X, which is the dual $\mathcal{L}ie(\mathsf{G}) = (\mathcal{I}/\mathcal{I}^2)^\vee$. In particular, when X is reduced and irreducible, it is a reflexive sheaf.

Reflexive Lie algebroids will appear in a few places in this thesis, particularly in the context of Poisson geometry, where they help us to understand the structure of the symplectic leaves. They seem to be an interesting class worthy in their own right. We therefore pose the following

Problem 2.8.1. *Suppose that X is reduced and irreducible, and that $\mathcal{A}$ is a Lie algebroid on X that is a reflexive sheaf. Under what conditions does there exist a groupoid in analytic spaces that integrates $\mathcal{A}$? In other words, is there an analogue of the Crainic–Fernandes obstruction theory [42] in this context?*

To see that this problem is potentially interesting, let us return to the non-locally free Lie algbroid $\mathcal{F}$ of vector fields tangent to the symplectic leaves of $\mathbb{C}^3 \cong \mathfrak{sl}(2,\mathbb{C})$ from The Example. In Section 2.5, we saw that $\mathcal{F}$ is reflexive, and as a result, we were able use the Hartogs phenomenon to show that it carried a natural symplectic structure. We will now exhibit the corresponding groupoid.

Consider the map $\mathbb{C}^3 \to \mathbb{C}$ defined by the Casimir function $f = x^2 + 4yz$. We can then form the fibre product

$$\mathsf{G} = \mathbb{C}^3{}_f\!\times_f \mathbb{C}^3 = \left\{ (p,q) \in \mathbb{C}^3 \times \mathbb{C}^3 \,\middle|\, f(p) - f(q) = 0 \right\},$$

which is a five-dimensional analytic space with an isolated singularity at $(0,0)$. It is the cone over a smooth quadric hypersurface in $\mathbb{P}^5$.

Notice that with this description, G is a subgroupoid of the pair groupoid $\mathsf{Pair}(\mathbb{C}^3) = \mathbb{C}^3 \times \mathbb{C}^3$. It thus has the structure of a groupoid in analytic spaces with $\mathbb{C}^3$ as its space of objects. The orbits of this groupoid are precisely the level sets of f, which are the symplectic leaves. Away from the singular point, it is easy to see that G is a Lie groupoid, and that its Lie algebroid $\mathcal{A} = \mathcal{L}ie(\mathsf{G})$ is the tangent bundle to the symplectic leaves. Thus $\mathcal{A}|_{\mathbb{C}^3\setminus\{0\}} = \mathcal{F}|_{\mathbb{C}^3\setminus\{0\}}$. Since both of these Lie algebroids are reflexive and the origin has codimension three, it follows that $\mathcal{L}ie(\mathsf{G}) = \mathcal{F}$ on all of $\mathbb{C}^3$.

The fact that G is singular at $(0,0)$ is detected by the fact that the rank of $\mathcal{F}$ jumps at the origin. Similarly, the symplectic structure on $\mathcal{F}$ is reflected on G as follows: if we equip $\mathbb{C}^3 \times \mathbb{C}^3$ with the Poisson structure $\sigma \times (-\sigma)$, then G is a Poisson subspace (see Section 4.3), and the source and target maps to $\mathbb{C}^3$ are anti-Poisson and Poisson, respectively. Thus G is a Poisson groupoid and its source fibres are symplectic.

Consider the action groupoid $\mathsf{H} = \mathsf{SL}(2,\mathbb{C}) \ltimes \mathbb{C}^3$, where $\mathsf{SL}(2,\mathbb{C})$ acts on $\mathbb{C}^3 \cong \mathfrak{sl}(2,\mathbb{C})$ by the adjoint action. The source and target maps of this groupoid give a morphism $\mathsf{SL}(2,\mathbb{C}) \ltimes \mathbb{C}^3 \to \mathsf{Pair}(\mathbb{C}^3)$ whose image is the subgroupoid G. Thus G, although singular, arises very naturally as the image of a morphism of smooth Lie groupoids.

Furthermore, G can be used to produce Lie groupoid structures on some more interesting spaces and to get a good understanding of their geometry, as follows. Notice that the standard action of $\mathbb{C}^*$ on $\mathbb{C}^3$ induces an action on $\mathsf{Pair}(\mathbb{C}^3)$ for which all of the groupoid structure maps are equivariant. Moreover, $\mathsf{G} \subset \mathsf{Pair}(\mathbb{C}^3)$ is preserved by the action because it is defined by a homogeneous quadratic polynomial. The restriction

$$\mathsf{G}' = \mathsf{G} \setminus (\{0\} \times \mathsf{Y} \cup \mathsf{Y} \times \{0\})$$

of G to the open set $\mathbb{C}^3 \setminus \{0\}$ is therefore a Lie groupoid whose structure maps are $\mathbb{C}^*$-equivariant, and hence the quotient $\mathsf{H} = \mathbb{P}(\mathsf{G}') = \mathsf{G}'/\mathbb{C}^*$ is a Lie groupoid over the quotient $(\mathbb{C}^3 \setminus \{0\})/\mathbb{C}^* \cong \mathbb{P}^2$. Notice that H is simply a smooth quadric hypersurface with the pair $\mathsf{C}_1 = \mathbb{P}(\mathsf{Y} \times \{0\})$ and $\mathsf{C}_2 = \mathbb{P}(\{0\} \times \mathsf{Y})$ of smooth disjoint rational curves removed (a codimension three locus).

Proposition 2.8.2. *Let* $\mathsf{C} \subset \mathbb{P}^2$ *be the smooth conic obtained by projectivizing* Y. *Then the groupoid* H *constructed above is the source-simply connected Lie groupoid integrating* $\mathcal{T}_{\mathbb{P}^2}(-\log \mathsf{C})$.

Proof. One obtains the identification $\mathcal{L}ie(\mathsf{H}) \cong \mathcal{T}_{\mathbb{P}^2}(-\log \mathsf{C})$ by looking at the projections of the vector fields in $\mathcal{F}$ to $\mathbb{P}^2$.

To see that the groupoid is source-simply connected, we proceed as follows. Suppose first that $p \in \mathbb{P}^2 \setminus \mathsf{C}$ and consider the line $\mathsf{L} \subset \mathbb{C}^3$ corresponding to p. The source preimage of L in G is the intersection of the four-plane $\mathsf{L} \oplus \mathbb{C}^3$ with G, and therefore its projectivization is a smooth quadric surface. According to the construction, this surface is $s^{-1}(p) \sqcup \mathsf{C}_2$, and so we see that we see that t maps $s^{-1}(p)$ two-to-one onto $\mathbb{P}^2 \setminus \mathsf{C}$. Now, the complement of any curve in a smooth connected surface is necessarily connected. Hence $s^{-1}(p)$ is connected. Since the fundamental group of $\mathbb{P}^2 \setminus \mathsf{C}$ is $\mathbb{Z}/2\mathbb{Z}$, this map must be the universal cover, and hence the source fibre is simply connected.

On the other hand, if $p \in \mathsf{C}$, then the line L is contained in G and the source preimage is $\mathsf{L} \times \mathsf{Y}$. Its projectivization is a quadric surface $\mathsf{Q} \subset \mathbb{P}(\mathsf{L} \oplus \mathbb{C}^3) \cong \mathbb{P}^3$ with an isolated singularity at $q = \mathbb{P}(\mathsf{L} \setminus \{0\})$. Notice that $q = \mathsf{Q} \cap \mathsf{C}_1$ does not lie in H, and hence the source fibre is smooth. If we embed C in $\mathbb{P}^3$ as $\mathbb{P}(\{0\} \times \mathsf{Y})$, then Q is identified with the cone over C in $\mathbb{P}^3$ with vertex q, and the linear projection $\mathsf{Q} \setminus \{q\} \to \mathsf{C}$ gives the target map. In this way, the source fibre is identified with the the total space of a degree-two line bundle over C. Since $\mathsf{C} \cong \mathbb{P}^1$ is simply connected, so is $s^{-1}(p)$. $\square$

2.9 The question of algebraicity

The discussion so far has focussed on Lie algebroids in the complex analytic category, but one could also define a Lie algebroid in the algebraic category by replacing $\mathcal{O}_{\mathsf{X}}$ everywhere with the sheaf of regular functions on a complex algebraic variety X. We then require that the anchor map $a : \mathcal{A} \to \mathcal{T}_\mathsf{X}$ be an algebraic map between algebraic vector bundles or sheaves.

Although most of the general discussion about Lie algebroids carries through, there is one key difference: when we find the orbits of a Lie algebroid, we are looking for functions whose level sets define the integral submanifolds of a distribution $\mathcal{F} \subset \mathcal{T}_\mathsf{X}$, and we must therefore solve a system of differential equations. In general, the resulting functions need not be algebraic and so they may not describe algebraic subvarieties.

Knowing Chow's Theorem [38] that every closed complex analytic subspace Y of a projective variety X is necessarily algebraic, one might hope that the problem of finding algebraic orbits goes away if we restrict our attention to projective varieties. However, it is not so:

Example 2.9.1. Suppose that $\mathsf{X} = \mathbb{P}^2$ is the projective plane and let $\mathcal{A}$ be the line bundle $\mathcal{O}_{\mathbb{P}^2}(-d)$ with $d > 0$. Any nonzero map $a : \mathcal{A} \to \mathcal{T}_\mathsf{X}$ is necessarily algebraic. Since $\mathcal{A}$ has rank one, this map embeds $\mathcal{A}$ as an involutive subsheaf and therefore gives $\mathcal{A}$ the structure of a Lie algebroid. However, it follows from a theorem of Jouanolou [84, Theorem 4.1.1] that if a is generic, then the only $\mathcal{A}$-invariant algebraic subvarieties of $\mathbb{P}^2$ are the isolated points where a vanishes. □

Notice, however, that there are nevertheless some important $\mathcal{A}$-invariant algebraic subvarieties: in particular, if $a : \mathcal{A} \to \mathcal{T}_\mathsf{X}$ is an algebraic map of sheaves, then so are its exterior powers, and hence the degeneracy loci $\mathsf{Dgn}_k(\mathcal{A})$ are algebraic subvarieties.

As a result, we can prove the following

Proposition 2.9.2. *Suppose that X is a smooth algebraic variety and that $\mathcal{A}$ is a Lie algebroid on X with an algebraic anchor map. If X has only finitely many orbits of a given dimension k, then they are locally closed algebraic subvarieties.*

Proof. The space $\mathsf{Y} = \mathsf{Dgn}_k(\mathcal{A}) \setminus \mathsf{Dgn}_{k-1}(\mathcal{A})$ is smooth and $\mathcal{A}$-invariant, and hence $\mathcal{A}$ restricts to a Lie algebroid on Y of constant rank k. Since there are only finitely many orbits of dimension k, we must have that $\dim \mathsf{Y} = k$ and hence the orbits must coincide with the irreducible components of Y, which are algebraic. □

Similarly, Lie algebroids that come from algebraic Lie groupoids have algebraic leaves:

Lemma 2.9.3. *Let $\mathsf{G} \rightrightarrows \mathsf{X}$ be a Lie groupoid for which G and X are algebraic varieties and the structure maps are algebraic, and let $\mathcal{A}$ be its Lie algebroid. Then the orbits of $\mathcal{A}$ are locally closed algebraic subvarieties of X.*

Proof. The orbits of $\mathcal{A}$ are the connected components of the orbits of G, which in turn are the images of the source fibres of G under the target map. Since the source fibres are algebraic subvarieties and the target map is algebraic, the result follows. □

It would be interesting to know to which extent the converse holds:

Problem 2.9.4. *Give conditions that ensure that a Lie algebroid on an algebraic variety* X *is the Lie algebroid of some algebraic Lie groupoid.*

Remark 2.9.5. The related problem for Lie algebras was addressed in [37, 63]. □

We note that many interesting Lie algebroids in the smooth category—such as the Poisson structure on the dual of the Lie algebra of an algebraic group—integrate to algebraic Lie groupoids, even though the canonical integration described by Theorem 2.7.14 may not be algebraic. We will see more examples of this phenomenon in Chapter 3.

For now, we state the following

Conjecture 2.9.6. *If* D *is a free divisor on the algebraic variety* X*, then there exists an algebraic Lie groupoid* G *with Lie algebroid* $\mathcal{T}_{\mathsf{X}}(-\log \mathsf{D})$ *such that the natural map* $\mathsf{G} \to \mathsf{X} \times \mathsf{X}$ *is birational.*

We have several reasons for believing this conjecture to be true:

1. We saw already in Corollary 2.7.12 that such a Lie algebroid is integrable to some holomorphic Lie groupoid. The issue is that the canonical integration will not usually be algebraic: indeed, the source fibre over a point $x \in \mathsf{U} \cong \mathsf{X} \setminus \mathsf{D}$ is necessarily isomorphic to the universal cover of U, which need not be an algebraic variety.

2. There are many examples—for example, linear free divisors [66]—in which the Lie algebroids come from the action of an algebraic group with an open orbit. In these cases, the map $\mathsf{G} \to \mathsf{X} \times \mathsf{X}$ is finite but need not be birational; one might hope to take a quotient of this groupoid to get a new one that is birational.

3. Recent work of Li [97] in the smooth category explains how to construct an algebraic groupoid that is birational to $\mathsf{X} \times \mathsf{X}$ in the case when D is a union of smooth hypersurfaces with normal crossings.

4. The methods discussed in Section 3.3.2 involve blowing up $\mathsf{X} \times \mathsf{X}$ to obtain new groupoids. It is therefore tempting to imagine that there might exist an algorithm similar to Hironaka's theorem on resolution of singularities [81] that would produce the desired groupoid.

Chapter 3

Lie theory on curves and meromorphic connections

In this chapter we apply the theory of Lie algebroids and Lie groupoids to a classical notion in geometry and analysis: that of a meromorphic connection on a vector bundle. To motivate the discussion, we begin with a brief review of asymptotic analysis and the Stokes phenomenon. This subject has a rich history and has experienced a resurgence of late due to its relevance in studying various wall-crossing phenomena in geometry and physics.

We then introduce the relevant Lie algebroids, explore their basic properties and explain their relationship with meromorphic connections and ordinary differential equations, and give several concrete descriptions of the corresponding Lie groupoids. Finally, we examine how the Lie groupoids give an apparently new way of summing the divergent asymptotic series that arise when one attempts to expand the solutions of a certain differential equations in a power series centred at an irregular singular point. Much of this material appears in slightly altered form in the joint work [72] with Gualtieri and Li.

3.1 Invitation: divergent series and the Stokes phenomenon

Consider a complex curve X (a Riemann surface) and an effective divisor D. If E is a holomorphic vector over X, a ***meromorphic connection on*** E ***with poles bounded by*** D is a differential operator $\nabla : \mathcal{E} \to \Omega^1_{\mathsf{X}}(\mathsf{D}) \otimes \mathcal{E}$ on its sheaf of holomorphic sections that satisfies the Leibniz rule $\nabla(fs) = f\nabla s + df \otimes s$. Thus, in a local trivialization of E near a point $p \in \mathsf{D}$ of multiplicity k and a coordinate z centred at p, we may write

$$\nabla = d - \tfrac{1}{z^k} A(z)\, dz$$

for a holomorphic matrix $A(z)$.

We are interested in knowing when two such connections are isomorphic. Let us consider a simple example, where $\mathsf{X} = \mathbb{C}$, $\mathsf{D} = k \cdot (0)$ and $\mathsf{E} = \mathsf{X} \times \mathbb{C}^2$ is the trivial rank-two bundle. Consider the connections ∇_1 and ∇_2 given by

$$\nabla_1 = d - \frac{1}{2z^k} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} dz$$

and

$$\nabla_2 = d - \frac{1}{2z^k} \begin{pmatrix} -1 & z \\ 0 & 1 \end{pmatrix} dz$$

A similar example was considered by Witten in [143]. We wish to know if there is a holomorphic bundle automorphism $g : \mathsf{E} \to \mathsf{E}$ such that

$$\nabla_2 g = g \nabla_1$$

In other words, we are looking for a matrix-valued function g that is a solution of the differential equation

$$2z^k \frac{dg}{dz} = \begin{pmatrix} -1 & z \\ 0 & 1 \end{pmatrix} g - g \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

We seek a solution by expanding g as a power series

$$g(z) = \sum_{j=0}^{\infty} g_j z^j,$$

where $g_j \in \mathbb{C}^{2\times 2}$ are constant matrices and g_0 is invertible. Substituting this expression into the differential equation (or following the algorithm in [7]), one obtains a recurrence relation for the coefficients that can be solved explicitly.

If $k = 1$, we find that

$$g(z) = \begin{pmatrix} a & \frac{bz}{2} \\ 0 & b \end{pmatrix},$$

for some $a, b \in \mathbb{C}$, giving a polynomial gauge transformation that makes the connections equivalent.

Meanwhile, for $k = 2$, we obtain

$$g(z) = \begin{pmatrix} a & \frac{bz}{2} \sum_{n=0}^{\infty} (-1)^n n! z^n \\ 0 & b \end{pmatrix},$$

with $a, b \in \mathbb{C}$. We therefore encounter a problem: the series in the upper right corner diverges. We conclude that the only way for g to be holomorphic is if $b = 0$, in which case it is not invertible. So, we conclude that ∇_1 and ∇_2 cannot be equivalent.

However, the connections ∇_1 and ∇_2 are holomorphic on the punctured line $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, so it must therefore be possible to find gauge equivalences between them on simply-connected open subsets of $\mathbb{C}^*$. One then wonders if this divergent series has some interpretation in terms of honest, holomorphic gauge transformations.

One such interpretation is provided by the theory of ***Borel summation***. The basic idea is that the divergence of the formal power series

$$\hat{f}(z) = \sum_{n=0}^{\infty} (-1)^n n!\, z^n$$

can be tamed if we divide the n^{th} coefficient by $n!$. This procedure defines a map

$$\begin{array}{ccccc} \mathcal{B} & : & \mathbb{C}[[z]] & \to & \mathbb{C}[[x]] \\ & & \sum_{n=0}^{\infty} a_n z^n & \mapsto & \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} \end{array}$$

called the ***Borel transform***. In our example, the Borel transform is

$$\mathcal{B}\hat{f}(x) = \sum_{n=0}^{\infty} (-1)^n x^n.$$

This series converges in the unit disk and it can be analytically continued to all of $\mathbb{C} \setminus \{-1\}$, giving the rational function

$$\mathcal{B}\hat{f}(x) = \frac{1}{1+x}.$$

The theory tells us that we should assign the sum

$$\sum_{n=0}^{\infty} (-1) n! z^n \sim f(z) = \frac{1}{z}\int_{\gamma} \mathcal{B}\hat{f} e^{-x/z}\, dx = \frac{1}{z}\int_{\gamma} \frac{e^{-x/z}}{1+x}\, dx$$

where the contour γ is an infinite ray emanating from 0. This integral is essentially the Laplace transform, which gives an inverse to the Borel transform when the latter is restricted to convergent series.

For the integral to converge, we must choose γ so that $e^{-x/z}$ is decaying rapidly as $x \to \infty$ along γ, and we must avoid the singular point $x = -1$ of the integrand. Let us focus on the right half-plane $\mathsf{S}_+ = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$. For $z \in \mathsf{S}_+$, we can take γ to be the positive real axis; one can check that any other ray in this sector will produce the same result. We obtain the function

$$f_+(z) = \frac{1}{z}\int_0^{\infty} \frac{e^{-x/z}}{1+x}\, dx = \frac{e^{1/z}\Gamma(0,1/z)}{z},$$

where Γ is the incomplete gamma function, which has a branch cut along the negative real axis. This function has the divergent series $\hat{f}$ as an asymptotic expansion in the sector S_+, meaning that

$$\left| f_+(z) - \sum_{n=0}^{N} (-1)^n n! z^n \right| = O(|z|^{N+1})$$

as $z \to 0$ in S_+. In other words, while the series diverges, its partial sums nevertheless give a good approximation for f_+. Moreover, by "differentiating under the integral sign" one can verify that the corresponding gauge transformation

$$g_+(z) = \begin{pmatrix} a & \frac{bzf_+(z)}{2} \\ 0 & b \end{pmatrix}$$

gives an isomorphism between the connections ∇_1 and ∇_2 away from the branch cut. We conclude that while the series expansion for g is divergent, it can still be used to obtain an honest solution of the differential equation.

This appearance of divergent asymptotic expansions is typical when we study connections with poles of order $k \geq 2$; it is an essential feature of ***irregular singularities***. The remarkable and subtle analytic theory of ***multi-summation*** explains how to obtain holomorphic gauge transformations from divergent series such as $g(z)$ in great generality. We refer the reader to [7, 83, 124, 130, 139, 141] for various perspectives on this problem.

One of the major outputs of this analytic programme is the irregular Riemann–Hilbert correspondence. This correspondence gives a local classification of meromorphic connections in terms of "generalized monodromy data", in which one takes into account not only the possible multi-valuedness of the solutions, arising from analytic continuation around the poles of the connection, but also the way in which the asymptotic properties change in different angular sectors emanating from the poles. The latter information is encoded in the so-called ***Stokes matrices***.

Using the analytic results as input, Boalch [14] has explained how this data can be assembled into a Riemann–Hilbert map, giving an isomorphism between the moduli spaces of meromorphic connections, and the corresponding spaces of generalized monodromy data. This result generalizes the more standard description of the moduli space of flat connections on a surface X as the character variety $\mathsf{Hom}(\pi_1(\mathsf{X}, p), \mathsf{GL}(n, \mathbb{C}))/\mathsf{GL}(n, \mathbb{C})$.

In this chapter, we work towards a more geometric understanding of the analytic theory itself. Our perspective is that meromorphic connections on a curve X with poles bounded by an effective divisor D are precisely the same as modules over the Lie algebroid $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ of vector fields vanishing to prescribed order on D. It is therefore natural to integrate them to representations of the corresponding Lie groupoid. In this way, the groupoid serves as the universal domain for the parallel transport of these singular connections.

We close the chapter with an application of the groupoids: we explain that the reason we encounter divergent series when seeking gauge transformations as above is simply that the series live on the wrong space. By pulling them back to the groupoid in an appropriate way, we obtain convergent series that can be used to recover holomorphic solutions to the equations. Remarkably, the proof of convergence is purely geometric: it uses no analysis beyond the classical theorem on existence and uniqueness of solutions to analytic ordinary differential equations at a nonsingular point.

3.2 Lie algebroids on curves

3.2.1 Basic properties

Let X be a smooth connected complex curve, and let $\mathsf{D} \subset \mathsf{X}$ be an effective divisor. The subsheaf $\mathcal{T}_{\mathsf{X}}(-\mathsf{D}) \subset \mathcal{T}_{\mathsf{X}}$ consisting of vector fields vanishing to prescribed order on D is locally free and involutive, and therefore defines a Lie algebroid on X. The anchor map, given by the inclusion, is an isomorphism away from D, and vanishes on D, giving a canonical identification $\mathsf{Dgn}_0(\mathcal{T}_{\mathsf{X}}(-\mathsf{D})) = \mathsf{D}$ as analytic spaces. The orbits of $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ are precisely the complement $\mathsf{X} \setminus \mathsf{D}$ and the individual points of D. It is clear that these Lie algebroids are the only nontrivial examples of rank one on X:

Lemma 3.2.1. *Let $\mathcal{A}$ be a locally free Lie algebroid of rank one on a smooth connected curve X. Then either the anchor and bracket of $\mathcal{A}$ are trivial, or the anchor map defines a canonical isomorphism $\mathcal{A} \cong \mathcal{T}_{\mathsf{X}}(-\mathsf{D})$, where $\mathsf{D} = \mathsf{Dgn}_0(\mathcal{A})$.*

At a point $p \in \mathsf{D}$ of multiplicity k, the $(k-1)$-jet of the anchor map $\mathcal{A} \to \mathcal{T}_{\mathsf{X}}$ vanishes, but the k-jet does not. It therefore defines an element of the one-dimensional vector space $(\mathsf{T}_p^*\mathsf{X})^k \otimes \mathsf{A}_p^\vee \otimes \mathsf{T}_p\mathsf{X}$, giving a canonical identification

$$\mathsf{A}_p \cong (\mathsf{T}_p^*\mathsf{X})^{\otimes(k-1)}$$

of the isotropy Lie algebra, which is abelian.

The dual of the Lie algebroid $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ is the sheaf $\Omega^1_{\mathsf{X}}(\mathsf{D})$ of meromorphic forms with poles bounded by D. The corresponding de Rham complex is simply the derivative

$$\mathcal{O}_{\mathsf{X}} \xrightarrow{d} \Omega^1_{\mathsf{X}}(\mathsf{D}),$$

and hence a $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$-connection on a coherent sheaf $\mathcal{E}$ is exactly the same thing as a meromorphic connection $\nabla : \mathcal{E} \to \Omega^1_{\mathsf{X}}(\mathsf{D}) \otimes \mathcal{E}$. Because the rank of $\mathcal{A}$ is one, such a connection is automatically flat.

The canonical module for this Lie algebroid is given by $\omega_{\mathsf{D}} = \mathcal{T}_{\mathsf{X}}(-\mathsf{D}) \otimes \Omega^1_{\mathsf{X}} = \mathcal{O}_{\mathsf{X}}(-\mathsf{D})$, the ideal of functions vanishing to prescribed order on D. We therefore have the following

Proposition 3.2.2. *Suppose that* X *is a smooth, compact curve and let* $\mathcal{A} = \mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ *for an effective divisor* D*. Let* $(\mathcal{E}, \nabla)$ *be a vector bundle equipped with a* $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$*-connection. Then there is a perfect pairing*

$$\mathsf{H}^k(\mathcal{A}, \mathcal{E}) \otimes_{\mathbb{C}} \mathsf{H}^{2-k}(\mathcal{A}, \mathcal{E}^\vee(-\mathsf{D})) \to \mathbb{C}.$$

for $0 \leq k \leq 2$*. Moreover, the Euler characteristic is given by*

$$\chi(\mathcal{A}, \mathcal{E}) = \operatorname{rank}(\mathcal{E})\,(2 - 2g - \deg(\mathsf{D})),$$

where g *is the genus of* X*, and we have the following inequalities for the Betti numbers:*

$$\begin{aligned} h^0(\mathcal{A}, \mathcal{E}) &\leq \operatorname{rank}(\mathcal{E}) \\ h^1(\mathcal{A}, \mathcal{E}) &\leq \operatorname{rank}(\mathcal{E})\,(2g + \deg(\mathsf{D})) \\ h^2(\mathcal{A}, \mathcal{E}) &\leq \operatorname{rank}(\mathcal{E})\,. \end{aligned}$$

Proof. The statement about the duality is an immediate consequence of the result for general Lie algebroids (Theorem 2.5.2).

The Lie algebroid cohomology is the hypercohomology of the complex

$$\mathcal{E} \to \Omega^1_{\mathsf{X}}(\mathsf{D}) \otimes \mathcal{E},$$

Let $d = \deg \mathsf{D}$, $e = \deg \mathcal{E}$ and $r = \operatorname{rank}(\mathcal{E})$. We may compute the Euler characteristic using the Hirzebruch–Riemann–Roch theorem:

$$\begin{aligned} \chi(\mathcal{A}, \mathcal{E}) &= \chi(\mathcal{E}) - \chi(\Omega^1_{\mathsf{X}}(\mathsf{D}) \otimes \mathcal{E}) \\ &= (e + r(1-g)) - ((2g - 2 + d)r + e + r(1-g)) \\ &= r(2 - 2g - d). \end{aligned}$$

For the statement about the Betti numbers, simply notice that an element of $\mathsf{H}^0(\mathcal{A}, \mathcal{E})$ is a global holomorphic section s of $\mathcal{E}$ that is flat ($\nabla s = 0$). By parallel transport, such a section is completely determined by its value at a single point $p \in \mathsf{X} \setminus \mathsf{D}$, and hence we must have $h^0(\mathcal{A}, \mathcal{E}) \leq \dim \mathsf{E}_p = \operatorname{rank}(\mathcal{E})$. The statement about h^2 now follows from duality, and the remaining inequality for h^1 follows from those for h^0 and h^2 together with the computation of the Euler characteristic. □

Corollary 3.2.3. *Suppose that* X *is compact and* $k \in \mathbb{Z}$*, and consider the natural modules*

$\mathcal{O}_\mathsf{X}(k\mathsf{D})$. *We have*

$$h^0(\mathcal{A}, \mathcal{O}_\mathsf{X}(k\mathsf{D})) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$$
$$h^1(\mathcal{A}, \mathcal{O}_\mathsf{X}(k\mathsf{D})) = 2g + \deg(\mathsf{D}) - 1$$
$$h^2(\mathcal{A}, \mathcal{O}_\mathsf{X}(k\mathsf{D})) = \begin{cases} 0 & k \geq 0 \\ 1 & k < 0. \end{cases}$$

Proof. Since D is effective $\mathcal{O}_\mathsf{X}(k\mathsf{D})$ has no global sections for $k < 0$. Meanwhile, for $k \geq 0$, the natural inclusion $\mathcal{O}_\mathsf{X} \to \mathcal{O}_\mathsf{X}(k\mathsf{D})$ defines a flat section for the connection and hence we have $h^0 \geq 1$. Since $\mathcal{O}_\mathsf{X}(k\mathsf{D})$ is a line bundle, we have $h^0 \leq 1$, giving the first equality. The value of h^2 now follows from duality, and then the value of h^1 follows from the Euler characteristic. □

Notice that the case $k = 0$ is just the usual de Rham cohomology of $\mathcal{A}$. In particular, we see that the dimension of the Picard group $\mathsf{Pic}(\mathcal{A})$ is $2g + \deg \mathsf{D} - 1$.

3.2.2 Higher order connections and singular differential equations

The Lie algebroid $\mathcal{A} = \mathcal{T}_\mathsf{X}(-\mathsf{D})$ provides a useful way of describing ordinary differential equations with singularities. Recall from Section 2.6 that a vector bundle or coherent sheaf $\mathcal{E}$ has a sheaf of n-jets $\mathcal{J}^n_\mathcal{A}\mathcal{E}$ along $\mathcal{A}$, and that an nth-order connection on $\mathcal{E}$ is a splitting $\mathcal{J}^n_\mathcal{A}\mathcal{E} \to (\mathcal{A}^\vee)^n \otimes \mathcal{E}$ of the jet sequence. Recall further that such a connection defines an $\mathcal{A}$-connection on $\mathcal{J}^{n-1}_\mathcal{A}\mathcal{E}$ in a canonical way. This structure is nothing but the usual interplay between higher-order ordinary differential equations (ODEs) and linear systems as we now describe.

Let us consider the structure of an nth-order connection on $\mathcal{O}_\mathsf{X}$ in the neighbourhood of a point $p \in \mathsf{D}$ of multiplicity k. In a local coordinate z centred at p, the vector field $\delta = z^k\partial_z$ generates $\mathcal{A}$ and the meromorphic form $\alpha = \frac{dz}{z^k}$ generates $\mathcal{A}^\vee$. We therefore obtain a basis $1, \alpha, \alpha^2, \cdots, \alpha^n$ for $\mathcal{J}^n_\mathcal{A}\mathcal{O}_\mathsf{X}$. The n-jet of a function $f \in \mathcal{O}_\mathsf{X}$ is given by

$$f \cdot 1 + \delta(f) \cdot \alpha + \cdots + \delta^n(f) \cdot \alpha^n,$$

and a kth-order connection $\mathcal{J}^n_\mathcal{A}\mathcal{O}_\mathsf{X} \to \mathcal{O}_\mathsf{X}$ acts on f by

$$\Delta(f) = (\delta^n(f) + p_{n-1}\delta^{n-1}(f) + \cdots + p_0 f)\alpha^n.$$

for some fixed $p_0, \ldots, p_{n-1} \in \mathcal{O}_{\mathsf{X}}$. For example, with $n = 2$, we have the formula

$$\Delta(f) = \left(z^{2k} \frac{d^2 f}{dz^2} + (p_1 + kz^{k-1}) z^k \frac{df}{dz} + p_0 f \right) \alpha^2.$$

In the basis $1, \alpha, \cdots, \alpha^{n-1}$ for $\mathcal{J}^{n-1}_{\mathcal{A}} \mathcal{O}_{\mathsf{X}}$, the corresponding connection is given by

$$\nabla = d + \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ p_0 & p_1 & p_2 & \cdots & p_{n-1} \end{pmatrix} \frac{dz}{z^n} \tag{3.1}$$

Thus

$$\nabla \begin{pmatrix} f_0 \\ f_1 \\ \vdots \\ f_{n-1} \end{pmatrix} = 0$$

if and only if $\delta f_j = f_{j+1}$ for $j < n - 1$ and

$$\delta f_{n-1} + p_{n-1} f_{n-1} + \cdots + p_0 f_0 = 0,$$

which is clearly equivalent to the single nth-order equation $\Delta(f_0) = 0$.

It is often useful to examine an ODE on the complex line $\mathbb{C}$ near the point at infinity in $\mathbb{P}^1$. The advantage of the geometric formulation using Lie algebroids is that it clarifies which bundle should be used for the extension to $\mathbb{P}^1$.

Example 3.2.4. The hypergeometric equation is the second-order ODE

$$z(1 - z) \frac{d^2 f}{dz^2} + [c - (a + b + 1)z] \frac{df}{dz} - abf = 0,$$

with $a, b, c \in \mathbb{C}$. Its singular points form the divisor $\mathsf{D} = 0 + 1 + \infty \subset \mathbb{P}^1$. Let $\delta = z(1 - z)\partial_z$ be a generator for $\mathcal{A}$ on the chart $\mathbb{C} \subset \mathbb{P}^1$. Multiplying the hypergeometric equation by $z(1 - z)$ puts it in the form

$$\delta^2(f) + p_1 \delta(f) + p_0 f = 0$$

where $p_1 = c - 1 - (a + b - 1)z$ and $p_0 = -abz(1 - z)$. This operator extends to second-order connection $\mathcal{J}^2_{\mathcal{A}} \mathcal{O}_{\mathbb{P}^1} \to \mathcal{A}^2$. It is the unique such operator for which the solutions of the given hypergeometric equation are flat sections. □

Example 3.2.5. An important early example in the study of differential equations with irregular singularities was Stokes' analysis [136] of the Airy functions. Recall that the Airy equation is the second-order differential equation

$$\frac{d^2f}{dx^2} = xf.$$

This equation corresponds to the second-order connection

$$\Delta : f \mapsto \left(\frac{d^2f}{dx^2} - xf\right) dx^2$$

on $\mathcal{O}_{\mathbb{C}}$. In the coordinate $z = 1/x$ at infinity in $\mathbb{P}^1$, we have

$$\frac{d}{dx} = -z^2\frac{d}{dz}$$

and so

$$\begin{aligned}
\Delta(f) &= \left(z^4\frac{d^2f}{dz^2} + 2z^3\frac{df}{dz} - z^{-1}f\right)\left(\frac{dz}{z^2}\right)^2 \\
&= \left(z^6\frac{d^2f}{dz^2} + (2z^2)z^3\frac{df}{dz} - zf\right)\left(\frac{dz}{z^3}\right)^2 \\
&= \left(\delta^2(f) - z^2\delta(f) - zf\right)\left(\frac{dz}{z^3}\right)^2,
\end{aligned}$$

where $\delta = z^3\partial_z$. Therefore Δ extends to give a second-order connection

$$\Delta_{\text{Airy}} : \mathcal{O}_{\mathbb{P}^1} \to \mathcal{A}^{-2} \otimes \mathcal{O}_{\mathbb{P}^1}$$

for the Lie algebroid $\mathcal{A} = \mathcal{T}_{\mathbb{P}^1}(-3 \cdot \infty)$.

We can convert the Airy operator Δ_{Airy} into an $\mathcal{A}$-connection on the rank-two bundle

$$\mathcal{E} = \mathcal{J}^1_{\mathcal{A}}\mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1).$$

In the coordinate x and the corresponding basis $1, dx$ for $\mathcal{E}$, the connection is given by

$$\nabla^{\text{Airy}} = d - \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} dx.$$

In the coordinate z at infinity, with basis $1, dz/z^3$ for $\mathcal{E}$, the connection is given by

$$\nabla^{\text{Airy}} = d - \begin{pmatrix} 0 & 1 \\ z & z^2 \end{pmatrix} \frac{dz}{z^3},$$

giving a representation of $\mathcal{T}_{\mathbb{P}^1}(-3 \cdot \infty)$. $\square$

3.2.3 Meromorphic projective structures

Let X be a smooth, connected curve with an effective divisor D. In our description of the source-simply connected groupoid integrating the Lie algebroid $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$, we will make use of the uniformization theorem to identify the universal cover of U with the complex line $\mathbb{C}$ or the upper half-plane $\mathbb{H}$. From the universal covering map, U inherits an atlas of holomorphic charts whose overlap maps are constant projective transformations, and this data can be used to produce a flat connection on a $\mathbb{P}^1$-bundle over U, as described by Gunning [75]. Moreover, the behaviour of this connection near D is controlled, and hence it may described in terms of Lie algebroids.

We fix once and for all a square root $\omega_{\mathsf{X}}^{1/2}$ of the canonical bundle $\omega_{\mathsf{X}} = \Omega^1_{\mathsf{X}}$, and make the following

Definition 3.2.6. A ***projective structure on*** (X, D) is a second-order $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$-connection

$$\phi : \omega_{\mathsf{X}}^{-1/2} \to \omega_{\mathsf{X}}^2(2\mathsf{D}) \otimes \omega_{\mathsf{X}}^{-1/2}$$

for which the induced connection on the determinant $\det \mathcal{J}^1_{\mathcal{A}} \omega_{\mathsf{X}}^{-1/2} \cong \mathcal{O}_{\mathsf{X}}(\mathsf{D})$ is the canonical one.

Amongst all of the projective structures on (X, D) there is a privileged one that is determined by uniformization.

In a coordinate z centred at a point $x \in \mathsf{D}$ of multiplicity k, such a ϕ has the form

$$\phi(f dz^{-1/2}) = \left(\frac{d^2 f}{dz^2} + q_{\phi,z} f \right) dz^{3/2}$$

with

$$q_{\phi,z} = \frac{p_0}{z^{2k}}.$$

for some $p_0 \in \mathcal{O}_{\mathsf{X}}$. We call $q_{\phi,z}$ the ***connection coefficient of*** ϕ ***in the coordinate*** z.

If $w = g(z)$ is another coordinate, then

$$q_{\phi,w} = q_{\phi,z} + S(g) \circ z$$

where

$$S(g) = \frac{g'''}{g'} - \frac{3}{2} \left(\frac{g''}{g'} \right)^2$$

is the Schwarzian derivative of g.

A projective structure on (X, D) induces a $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$-connection on the rank-two bundle $\mathcal{J} = \mathcal{J}^1_{\mathcal{T}_{\mathsf{X}}(-\mathsf{D})} \omega_{\mathsf{X}}^{-1/2}$, and thus an algebroid $\mathsf{PSL}(2, \mathbb{C})$-connection on the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{J})$. It is this $\mathbb{P}^1$-bundle that will be used in our description of the groupoid.

3.3 Lie groupoids on curves

3.3.1 Motivation: integration of representations

If D is an effective divisor on the curve X, the Lie algebroid $\mathcal{A} = \mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ is locally free, and the anchor map is generically an isomorphism. It therefore follows from the general theory of Lie groupoids that $\mathcal{A}$ is integrable to a source-simply connected holomorphic Lie groupoid $\Pi_1(\mathsf{X}, \mathsf{D})$, which is unique up to isomorphism; see Section 2.7.1. Moreover, the groupoid has the following property:

Theorem 3.3.1. *Parallel transport establishes an equivalence between the category of meromorphic $\mathsf{GL}(n, \mathbb{C})$-connections on X with poles bounded by D and the category of equivariant vector bundles on $\Pi_1(\mathsf{X}, \mathsf{D})$.*

In other words, the groupoids $\Pi_1(\mathsf{X}, \mathsf{D})$ are the natural domains for the parallel transport of meromorphic connections; they give a canonical way to extend the usual parallel transport holomorphically over the singular points. Identical statements are true for principal bundles with arbitrary structure groups.

The equivalence can be described in concrete terms as follows. On the groupoid $\mathsf{G} = \Pi_1(\mathsf{X}, \mathsf{D})$, there is a natural foliation given by the fibres of the source map. Let $\mathcal{F} \subset \mathcal{T}_{\mathsf{G}}$ be the corresponding involutive subbundle. We immediately have the following

Lemma 3.3.2. *Suppose that $\mathcal{E}_{\mathsf{G}}$ is an $\mathcal{F}$-module. Then restriction of flat sections gives an isomorphism*

$$\mathsf{H}^0(\mathsf{G}, \mathcal{E}_{\mathsf{G}})^{\nabla} \cong \mathsf{H}^0(\mathsf{X}, \mathcal{E}_{\mathsf{G}}|_{\mathsf{X}}) \, .$$

where $\mathsf{X} \subset \mathsf{G}$ is the manifold of identities. In other words, a flat section of $\mathcal{E}$ is uniquely determined by its value at the identity.

Proof. Each source fibre is simply connected and intersects X at a single point. Hence parallel transport along the source fibres uniquely specifies the section from its restriction to X. $\square$

Now if $\mathcal{E}$ is a representation of $\mathcal{A} = \mathcal{T}_{\mathsf{X}}(-\mathsf{D})$, the pullback $s^*\mathcal{E}$ is canonically trivial along each source fibre of G, and so $s^*\mathcal{E}$ inherits in a natural way the structure of an $\mathcal{F}$-module. Meanwhile, the pullback $(t^*\mathcal{E}, t^*\nabla)$ is a meromorphic $\mathcal{T}_{\mathsf{G}}$-connection with singularities along the subgroupoid $\mathsf{G}|_{\mathsf{D}}$. However, its restriction to the subsheaf $\mathcal{F} \subset \mathcal{T}_{\mathsf{G}}$ is holomorphic, giving $t^*\mathcal{E}$ the structure of an $\mathcal{F}$-module as well. Notice that we have $s^*\mathcal{E}|_{\mathsf{X}} \cong t^*\mathcal{E}|_{\mathsf{X}} \cong \mathcal{E}$. Hence the identity map on $\mathcal{E}$ defines a canonical section of $\mathcal{H}om(s^*\mathcal{E}, t^*\mathcal{E})\,|_{\mathsf{X}}$. Applying the lemma, we see that there is a unique morphism $\rho : s^*\mathcal{E} \to t^*\mathcal{E}$ of $\mathcal{F}$-modules whose restriction to X is the identity. One can verify that this section is multiplicative, i.e., $\rho(gh) = \rho(g)\rho(h)$ for all $g, h \in \mathsf{G}$, giving the unique equivariant structure on $\mathcal{E}$ whose derivative along X is the Lie algebroid connection ∇.

Suppose now that we have a morphism $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ of $\mathcal{A}$-modules and let $\rho_i : s^*\mathcal{E}_i \to t^*\mathcal{E}_i$ be the corresponding equivariant structures. Then we have a diagram

$$\begin{tikzcd} s^*\mathcal{E}_1 \arrow[r, "\rho_1"] \arrow[d, "s^*\phi"'] & t^*\mathcal{E}_1 \arrow[d, "t^*\phi"] \\ s^*\mathcal{E}_2 \arrow[r, "\rho_2"] & t^*\mathcal{E}_2 \end{tikzcd}$$

where each map is a morphism of $\mathcal{F}$-modules. Since $t^*\phi \circ \rho_1$ and $\rho_2 \circ s^*\phi$ have the same restriction to X, namely ϕ, the two must be equal on all of G. In other words, the diagram must commute, so that ϕ is a G-equivariant map. We have therefore obtained the desired isomorphism $\mathsf{Hom}_{\mathcal{A}}(\mathcal{E}_1, \mathcal{E}_2) \cong \mathsf{Hom}_{\mathsf{G}}(\mathcal{E}_1, \mathcal{E}_2)$, giving an equivalence of categories.

3.3.2 Blowing up: the adjoint groupoids

The first method for constructing an integration of $\mathcal{A} = \mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ is motivated as follows: the pair groupoid $\mathsf{Pair}(\mathsf{X})$ is an integration of the Lie algebroid $\mathcal{T}_{\mathsf{X}}$, and $\mathcal{A}$ is obtained from $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ simply by twisting along the divisor D. We therefore wish to modify $\mathsf{Pair}(\mathsf{X})$ along D in some appropriate manner. The correct formulation was developed by Gualtieri and Li in the smooth category and the construction is equally valid in the holomorphic setting:

Theorem 3.3.3 ([71]). *Let X be a complex manifold and let $\mathsf{G} \rightrightarrows \mathsf{X}$ be a Lie groupoid. Suppose that $\mathsf{H} \rightrightarrows \mathsf{Y}$ is a closed Lie subgroupoid supported on a smooth closed hypersurface $\mathsf{Y} \subset \mathsf{X}$. Denote by $p : \mathsf{Bl}_{\mathsf{H}}(\mathsf{G}) \to \mathsf{G}$ the blowup of G along H, and let $\mathsf{S}, \mathsf{T} \subset \mathsf{Bl}_{\mathsf{H}}(\mathsf{G})$ be the proper transforms of $s^{-1}(\mathsf{Y}), t^{-1}(\mathsf{Y}) \subset \mathsf{G}$. Then the manifold*

$$[\mathsf{G} : \mathsf{H}] = \mathsf{Bl}_{\mathsf{H}}(\mathsf{G}) \setminus (\mathsf{S} \cup \mathsf{T})$$

inherits in a unique way the structure of a Lie groupoid over X for which the blowdown map $p|_{[\mathsf{G}:\mathsf{H}]} : [\mathsf{G} : \mathsf{H}] \to \mathsf{G}$ is a morphism of groupoids.

Moreover, the Lie algebroid of $[\mathsf{G} : \mathsf{H}]$ is canonically identified with the subsheaf of $\mathcal{L}ie(\mathsf{G})$ consisting of sections whose restriction to Y lies in $\mathcal{L}ie(\mathsf{H})$, i.e., we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{L}ie([\mathsf{G} : \mathsf{H}]) \longrightarrow \mathcal{L}ie(\mathsf{G}) \longrightarrow \mathcal{L}ie(\mathsf{G})\,|_{\mathsf{Y}} / \mathcal{L}ie(\mathsf{H}) \longrightarrow 0.$$

Using this approach, we have a notion of twisting a Lie groupoids along a divisor supported on a union of smooth hypersurfaces:

Definition 3.3.4. Let $\mathsf{G} \rightrightarrows \mathsf{X}$ be a Lie groupoid and let $\mathsf{Y} \subset \mathsf{X}$ be a smooth closed hypersurface. The ***twist of*** G ***along*** Y is the Lie groupoid

$$\mathsf{G}(-\mathsf{Y}) = [\mathsf{G} : \mathrm{id}_{\mathsf{G}}(\mathsf{Y})]$$

obtained by blowing up G along the identity arrows over Y. More generally, if $\mathsf{D} = k_1\mathsf{Y}_1 + \cdots + k_n\mathsf{Y}_n$ is an effective divisor such that the reduced hypersurfaces $\mathsf{Y}_1, \ldots, \mathsf{Y}_n$ are smooth and disjoint, the ***twist of*** G ***along*** D is the Lie groupoid obtained from G by performing a k_i-fold twist of G along each hypersurface Y_i.

Notice that $\mathcal{L}ie(\mathsf{G}(-\mathsf{Y})) = \mathcal{L}ie(\mathsf{G})\,(-\mathsf{Y})$ consists of all sections that vanish identically along Y, justifying the nomenclature. Applied to the particular case when X is a curve, D is an effective divisor and $\mathsf{G} = \mathsf{Pair}(\mathsf{X})$ is the pair groupoid, we obtain a new groupoid

$$\mathsf{Pair}(\mathsf{X}, \mathsf{D}) = \mathsf{Pair}(\mathsf{X})\,(-\mathsf{D})$$

whose Lie algebroid is $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$. This groupoid has the following universal property: let $\mathsf{H} \rightrightarrows \mathsf{X}$ be another Lie groupoid and suppose that the image of the anchor of $\mathcal{L}ie(\mathsf{H})$ is contained in $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$. Then there is a unique morphism $\phi : \mathsf{H} \to \mathsf{Pair}(\mathsf{X}, \mathsf{D})$ that makes the following diagram commute:

$$\begin{tikzcd} & \mathsf{Pair}(\mathsf{X}, \mathsf{D}) \arrow[d] \\ \mathsf{H} \arrow[ur, "\phi"] \arrow[r, "(s,t)"'] & \mathsf{Pair}(\mathsf{X})\,. \end{tikzcd}$$

In particular, every Lie groupoid integrating $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ has a unique morphism to $\mathsf{Pair}(\mathsf{X}, \mathsf{D})$. In this way, $\mathsf{Pair}(\mathsf{X}, \mathsf{D})$ plays a role analogous to the adjoint group of a Lie algebra.

Notice that this construction is local: if we take a connected open set $\mathsf{U} \subset \mathsf{X}$, then the restriction $\mathsf{Pair}(\mathsf{X}, \mathsf{D})\,|_{\mathsf{U}}$ is canonically identified with $\mathsf{Pair}(\mathsf{U}, \mathsf{D} \cap \mathsf{U})$. Thus, in order to understand the local structure of $\mathsf{Pair}(\mathsf{X}, \mathsf{D})$, it suffices to examine the groupoids $\mathsf{Pair}(\Delta, k \cdot 0)$, for $k \geq 1$ where $\Delta \subset \mathbb{C}$ is the unit disk. We delay the explicit description of these local normal forms until later in the chapter, where they will be obtained as a consequence of some of the following simple examples on $\mathbb{P}^1$.

3.3.3 Examples on $\mathbb{P}^1$

Degree one divisors on $\mathbb{P}^1$

Since all degree-one divisors on $\mathbb{P}^1$ are related by an automorphism of $\mathbb{P}^1$, we may fix our attention on the case when D is the point $\infty \in \mathbb{P}^1$. To integrate $\mathcal{T}_{\mathbb{P}^1}(-\infty)$ we must twist the pair groupoid $\mathsf{Pair}\big(\mathbb{P}^1\big) = \mathbb{P}^1 \times \mathbb{P}^1$ along ∞. This groupoid contains three important rational curves: the diagonal $\Delta = \mathrm{id}(\mathbb{P}^1)$, together with the source and target fibres

$$\mathsf{S} = s^{-1}(\infty) = \mathbb{P}^1 \times \{\infty\}$$

and

$$\mathsf{T} = t^{-1}(\infty) = \{\infty\} \times \mathbb{P}^1.$$

The groupoid $\mathsf{Pair}\big(\mathbb{P}^1, \infty\big)$ is obtained by blowing up the point $\mathrm{id}(\infty) = (\infty, \infty)$ at which S and T intersect, and removing the proper transforms of S and T. The identity embedding $\mathrm{id}(\mathbb{P}^1)$ in the blowup is simply the proper transform of Δ. The situation is illustrated in Figure 3.1.

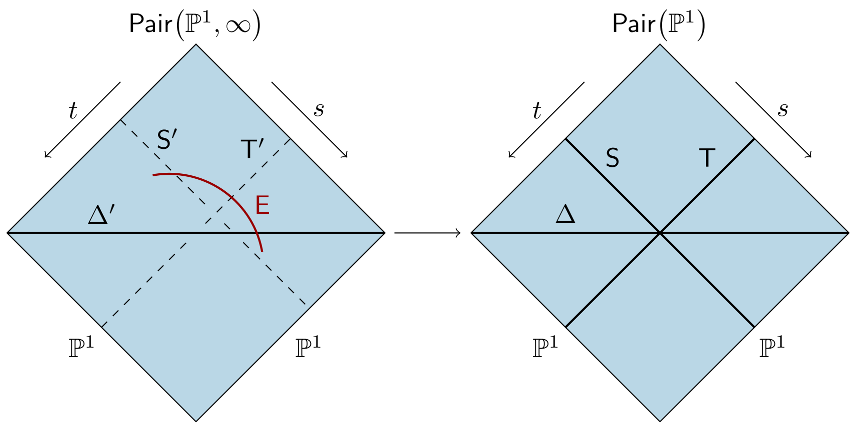


Figure 3.1: The twist of $\mathsf{Pair}\big(\mathbb{P}^1\big)$ along $\mathsf{D} = \infty$, presented as the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ with the rational curves S', T' removed. The isotropy group at ∞ is the exceptional divisor E, shown in red, punctured where it meets S' and T' to give a copy of $\mathbb{C}^*$.

This groupoid has the following elegant description, obtained by Gualtieri. The rational curves S and T have self-intersection zero, and hence their proper transforms, which are disjoint, have self-intersection -1. Blowing down these two curves we obtain a copy of $\mathbb{P}^2$ with two privileged points $p_s, p_t \in \mathbb{P}^2$, giving an isomorphism

$$\mathsf{Pair}\big(\mathbb{P}^1, \infty\big) \cong \mathbb{P}^2 \setminus \{p_s, p_t\}.$$

The manifold of identities is given by the proper transform Δ' of Δ, which is a $+1$-curve intersecting neither S' nor T'. It therefore maps to a line in $\mathbb{P}^2$ disjoint from p_s and p_t. Meanwhile, the exceptional divisor E, a -1-curve, intersects both S' and T' with multiplicity one, and hence it also maps to a line in $\mathbb{P}^2$: the one spanned by p_s and p_t. Similarly, the source and target fibres through a point $x \in \Delta' \cong \mathbb{P}^1$ map to the lines $\overline{xp_s}$ and $\overline{xp_t}$. This analysis shows that the groupoid composition is given by intersecting lines in $\mathbb{P}^2$, as illustrated in Figure 3.2.

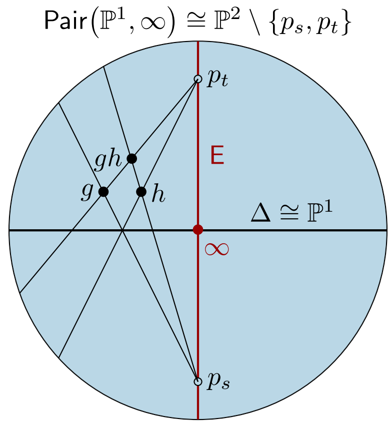


Figure 3.2: The groupoid $\mathsf{Pair}(\mathbb{P}^1, \infty)$, presented as $\mathbb{P}^2$ with two points removed. Projection from these points onto the line $\Delta \cong \mathbb{P}^1$ give the source and target of an element in the groupoid, and the multiplication gh is obtained by intersecting the lines $\overline{gp_s}$ and $\overline{hp_t}$.

As a result of the latter description, we see that the groupoid $\mathsf{Pair}(\mathbb{P}^1, \infty)$ obeys a version of the GAGA principle; we note that this property will not hold for some of the other groupoids that we shall consider:

Theorem 3.3.5. *Every analytic equivariant vector bundle $\mathcal{E}$ for the groupoid $\mathsf{Pair}(\mathbb{P}^1, \infty)$ is isomorphic to an algebraic one, i.e., one for which the structure map $s^*\mathcal{E} \to t^*\mathcal{E}$ is algebraic.*

Proof. Consider the open embedding $j : \mathsf{G} \to \mathbb{P}^2$. By the Birkhoff–Grothendieck theorem, the bundle $\mathcal{E}$ on $\mathbb{P}^1$ splits as a sum of line bundles. Since the source and target maps are linear projections, we have $j_*s^*\mathcal{O}_{\mathbb{P}^1}(1) \cong j_*t^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathbb{P}^2}(1)$, and hence the sheaves $j_*s^*\mathcal{E}$ and $j_*t^*\mathcal{E}$ must be locally free. Since the complement of G has codimension two, the equivariant structure $\rho : s^*\mathcal{E} \to t^*\mathcal{E}$ defines a global section of $\mathcal{H}om(j_*s^*\mathcal{E}, j_*s^*\mathcal{E})$. Since $\mathbb{P}^2$ is projective, this section must be algebraic. $\square$

Remark 3.3.6. This theorem is closely related to the fact that a meromorphic connection with first order poles has regular singularities: the flat sections have at most polynomial growth near the poles. $\square$

We note that every source fibre of this groupoid is simply connected, except for the isotropy group over $\infty \in \mathbb{P}^1$, which is a copy of $\mathbb{C}^*$. Even though this groupoid is not source-simple connected, one can verify directly that every $\mathcal{T}_{\mathbb{P}^1}(-\infty)$-connection on a vector bundle over $\mathbb{P}^1$ integrates to a representation of the groupoid. Hence, in this case, the source-simply connected integration, which fails to be Hausdorff, is not required for the study $\mathcal{T}_{\mathbb{P}^1}(-\infty)$-connections.

Degree two divisors on $\mathbb{P}^1$ (reduced)

Up to automorphisms of $\mathbb{P}^1$, the only reduced divisor of degree two is $\mathsf{D} = 0 + \infty$. In this case, the pair groupoid $\mathsf{Pair}(\mathbb{P}^1)$ contains four distinguished lines, the source and target fibres $\mathsf{S}_0, \mathsf{S}_\infty$ and $\mathsf{T}_0, \mathsf{T}_\infty$ through both 0 and ∞, which all have zero self-intersection.

To twist the pair groupoid along D, we form the blowup

$$\mathsf{H} = \mathsf{Bl}_{\{(0,0),(\infty,\infty)\}}\left(\mathbb{P}^1 \times \mathbb{P}^1\right).$$

The proper transforms $\mathsf{S}'_0, \mathsf{S}'_\infty, \mathsf{T}'_0, \mathsf{T}'_\infty$ are all -1-curves, and the groupoid is

$$\mathsf{Pair}\left(\mathbb{P}^1, 0+\infty\right) = \mathsf{H} \setminus (\mathsf{S}'_0 \cup \mathsf{S}'_\infty \cup \mathsf{T}'_0 \cup \mathsf{T}'_\infty).$$

Blowing down S'_0 and S'_∞, we identify

$$\mathsf{Pair}\left(\mathbb{P}^1, 0+\infty\right) \cong \mathbb{P}^1 \times \mathbb{P}^1 \setminus (\mathsf{T}''_0 \cup \mathsf{T}''_\infty)$$

where T''_0 and T''_∞ are the images of T'_0 and T'_∞ under the blow-down. These curves are non-intersecting and have zero self-intersection. Thus, they may be identified with the fibres over 0 and ∞ of the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. In this way, we identify

$$\mathsf{Pair}\left(\mathbb{P}^1, 0+\infty\right) \cong \mathbb{C}^* \ltimes \mathbb{P}^1$$

where $\mathbb{C}^* \ltimes \mathbb{P}^1 = \mathbb{C}^* \times \mathbb{P}^1 \rightrightarrows \mathbb{P}^1$ is the groupoid induced by the action of $\mathbb{C}^*$ on $\mathbb{P}^1$ by automorphisms fixing 0 and ∞ (cf. Example 2.7.4). This groupoid is not source-simply connected, as its the source fibres are all isomorphic to $\mathbb{C}^*$. However, using the exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$, we obtain an action of $\mathbb{C}$ on $\mathbb{P}^1$, and the source-simply connected integration is

$$\Pi_1\left(\mathbb{P}^1, 0+\infty\right) \cong \mathbb{C} \ltimes \mathbb{P}^1.$$

Degree two divisors on $\mathbb{P}^1$ (confluent)

We now consider our first example for which the divisor is not reduced: $\mathsf{D} = 2 \cdot \infty \subset \mathbb{P}^1$. The Lie algebroid $\mathcal{A} = \mathcal{T}_{\mathbb{P}^1}(-2 \cdot \infty)$ has a one-dimensional space of global sections, given by the infinitesimal generators for the group of parabolic automorphisms of $\mathbb{P}^1$ that fix ∞. Therefore, the exponential map

$$\mathsf{H}^0\left(\mathbb{P}^1, \mathcal{A}\right) \to \mathsf{Aut}\left(\mathbb{P}^1\right)$$

is an embedding, giving rise to an algebraic action of the one-dimensional additive group of sections of $\mathcal{A}$ on $\mathbb{P}^1$. This group acts freely and transitively on the complement of ∞, and hence we have an identification

$$\mathsf{Pair}\big(\mathbb{P}^1, 2\cdot\infty\big) \cong \mathsf{H}^0\big(\mathbb{P}^1, \mathcal{A}\big) \ltimes \mathbb{P}^1$$

with the action groupoid. The natural map

$$\mathsf{H}^0\big(\mathbb{P}^1, \mathcal{A}\big) \otimes_{\mathbb{C}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1} \to \mathcal{A}$$

is an isomorphism in this case. Hence, the groupoid is identified with the total space of the trivial bundle $\mathcal{T}_{\mathbb{P}^1}(-2\cdot\infty)$ in such a way that the source map is the bundle projection and the identity map is the zero section. In particular, this groupoid is source-simply connected.

Divisors of arbitrary degree supported on a single point

To obtain $\mathsf{Pair}\big(\mathbb{P}^1, k\cdot\infty\big)$ with $k > 2$, we must perform a $(k-2)$-fold iterated twist of $\mathsf{Pair}\big(\mathbb{P}^1, 2\cdot\infty\big)$ at ∞. Using this construction we can prove the following

Theorem 3.3.7. *For $k \geq 2$ and $p \in \mathbb{P}^1$, there is a canonical isomorphism*

$$\mathsf{Pair}\big(\mathbb{P}^1, k\cdot p\big) \cong \mathsf{Tot}(\mathcal{T}_{\mathbb{P}^1}(-k\cdot p))$$

for which the source map is the bundle projection and the identity bisection is the zero section. Moreover, the groupoid homomorphisms

$$\mathsf{Pair}\big(\mathbb{P}^1, k\cdot p\big) \to \mathsf{Pair}\big(\mathbb{P}^1, k'\cdot p\big)$$

for $k \geq k' \geq 2$ are given by the natural maps

$$\mathsf{Tot}(\mathcal{T}_{\mathbb{P}^1}(k\cdot\infty)) \to \mathsf{Tot}(\mathcal{T}_{\mathbb{P}^1}(k'\cdot\infty))\,.$$

Proof. We saw above that the statement is true for $k = 2$. So, suppose by way of induction that the statement is true for some $k \geq 2$. Since p is an orbit of $\mathsf{Pair}(\mathsf{X}, k\cdot p)$, the source and target fibres over p are equal to the fibre of $\mathcal{T}_{\mathbb{P}^1}(-k\cdot p)$ over p. Therefore, $\mathsf{Pair}(\mathsf{X}, (k+1)\cdot p)$ is obtained by blowing up $0_p \in \mathsf{Tot}(\mathcal{T}_{\mathbb{P}^1}(-k\cdot p))$ and removing the proper transform of the fibre. By Lemma 3.3.8 below, the result is $\mathsf{Tot}(\mathcal{T}_{\mathbb{P}^1}(-(k+1)\cdot p))$. The statements about the source and identity maps are true by continuity, since the blow-down is an isomorphism of line bundles away from the fibre over p. ∎

Lemma 3.3.8. *Let X be a curve, $\mathcal{L}$ an invertible sheaf on X and $p \in \mathsf{X}$. Let $\mathsf{Y} = \mathsf{Bl}_{0_p}(\mathsf{Tot}(\mathcal{L}))$ be the blowup of the total space of $\mathcal{L}$ at the zero element over p, and let $\overline{\mathcal{L}|_p} \subset \mathsf{Y}$ be the proper* *transform of the fibre over p. Then there is a canonical identification*

$$\mathsf{Y} \setminus \overline{\overline{\mathcal{L}|_p}} \cong \mathsf{Tot}(\mathcal{L}(-p))$$

for which the blow down is the natural map $\mathsf{Tot}(\mathcal{L}(-p)) \to \mathsf{Tot}(\mathcal{L})$.

Proof. The map $\phi : \mathsf{Tot}(\mathcal{L}(-p)) \to \mathsf{Tot}(\mathcal{L})$ has the property that $\phi^{-1}(0_p) = \mathcal{L}(-p)|_p$ is a Cartier divisor on $\mathsf{Tot}(\mathcal{L}(-p))$. Hence the universal property of blowing up gives a natural morphism

$$\phi' : \mathsf{Tot}(\mathcal{L}(-p)) \to \mathsf{Y}.$$

We claim that ϕ' maps $\mathsf{Tot}(\mathcal{L}(-p))$ isomorphically onto $\mathsf{Y} \setminus \overline{\overline{\mathcal{L}_p}}$. Since $\phi(\mathcal{L}(-p)|_p) = \{0_p\}$, the image of ϕ' may only intersect $\overline{\overline{\mathcal{L}|_p}}$ along the exceptional divisor E. The tangent space to $\mathsf{Tot}(\mathcal{L})$ at 0_p has a natural splitting, resulting in the identification

$$\mathsf{E} = \mathbb{P}(\mathsf{T}_p\mathsf{X} \oplus \mathcal{L}|_p).$$

Meanwhile, the fibre of $\mathcal{L}(-p)$ at p is

$$\mathcal{L}(-p)|_p \cong \mathsf{Hom}_{\mathbb{C}}(\mathsf{T}_p\mathsf{X}, \mathcal{L}|_p)$$

Sending a linear map $\mathsf{T}_p\mathsf{X} \to \mathcal{L}|_p$ to its graph defines an embedding

$$\mathcal{L}(-p)|_p \to \mathbb{P}(\mathsf{T}_p\mathsf{X} \oplus \mathcal{L}|_p)$$

which is the restriction of ϕ' to the fibre. The image of this embedding is the complement of $\mathbb{P}(\mathcal{L}|_p) = \overline{\overline{\mathcal{L}|_p}} \cap \mathsf{E}$, giving the required identification of the image of ϕ'. □

3.3.4 Uniformization

The blowup construction described in the previous sections produces the adjoint-type groupoid $\mathsf{Pair}(\mathsf{X}, \mathsf{D})$, to which all other integrations of $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ map. However, as we have seen, these groupoids are usually not source-simply connected. Meanwhile, the procedure of Crainic–Fernandes [42], which produces the canonical source-simply connected integration, relies on an infinite-dimensional quotient. In this section, we give two explicit finite-dimensional descriptions of the source-simply connected integrations which rely on the uniformization theorem for Riemann surfaces. We assume without loss of generality that X is connected, and for both constructions, we focus on the case when D is reduced, as the other cases may be obtained from this one by iterated blowups.

The first construction express the groupoid as suitably completed quotient of the product $\mathbb{H} \times \mathbb{H}$ of two copies of the upper half-plane by the action of a Fuchsian group. The second description uses the uniformizing projective connection to embed the groupoid as an explicit open set in a $\mathbb{P}^1$-bundle over X.

The basic observation underlying both approaches is that the restriction of $\Pi_1(\mathsf{X},\mathsf{D})$ to the punctured curve $\mathsf{U} = \mathsf{X} \setminus \mathsf{D}$ must be isomorphic to the fundamental groupoid of U. Let $\tilde{\mathsf{U}}$ be the universal cover, so that $\mathsf{U} \cong \tilde{\mathsf{U}}/\Gamma$, where $\Gamma = \pi_1(\mathsf{U}, p)$ for some $p \in \mathsf{U}$. By the uniformization theorem, $\tilde{\mathsf{U}}$ must be isomorphic to $\mathbb{C}, \mathbb{P}^1$ or the upper half-plane $\mathbb{H}$. We shall focus on the case when $\tilde{\mathsf{U}} = \mathbb{H}$ since it is the most prevalent: for example, if X is compact of genus g and D consists of n distinct points, then the universal cover of U will be $\mathbb{H}$ whenever $2-2g-n < 0$. Thus if D is non-empty, the only exceptions are when $g = 0$, so that $\mathsf{X} \cong \mathbb{P}^1$, and $n \leq 2$. These cases were studied already in Section 3.3.3.

Quotient construction

With the notation above, let $f_0 : \mathbb{H} \to \mathsf{U}$ be the covering map. The fundamental groupoid of U may be presented as

$$\Pi_1(\mathsf{U}) \cong (\mathbb{H} \times \mathbb{H})/\Gamma,$$

where Γ acts diagonally. The isomorphism is as follows: since $\mathbb{H}$ is simply connected, we have an isomorphism $\mathbb{H} \times \mathbb{H} \to \Pi_1(\mathbb{H})$, and the map $(f_0)_* : \Pi_1(\mathbb{H}) \to \Pi_1(\mathsf{U})$, sending a curve in $\mathbb{H}$ to its image via f_0, identifies $\Pi_1(\mathsf{U})$ with the quotient. In particular, the target and source maps are given by applying f_0 to the left and right factors of $\mathbb{H} \times \mathbb{H}$, respectively. Our goal in this section is to explain how to "complete" $\Pi_1(\mathsf{U})$ to $\Pi_1(\mathsf{X},\mathsf{D})$ by adding in the isotropy groups over D.

We shall use a procedure that mimics the standard completion of U to X by adding the cusps, which we now briefly recall. For details, see [55, 106]. Let $\partial\mathbb{H} \subset \mathbb{P}^1$ denote the boundary of $\mathbb{H}$, and let $\Lambda \subset \partial\mathbb{H}$ be the cusps of Γ—that is, the collection of fixed points of parabolic elements of Γ. We give $\mathbb{H}^* = \mathbb{H} \cup \Lambda$ a topology by declaring that the open neighbourhoods of a point $p \in \Lambda$ are the open disks in $\mathbb{P}^1$ that are contained in $\mathbb{H}$ and tangent to $\partial\mathbb{H}$ at p. With this topology (which is different from the subspace topology in $\mathbb{P}^1$), the map f_0 extends to a continuous surjective map $f : \mathbb{H}^* \to \mathsf{X}$. In this way, the points of the divisor D are identified with the points of Λ/Γ.

Suppose that $p \in \Lambda$. Acting by an element of $\mathsf{PSL}(2,\mathbb{R})$, we may assume that $p = \infty$. Then the stabilizer of p is freely generated by an element of the form

$$T_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

with $a > 0$, i.e., a translation $\tau \mapsto \tau + a$. The shifted half-planes

$$\mathbb{H}_c = \{\tau \in \mathbb{H} \,|\, \mathrm{Im}(\tau) > c\}$$

for $c \gg 0$ give a basis of neighbourhoods for $\infty \in \mathbb{H}^*$ and the quotient $\Delta = (\mathbb{H}_c \cup \{\infty\})/\langle T_a \rangle$ gives an open neighbourhood of the point $x = [\infty]$ in the quotient $\mathsf{X} = \mathbb{H}^*/\Gamma$. Moreover, the function $\tau \mapsto \exp(2\pi i \tau/a)$, which is invariant under T_a, descends to the quotient to give a coordinate on Δ that vanishes at x.

Example 3.3.9. Suppose that $(\mathsf{X}, \mathsf{D}) = (\mathbb{P}^1, 0 + 1 + \infty)$. For $\tau \in \mathbb{H}$, let

$$\wp_\tau(z) = \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(z+m+n\tau)^2} - \frac{1}{(m+n\tau)^2}$$

be the Weierstrass $\wp$-function of modulus τ. Define the modular function

$$\begin{array}{rccc} \lambda & : & \mathbb{H} & \to & \mathbb{P}^1 \\ & & \tau & \mapsto & \frac{\wp_\tau(\frac{1+\tau}{2}) - \wp_\tau(\frac{\tau}{2})}{\wp_\tau(\frac{1}{2}) - \wp_\tau(\frac{\tau}{2})}. \end{array}$$

Then λ is invariant under the principal congruence subgroup

$$\begin{aligned} \Gamma(2) &= \{ g \in \mathsf{SL}(2,\mathbb{Z}) \mid g \equiv 1 \mod 2\mathbb{Z} \} \\ &= \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle, \end{aligned}$$

and gives the universal cover $\lambda : \mathbb{H} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$ (see [2, Section 7.3.4]). As $\tau \to +i\infty$, we have $\lambda e^{-\pi i \tau} \to 16$, so that $\lambda \to 0$ and $e^{\pi i \tau}$ descends to a coordinate on $\mathbb{P}^1$ in a neighbourhood of 0. □

Let us now explain how to extend the fundamental groupoid of U to the groupoid $\Pi_1(\mathsf{X}, \mathsf{D})$ when D is reduced. By exponentiation, the isotropy group of $\Pi_1(\mathsf{X}, \mathsf{D})$ at $p \in \mathsf{D}$ is identified with the fibre of $\mathcal{T}_{\mathsf{X}}(-p)$ at p, which is *canonically* isomorphic to $\mathbb{C}$ (see Section 3.2.1). We must therefore glue a copy of $\mathbb{C}$ to every cusp. Consider the set

$$\mathsf{W}(\Gamma) = \mathbb{H} \times \mathbb{H} \coprod \Lambda \times \mathbb{C}.$$

We give $\mathsf{W}(\Gamma)$ a topology as follows. The topology on $\mathbb{H} \times \mathbb{H}$ is the usual one, so it remains to define the open neighbourhoods of a point $(p, u) \in \Lambda \times \mathbb{C}$. If $p = \infty$, we take as a basis of open neighbourhoods the sets

$$\mathsf{V}(c, r, u) = \{ (\tau_1, \tau_2) \in \mathbb{H}_c \times \mathbb{H}_c \mid |\tau_1 - \tau_2 - u| < r \} \cup \{ (p, u') \in \Lambda \times \mathbb{C} \mid |u - u'| < r \}$$

for $r, c > 0$. If $p \neq \infty$, we pick an element $g \in \mathsf{PSL}(2, \mathbb{R})$ with $gp = \infty$ and use the sets $g^{-1}\mathsf{V}(c, r)$ as a basis of open sets of p. The resulting system of neighbourhoods is independent of the choice of g.

In this topology, a sequence $(\tau_{1,n}, \tau_{2,n})_{n \geq 0} \in \mathbb{H} \times \mathbb{H}$ limits to the point $(\infty, u) \in \Lambda \times \mathbb{C}$

if and only if

$$\lim_{n\to\infty} \mathrm{Im}(\tau_{1,n}) = \lim_{n\to\infty} \mathrm{Im}(\tau_{2,n}) = \infty \tag{3.2}$$

and

$$\lim_{n\to\infty} (\tau_{1,n} - \tau_{2,n}) = u \tag{3.3}$$

with respect to the usual topology of $\mathbb{C}$. Notice that since we may take r, c to be rational and still obtain a basis of neighbourhoods, the topology on $\mathsf{W}(\Gamma)$ is first countable.

The space $\mathsf{W}(\Gamma)$ carries a natural action of Γ: the action on $\mathbb{H}\times\mathbb{H} \subset \mathsf{W}(\Gamma)$ is the diagonal one

$$g \cdot (\tau_1, \tau_2) = (g\tau_1, g\tau_2)$$

and the action on $\Lambda \times \mathbb{C} \subset \mathsf{W}(\Gamma)$ is given by

$$g(p, u) = (gp, u).$$

This action is continuous since it preserves the basis of open neighbourhoods described above.

Furthermore, $\mathsf{W}(\Gamma)$ has in a canonical way the structure of a topological groupoid whose objects are $\mathbb{H}^*$. The groupoid operation is defined as follows: $\Lambda \times \mathbb{C}$ is a disjoint union of abelian groups indexed by Λ, and hence it forms a groupoid for which the source and target maps are the projection $\Lambda \times \mathbb{C} \to \Lambda$, and the groupoid operation is addition:

$$(p, u)(p, u') = (p, u + u')$$

for $p \in \Lambda$ and $u, u' \in \mathbb{C}$. Meanwhile $\mathbb{H} \times \mathbb{H} \rightrightarrows \mathbb{H}$ is the pair groupoid $\mathsf{Pair}(\mathbb{H})$. Taking the disjoint union of these groupoids we obtain the groupoid

$$\mathsf{W}(\Gamma) = \mathsf{Pair}(\mathbb{H}) \coprod \Lambda \times \mathbb{C} \rightrightarrows \mathbb{H} \cup \Lambda,$$

and we leave to the reader the straightforward verification that the groupoid operations are continuous.

Now consider the quotient

$$\mathsf{G} = \mathsf{W}(\Gamma)/\Gamma = \mathsf{Pair}(\mathbb{H})\,/\Gamma \coprod (\Lambda \times \mathbb{C})/\Gamma = \mathsf{\Pi}_1(\mathsf{U}) \coprod (\mathsf{D} \times \mathbb{C}),$$

and notice that this set has the structure of a groupoid over X. By construction, this structure is compatible with the topology and hence $\mathsf{G} \rightrightarrows \mathsf{X}$ is a topological groupoid. Moreover, this description makes it clear that the source fibres of G are simply-connected.

It remains to give G a holomorphic structure, making it into a Lie groupoid isomorphic to $\Pi_1(\mathsf{X}, \mathsf{D})$. Once again, it is enough to consider the situation in a neighbourhood of ∞. Suppose that the stabilizer of ∞ in Γ is generated by the translation $T_a : \tau \mapsto \tau + a$ with $a > 0$. Then $\mathsf{V}' = \mathsf{V}(c, r, u)/\langle \mathsf{T}_a \rangle$ gives a neighbourhood of $[(\infty, u)] \in \mathsf{G}$. Let us give coordinates on this neighbourhood.

Let $\Delta_1 \subset \mathbb{C}$ and $\Delta_2 \subset \mathbb{C}$, be the product of the disks of radii e^{-c} and r centred at 0 and u, respectively, and consider the map

$$\phi_{r,c,u} : \mathsf{V}(c, r, u) \to \Delta_1 \times \Delta_2$$

defined by

$$\phi_{r,c,u}(\tau_1, \tau_2) = (e^{2\pi i \tau_1/a}, \tau_1 - \tau_2)$$

for $(\tau_2, \tau_2) \in \mathbb{H} \times \mathbb{H}$ and

$$\phi_{r,c,u}(\infty, u') = (0, u')$$

Since $e^{2\pi i \tau_1/a} \to 0$ as $\mathrm{Im}(\tau_1) \to \infty$, this map is continuous, and it descends to a homeomorphism from V' to $\Delta_1 \times \Delta_2$. Its restriction to $\mathsf{V}' \cap \Pi_1(\mathsf{U}) \subset \mathsf{G}$ is clearly holomorphic and hence it gives a coordinate chart.

Using this chart together with the corresponding coordinate $e^{2\pi i \tau/a}$ on the neighbourhood $\mathbb{H}_c/\langle T_a \rangle \subset \mathsf{X}$ of $[\infty] \in \mathsf{D}$, the target map t is just the projection $\Delta_1 \times \Delta_2 \to \Delta_1$. Hence it is a holomorphic submersion. Meanwhile, the source map is given by

$$s(z, w) = [\tau_2] = e^{2\pi i(\tau_1 - u)/a} = e^{2\pi i u/a} t.$$

which is also a submersion. Similarly, the multiplication and inversion are seen to be holomorphic. We have arrived at the following

Theorem 3.3.10. *The topological groupoid* $\mathsf{G} = \mathsf{W}(\Gamma)/\Gamma \rightrightarrows \mathsf{X}$ *is canonically isomorphic to the source-simply connected Lie groupoid groupoid* $\Pi_1(\mathsf{X}, \mathsf{D})$.

Proof. The arguments above show that $\mathsf{W}(\Gamma)/\Gamma$ is a source-simply connected Lie groupoid. It remains to check that the Lie algebroid is given $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$, but this fact follows easily from the local formulae for s and t above. □

Corollary 3.3.11. *If* $\mathsf{D} = p_1 + \cdots + p_n$ *is the divisor of punctures, then the groupoid* $\Pi_1(\mathsf{X}, k_1 p_1 + \cdots + k_n p_n)$ *is given by performing a* $(k_i - 1)$*-fold iterated blowup of* $\mathsf{W}(\Gamma)/\Gamma$ *at* $\mathrm{id}(p_i)$ *for each* i.

Remark 3.3.12. It should be straightforward to generalize the quotient construction to deal directly with the non-reduced case. □

This description of the groupoid naturally leads us to the following

Question 3.3.13. *What is the precise relationship between equivariant vector bundles for* $\Pi_1(\mathsf{X},\mathsf{D})$ *(i.e., meromorphic connections) and automorphic forms for the action of* Γ *on* $\mathbb{H}\times\mathbb{H}$*, with suitable holomorphicity conditions imposed at the cusps?*

Embeddings in projective bundles

Once again, we focus on the case in which D is reduced and the universal cover of the punctured curve $\mathsf{U}=\mathsf{X}\setminus\mathsf{D}$ is isomorphic to $\mathbb{H}$. The uniformization theorem gives rise to a natural projective connection on U; see [75]. We will use this projective connection to embed the groupoid in a $\mathbb{P}^1$-bundle over X.

Let τ be the standard coordinate on $\mathbb{H}$, and let $\mathcal{L}=\omega_{\mathbb{H}}^{-1/2}$ be a square-root of the anti-canonical sheaf on $\mathbb{H}$. The second-order differential operator

$$\begin{array}{cccccc}\Delta & : & \mathcal{L} & \to & (\Omega^1_{\mathbb{H}})^2\otimes\mathcal{L}=\mathcal{L}^{-3} \\ & & f(d\tau)^{-1/2} & \mapsto & \frac{d^2f}{d\tau^2}(d\tau)^{3/2}\end{array}$$

defines a projective structure on $\mathbb{H}$, which is invariant under the action of $\mathsf{PSL}(2,\mathbb{R})$. The kernel of this operator is generated by the sections

$$\mu_1 = d\tau^{-1/2}$$

and

$$\mu_2=\tau d\tau^{-1/2}.$$

This basis was chosen so that the ratio η_2/η_1 gives the standard coordinate on $\mathbb{H}$.

If $\mathsf{U}=\mathbb{H}/\Gamma$ is the quotient by the action of a Fuchsian group Γ, the operator Δ descends to give a projective structure on U. If we push μ_1 and μ_2 forward by the covering map, we obtain multivalued sections η_1 and η_2 of $\omega_{\mathsf{U}}^{-1/2}$ that are annihilated by the projective structure and whose ratio gives the inverse of the covering map.

Including the cusps of Γ, we obtain the curve X with a reduced effective divisor D, so that $\mathsf{U}=\mathsf{X}\setminus\mathsf{D}$. A straightforward calculation shows that the operator Δ extends to a meromorphic projective structure on (X,D) in the sense of Definition 3.2.6; see, e.g., [80, 92]. Near a point $p\in\mathsf{D}$, there is a coordinate z on X in which the uniformizing projective structure has the form

$$\begin{aligned}\Delta(f dz^{-1/2}) &= \left(\frac{d^2f}{dz^2}+\frac{f}{4z^2}\right)dz^{3/2}\\ &= \left(\delta^2(f)-\delta(f)+\frac{1}{4}f\right)\frac{dz^{3/2}}{z^2}\end{aligned}\tag{3.4}$$

where $\delta=z\partial_z$ is a generator for $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$ in this chart.

Any meromorphic projective structure on (X,D) induces a $\mathcal{T}_{\mathsf{X}}(-\mathsf{D})$-connection on the jet bundle $\mathcal{J} = \mathcal{J}^1_{\mathcal{T}_{\mathsf{X}}(-\mathsf{D})}\omega_{\mathsf{X}}^{-1/2}$, and thus a $\mathsf{PSL}(2,\mathbb{C})$-connection on the corresponding $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{J}) \to \mathsf{X}$. This connection integrates to an action of the source-simply connected groupoid $\Pi_1(\mathsf{X},\mathsf{D})$ on $\mathbb{P}(\mathcal{J})$, i.e., a homomorphism

$$\rho : \Pi_1(\mathsf{X},\mathsf{D}) \to \mathsf{PSL}(\mathcal{J})$$

of Lie groupoids.

Notice that $\mathbb{P}(\mathcal{J})$ has a canonical section $\sigma : \mathsf{X} \to \mathbb{P}(\mathcal{J})$, given by the embedding the line bundle $\Omega^1_{\mathsf{X}}(\mathsf{D}) \otimes \omega_{\mathsf{X}}^{-1/2}$ via the jet sequence. We therefore obtain a holomorphic map

$$\begin{array}{rcccc} \iota & : & \Pi_1(\mathsf{X},\mathsf{D}) & \to & \mathbb{P}(\mathcal{J}) \\ & & g & \mapsto & \rho(g)\cdot\sigma(s(g)) \end{array} \tag{3.5}$$

by acting on this section.

Lemma 3.3.14. *The map $\iota : \Pi_1(\mathsf{X},\mathsf{D}) \to \mathbb{P}(\mathcal{J})$ is an open embedding that makes the following diagrams commute:*

$$\begin{tikzcd} \Pi_1(\mathsf{X},\mathsf{D}) \arrow[rr, "\iota"] \arrow[dr, "t"'] & & \mathbb{P}(\mathcal{J}) \arrow[dl, "\pi"] \\ & \mathsf{X} & \end{tikzcd} \qquad \begin{tikzcd} \Pi_1(\mathsf{X},\mathsf{D}) \arrow[rr, "\iota"] & & \mathbb{P}(\mathcal{J}) \\ & \mathsf{X} \arrow[ul, "\mathrm{id}"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Proof. The commutativity of the diagrams is immediate from the definition. We must show that ι is an open embedding. Since $\Pi_1(\mathsf{X},\mathsf{D})$ and $\mathbb{P}(\mathcal{J})$ are complex manifolds of equal dimension and ι is holomorphic, it is sufficient to verify that ι is injective.

If $p \in \mathsf{U} = \mathsf{X} \setminus \mathsf{D}$, then the source fibre through p is mapped to the horizontal leaf of the connection on $\mathbb{P}(\mathcal{J})$ that contains $\sigma(p)$. Since the connection comes from the uniformizing projective structure, each horizontal leaf intersects the section σ in at most one point, and the monodromy action of $\pi_1(\mathsf{U},p)$ on the fibre $\mathsf{PSL}(\mathcal{J}_p)$ is free for all $p \in \mathsf{U}$. It follows easily that ι is injective on the open set $\Pi_1(\mathsf{U}) \subset \Pi_1(\mathsf{X},\mathsf{D})$.

Since $t = \pi \circ \iota$ the image of $\Pi_1(\mathsf{U})$ is disjoint from the image of the target fibres over D. Hence it remains to check injectivity on the individual isotropy groups over D. Using the local form (3.4) of the projective structure in the coordinate z near $p \in \mathsf{D}$, we compute using (3.1) the local form for the connection on $\mathcal{J}$:

$$\nabla = d + \begin{pmatrix} 0 & -1 \\ \frac{1}{4} & -1 \end{pmatrix} \frac{dz}{z}.$$

in the basis $dz^{-1/2}$, $\frac{dz}{z} \otimes dz^{-1/2}$. The projection of the connection matrix to $\mathfrak{sl}(2,\mathbb{C})$ is

$$A = \begin{pmatrix} \frac{1}{2} & -1 \\ \frac{1}{4} & \frac{-1}{2} \end{pmatrix}.$$

Recall that the isotropy group of $\Pi_1(\mathsf{X},\mathsf{D})$ over p is identified with additive group of complex numbers, and hence the action of the isotropy group is given by the map

$$\begin{array}{rcccc} \rho|_p & : & \mathbb{C} & \to & \mathsf{Aut}(\mathbb{P}(\mathcal{J})_p) \cong \mathsf{PSL}(2,\mathbb{C}) \\ & & t & \mapsto & [\exp(-tA)] \end{array} \tag{3.6}$$

Since A has rank one and

$$A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \neq 0,$$

the isotropy group acts freely on the orbit of $\sigma(p)$, as required. □

3.4 Local normal forms

In this section, we describe the local structure of the groupoids $\mathsf{Pair}(\mathsf{X},\mathsf{D})$ and $\Pi_1(\mathsf{X},\mathsf{D})$ associated to an effective divisor on a smooth curve.

3.4.1 Local normal form for twisted pair groupoids

If D is an effective divisor on the smooth curve X and $\mathsf{U} \subset \mathsf{X}$ is an open set, we have

$$\mathsf{Pair}(\mathsf{X},\mathsf{D})\,|_{\mathsf{U}} \cong \mathsf{Pair}(\mathsf{U},\mathsf{D}\cap\mathsf{U})$$

Taking U to be a coordinate disk centred at a point $p \in \mathsf{D}$ of multiplicity k, we see that $\mathsf{Pair}(\mathsf{X},\mathsf{D})$ is locally isomorphic to the restriction of the groupoid $\mathsf{Pair}\big(\mathbb{P}^1, k\cdot 0\big)$ to the unit disk $\Delta \subset \mathbb{C} \subset \mathbb{P}^1$. We may therefore use the description of $\mathsf{Pair}\big(\mathbb{P}^1, k\cdot 0\big)$ given in Theorem 3.3.7 to obtain the local normal form:

Theorem 3.4.1. *We have an isomorphism*

$$\mathsf{Pair}(\mathbb{C},0) \cong \mathbb{C}^* \ltimes \mathbb{C} \rightrightarrows \mathbb{C}$$

where the action of $\mathbb{C}^*$ on $\mathbb{C}$ is given by $\lambda\cdot z = \lambda z$ for $\lambda \in \mathbb{C}^*$ and $z \in \mathbb{C}$.

If $k \geq 1$, then

$$\mathsf{Pair}(\mathbb{C},k\cdot 0) \cong \left\{(\mu,z)\in\mathbb{C}^2 \,\middle|\, \mu z^{k-1} \neq 1\right\}$$

with structure maps

$$\begin{aligned}
s(\mu, z) &= z \\
t(\mu, z) &= \frac{z}{1 - \mu z^{k-1}} \\
\mathrm{id}(z) &= (0, z) \\
(\nu, w) \cdot (\mu, z) &= \left(\frac{\nu}{(1 - \mu z^{k-1})^{k-2}} + \mu, z \right) \text{ for } w = t(\mu, z) \\
(\mu, z)^{-1} &= \left(-(1 - \mu z^{k-1})^{k-2} \mu, \frac{z}{1 - \mu z^{k-1}} \right).
\end{aligned}$$

Proof. The case $k = 1$ is the restriction of $\mathsf{Pair}(\mathbb{P}^1, 0 + \infty)$ treated in Section 3.3.3, and is the action groupoid associated with action of $\mathbb{C}^*$ by automorphisms fixing 0 and ∞. This gives the first description of $\mathsf{Pair}(\mathbb{C}, 0)$ in the theorem. The second description is given by setting $\lambda = \frac{1}{1-\mu} \in \mathbb{C}^*$.

We now treat the case $k \geq 2$. By Theorem 3.3.7, the groupoids in question are given by removing the source and target fibres through ∞ from

$$\mathsf{Pair}(\mathbb{P}^1, k \cdot 0) \cong \mathsf{Tot}(\mathcal{T}_{\mathbb{P}^1}(-k \cdot 0)),$$

and when $k = 2$ the groupoid is given by the action of automorphisms of $\mathbb{P}^1$ which fix zero.

Let z be an affine coordinate centred at 0 and $u = z^{-1}$ the coordinate at ∞. Then the vector field

$$\delta_2 = -\partial_u = z^2 \partial_z$$

is a basis for $\mathcal{T}_{\mathbb{P}^1}(-2 \cdot 0)$. In the u-coordinate, the action is given by

$$(\mu\delta_2, u) \mapsto u - \mu$$

In the z-coordinate, the action is therefore

$$(\mu\delta_2, z) \mapsto \frac{1}{z^{-1} - \mu} = \frac{z}{1 - \mu z},$$

which gives the target map in the groupoid. Note that the source fibre through ∞ is given by $u = 0$, which is not in the z chart. Meanwhile, the target fibre through ∞ is given by $\mu z = 0$. Hence the formulae in the theorem are correct for $k = 2$.

Now let $\delta_k = z^k \partial_z$. To describe the groupoids for $k > 2$, consider the map

$$\begin{aligned}
\mathsf{Tot}(\mathcal{T}_{\mathbb{C}}(-k \cdot 0)) &\to \mathsf{Tot}(\mathcal{T}_{\mathbb{C}}(-2 \cdot 0)) \\
(\mu\delta_k, z) &\mapsto (z^{k-2}\delta_2, z)
\end{aligned}$$

which by Theorem 3.3.7 is the restriction of a groupoid homomorphism. We see immediately that

$$s(\mu\delta_k, z) = z$$

and

$$t(\mu\delta_k, z) = \frac{z}{1 - \mu z^{k-2} \cdot z} = \frac{z}{1 - \mu z^{k-1}}.$$

All of the other formulae in the theorem are straightforward consequences of these ones, and we therefore omit the calculations. □

Corollary 3.4.2. *We have*

$$\mathsf{Pair}(\Delta, 0) = \{(\mu, z) \in \mathbb{C}^* \times \Delta \mid |\mu z| < 1\}$$

and

$$\mathsf{Pair}(\Delta, k \cdot 0) = \left\{ (\mu, z) \in \mathbb{C} \times \Delta \,\middle|\, \mu z^{k-1} \neq 1, \frac{|z|}{|1 - \mu z^{k-1}|} < 1 \right\}$$

for $k \geq 2$, with the same structure maps as in the theorem.

3.4.2 Source-simply connected case

Suppose that D is an effective divisor on a smooth curve X and $p \in \mathsf{D}$ is a point of multiplicity k. If we choose a coordinate disk Δ centred at p, there will be a natural map

$$\Pi_1(\Delta, k \cdot p) \to \Pi_1(\mathsf{X}, \mathsf{D})\,|_\Delta,$$

but in general this map will not be an isomorphism. The reason is that the isotropy group of $\Pi_1(\mathsf{X}, \mathsf{D})$ at a point $q \in \Delta \setminus \{p\}$ is the fundamental group $\pi_1(\mathsf{X} \setminus \mathsf{D}, q)$ while the isotropy group of $\Pi_1(\Delta, k \cdot p)$ is the local fundamental group $\pi_1(\Delta \setminus \{p\}, q)$. However, if we take the connected component $\Pi_1(\mathsf{X}, \mathsf{D})\,|^0_\Delta \subset \Pi_1(\mathsf{X}, \mathsf{D})\,|_\Delta$ containing the identity bisection, we obtain a surjective map

$$\phi : \Pi_1(\Delta, k \cdot p) \to \Pi_1(\mathsf{X}, \mathsf{D})\,|^0_\Delta$$

that is locally biholomorphic. Moreover, if the homomorphism $\pi_1(\Delta \setminus \{p\}, q) \to \pi_1(\mathsf{X} \setminus \mathsf{D}, q)$ on fundamental groups is injective, then ϕ will be an isomorphism, giving a local normal form for the groupoid. This situation will always occur if X is compact with positive genus, or if $\mathsf{X} = \mathbb{P}^1$ and D contains at least two distinct points.

We now describe the source-simply connected groupoids $\Pi_1(\mathbb{C}, k \cdot 0)$ associated with a divisor supported at the origin in $\mathbb{C}$.

Theorem 3.4.3. *For $k \geq 1$, there is an isomorphism*

$$\Pi_1(\mathbb{C}, k \cdot 0) \cong \mathbb{C}^2$$

with structure maps

$$\begin{aligned} s(\mu, z) &= z \\ t(\mu, z) &= e^{\lambda z^{k-1}} z \\ \mathrm{id}(z) &= (0, z) \\ (\nu, w) \cdot (\mu, z) &= \left(e^{(k-1)\mu z^{k-1}} \nu + \mu, z \right) \textit{ for } w = t(\mu, z) \\ (\mu, z)^{-1} &= \left(-e^{(1-k)\mu z^{k-1}} \mu, e^{\mu z^{k-1}} z \right). \end{aligned}$$

Proof. When $k = 1$, these formulae reduce to the definition of the action groupoid for the action of $\mathbb{C}$ on itself given by $\mu \cdot z = e^{\mu} z$. Since $\mathbb{C}$ is simply connected, this action groupoid is source-simply connected. Moreover, the vector field $z\partial_z$ is an infinitesimal generator for the action, and hence the Lie algebroid is canonically isomorphic to $\mathcal{T}_{\mathbb{C}}(-0)$. The action groupoid is therefore identified with $\Pi_1(\mathbb{C}, 0)$. Since the exponential map $\mathcal{L}ie(\mathbb{C}) \to \mathbb{C}$ is the identity, the groupoid can be identified with the total space $\mathsf{Tot}(\mathcal{T}_{\mathbb{C}}(-0))$ in such a way that the source map is the bundle projection and the identity bisection is the zero section.

The groupoids for $k > 1$ are obtained by repeatedly blowing up $\Pi_1(\mathbb{C}, 0)$ at the identity element over 0 and discarding the proper transform of the fibre, just as in the proof of Theorem 3.3.7. Since the blowup only affects the source fibre at 0, the groupoids so obtained are source-simply connected. We can therefore identify

$$\Pi_1(\mathbb{C}, k \cdot 0) \cong \mathsf{Tot}(\mathcal{T}_{\mathbb{C}}(-k \cdot 0))$$

in such a way that the morphism

$$\Pi_1(\mathbb{C}, k \cdot 0) \to \Pi_1(\mathbb{C}, 0)$$

is given by the map

$$(\mu(z^k \partial_z), z) \mapsto (\mu z^{k-1}(z\partial_z), z).$$

The structure maps are uniquely determined by the fact that this map is a groupoid homomorphism and so a straightforward calculation yields the given formulae. ☐

Corollary 3.4.4. *We have*

$$\Pi_1(\Delta, k \cdot 0) \cong \left\{ (\mu, z) \in \mathbb{C} \times \Delta \,\middle|\, \mathrm{Re}(\lambda z^{k-1}) < \log |z|^{-1} \right\} \subset \Pi_1(\mathbb{C}, k \cdot 0)$$

with structure maps given by restricting those of $\Pi_1(\mathbb{C}, k \cdot 0)$.

Proof. Since the inclusion $\Delta \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ induces an isomorphism on fundamental groups, the restriction of $\Pi_1(\mathbb{C}, k \cdot 0)$ to Δ is source-simply connected; this restriction is the set described above. □

3.5 Summation of divergent series

At the outset of this chapter, we encountered divergent formal power series solutions of certain differential equations, and saw that they could be used to obtain holomorphic solutions by the analytic procedure of Borel summation. In this section, we explain how some of these divergent series can alternatively be summed directly using the appropriate groupoids, in a bid to answer the following motivating question from Chapter 1:

Question. *Do divergent asymptotic series expansions, such as*

$$\frac{1}{z}\int_0^\infty \frac{e^{-t/z}}{1+t}\,dt \sim \sum_{n=0}^{\infty}(-1)^n n! z^n$$

have some intrinsic geometric meaning?

The idea is simple: we will see that if we take the formal power series solution on the curve X and pull it back to the groupoid $\mathsf{G} = \Pi_1(\mathsf{X},\mathsf{D})$ in an appropriate way, we obtain a new power series in coordinates on the complex surface G that is necessarily convergent.

In order to make this procedure precise, it is convenient to use the language of formal geometry, in which we treat formal power series as though they were functions on some actual space. We refer the reader to [79, Section II.9] and [82] for an introduction to the subject. We note that it is possible to give a more elementary (but less geometric) account of the facts presented here without the use of this language; at a basic level, we are simply make some elementary claims about the algebraic relationship between certain power series.

Let D be an effective divisor on the smooth complex curve X, and let $\mathcal{A} = \mathcal{T}_\mathsf{X}(-\mathsf{D})$ be the associated Lie algebroid. We denote by $\hat{\mathsf{X}}$ the formal neighbourhood of D in X. Since the kth-order neighbourhood $k\mathsf{D}$ of D is $\mathcal{A}$-invariant for all $k \in \mathbb{N}$, the Lie algebroid $\mathcal{A}$ restricts to a Lie algebroid $\hat{\mathcal{A}}$ on $\hat{\mathsf{X}}$.

We can now give an intrinsic definition of a formal power series gauge transformation:

Definition 3.5.1. Let $(\mathcal{E}_1,\nabla_1)$ and $(\mathcal{E}_2,\nabla_2)$ be $\mathcal{A}$-modules. A ***formal isomorphism from*** $\mathcal{E}_1$ ***to*** $\mathcal{E}_2$ is an isomorphism

$$\hat{\phi} : (\hat{\mathcal{E}}_1,\hat{\nabla}_1) \to (\hat{\mathcal{E}}_2,\hat{\nabla}_2)$$

of $\hat{\mathcal{A}}$-modules.

Let G be a Lie groupoid integrating the Lie algebroid $\mathcal{A} = \mathcal{T}_\mathsf{X}(-\mathsf{D})$. We identify X with its embedding $\mathrm{id}(\mathsf{X}) \subset \mathsf{G}$. Let $\hat{\mathsf{X}}$ and $\hat{\mathsf{G}}$ be the formal neighbourhoods of D in X and G respectively. Then the groupoid structure maps for G restrict to give a groupoid $\hat{\mathsf{G}} \rightrightarrows \hat{\mathsf{X}}$ in the category of formal schemes. Notice that the natural maps $\Pi_1(\mathsf{X},\mathsf{D}) \to$

$\mathsf{G} \to \mathsf{Pair}(\mathsf{X}, \mathsf{D})$ are analytic isomorphisms in a neighbourhood of D, and hence we have isomorphisms $\widehat{\Pi_1(\mathsf{X}, \mathsf{D})} \cong \hat{\mathsf{G}} \cong \widehat{\mathsf{Pair}(\mathsf{X}, \mathsf{D})}$.

We have the following formal analogue of Lemma 3.3.2:

Proposition 3.5.2. *Let $\hat{\mathcal{F}} \subset \mathcal{T}_{\hat{\mathsf{G}}}$ be the formal Lie algebroid of vector fields tangent to the source fibres of $\hat{\mathsf{G}}$, and let $(\hat{\mathcal{E}}_{\hat{\mathsf{G}}}, \hat{\nabla})$ be a locally free $\hat{\mathcal{F}}$-module. Then restriction of flat sections to $\hat{\mathsf{X}} \subset \hat{\mathsf{G}}$ defines a canonical isomorphism*

$$\mathsf{H}^0\left(\hat{\mathsf{G}}, \hat{\mathcal{E}}_{\hat{\mathsf{G}}}\right)^{\hat{\nabla}} = \mathsf{H}^0\left(\hat{\mathsf{X}}, \hat{\mathcal{E}}_{\hat{\mathsf{G}}}|_{\hat{\mathsf{X}}}\right).$$

Proof. If we choose local coordinates (z, λ) on G so that the source map is $(z, \lambda) \mapsto z$ and we pick a trivialization of $\mathcal{E}$ near D, the statement reduces to the observation that an initial value problem of the form

$$\begin{aligned} \frac{df}{d\lambda} &= A(z, \lambda) f \\ f(z, 0) &= f_0(z) \end{aligned}$$

for $A(z, \lambda) \in \mathbb{C}[[z, \lambda]]^{n \times n}$ and $f_0(z) \in \mathbb{C}[[z]]^n$ has a unique solution $f \in \mathbb{C}[[z, \lambda]]^n$. $\square$

Therefore, arguing exactly as in Section 3.3.1, we can obtain an equivalence between the category of locally free $\hat{\mathcal{A}}$-modules and equivariant vector bundles on $\hat{\mathsf{G}}$. We observe that there is an obvious restriction functor

$$\hat{\ } : \mathscr{R}ep(\mathsf{G}) \to \mathscr{R}ep(\hat{\mathsf{G}}).$$

In particular, if $\mathcal{E}_1, \mathcal{E}_2 \in \mathscr{R}ep(\mathsf{G})$ then we have a $\mathbb{C}$-linear map

$$R(\mathcal{E}_1, \mathcal{E}_2) : \mathsf{Hom}_{\mathsf{G}}(\mathcal{E}_1, \mathcal{E}_2) \to \mathsf{Hom}_{\hat{\mathsf{G}}}\left(\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2\right)$$

Assuming that X is connected and D is non-empty, $R(\mathcal{E}_1, \mathcal{E}_2)$ will be injective, because an element of $\mathsf{Hom}_{\mathsf{G}}(\mathcal{E}_1, \mathcal{E}_2) \subset \mathsf{H}^0(\mathsf{X}, \mathcal{H}om(\mathcal{E}_1, \mathcal{E}_2))$ is uniquely determined by its Taylor expansion along D by analytic continuation. As we have seen, $R(\mathcal{E}_1, \mathcal{E}_2)$ will not in general be surjective, since there may exist formal homomorphisms that do not arise from analytic ones. Nevertheless, in some cases we can use the theory to produce convergent series from divergent ones:

Theorem 3.5.3. *Suppose that $\mathcal{E}_1$ and $\mathcal{E}_2$ are analytic G-equivariant bundles with structure maps $\rho_i : s^*\mathcal{E}_i \to t^*\mathcal{E}_i$, and let ∇_1, ∇_2 be the corresponding $\mathcal{A}$-connections. Suppose that $\hat{\phi} : \hat{\mathcal{E}}_1 \to \hat{\mathcal{E}}_2$ is a formal isomorphism $(\hat{\mathcal{E}}_1, \hat{\nabla}_1) \to (\hat{\mathcal{E}}_2, \hat{\nabla}_2)$ over $\hat{\mathsf{X}}$. Then we have the equality*

$$\hat{\rho}_2 = t^*\hat{\phi} \circ \hat{\rho}_1 \circ s^*\hat{\phi}^{-1}$$

on the formal groupoid $\hat{\mathsf{G}}$. In particular, the right hand side, which a priori is purely formal, is actually analytic.

Proof. The formal isomorphism $\hat{\phi}$ between the connections defines and isomorphism of $\hat{\mathsf{G}}$-representations, which exactly means that

$$t^*\hat{\phi}\hat{\rho}_1 = \hat{\rho}_2 s^*\phi,$$

as desired. Now $\hat{\rho}_2$ is just the Taylor expansion of ρ_2 at D, and hence it must converge because ρ_2 is analytic. It follows that the right hand side also converges. $\square$

Let us illustrate this phenomenon by considering the example from Section 3.1, where $(\mathsf{X},\mathsf{D}) = (\mathbb{C}, 2\cdot(0))$ and $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{O}_{\mathsf{X}}^{\oplus 2}$ with the connections

$$\nabla_1 = d + \frac{1}{2z^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} dz$$

and

$$\nabla_2 = d + \frac{1}{2z^2}\begin{pmatrix} -1 & z \\ 0 & 1 \end{pmatrix} dz.$$

We recall that the matrix-valued formal power series

$$\hat{\phi} = \begin{pmatrix} 1 & \frac{z}{2}\sum_{n=0}^{\infty}(-1)^n n! z^n \\ 0 & 1 \end{pmatrix},$$

defines a formal isomorphism $\hat{\nabla}_1 \to \hat{\nabla}_2$ that is not analytic. We will use Theorem 3.5.3 to extract analytic isomorphisms on small open sets from this divergent series.

We use the coordinates (λ, z) on $\mathsf{Pair}(\mathbb{C}, 2\cdot(0))$ with $s(u,z) = z$ and $t(u,z) = \frac{z}{1-uz}$ as coordinates on the formal groupoid $\hat{\mathsf{G}}$. Since the connection ∇_1 is diagonal, it is easy to compute its solutions: the matrix

$$\psi_1(z) = \begin{pmatrix} e^{-1/2z} & 0 \\ 0 & e^{1/2z} \end{pmatrix}$$

is a fundamental solution of $\nabla_1\psi_1 = 0$, and hence the representation corresponding to ∇_1 is readily computed as

$$\rho_1 = t^*\psi_1 \cdot s^*\psi_1^{-1} = \begin{pmatrix} e^{-u/2} & 0 \\ 0 & e^{u/2} \end{pmatrix}$$

According to Theorem 3.5.3, the Taylor expansion for the representation ρ_2 corresponding to ∇_2 is given by expanding the formal power series

$$\rho_2 = \begin{pmatrix} 1 & \frac{z}{2(1-uz)} \sum_{n=0}^{\infty} \frac{(-1)^n n! z^n}{(1-uz)^n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-u/2} & 0 \\ 0 & e^{u/2} \end{pmatrix} \begin{pmatrix} 1 & -\frac{z}{2} \sum_{n=0}^{\infty} (-1)^n n! z^n \\ 0 & 1 \end{pmatrix},$$

which seems likely to diverge at first glance. However, many cancellations occur, giving

$$\rho_2 = \begin{pmatrix} e^{-u/2} & f \\ 0 & e^{u/2} \end{pmatrix},$$

where

$$f = \tfrac{1}{2} zu + \left(\tfrac{1}{48} u^3 z + \tfrac{1}{4} u^2 z^2\right) + \tfrac{1}{24} u^3 z^2 + \left(\tfrac{1}{6} u^3 z^3 + \tfrac{1}{96} u^4 z^2 + \tfrac{1}{3840} u^5 z\right) + \cdots$$

converges in a neighbourhood of $(0,0)$. We note that the resulting analytic function is multivalued on $\mathsf{Pair}(\mathsf{X},\mathsf{D})$, but single-valued when pulled back to $\Pi_1(\mathsf{X},\mathsf{D})$.

Fix a base point $z_0 \in \mathbb{C}^*$. Thus the parallel transport of ∇_2 from z_0 to z is

$$\psi_2(z) = \rho_2\left(\tfrac{z_0^{-1} - z^{-1}}{2}, z_0\right) = \begin{pmatrix} e^{(z^{-1} - z_0^{-1})/2} & f(z_0^{-1} - z^{-1}) \\ 0 & e^{-(z^{-1} - z_0^{-1})/2} \end{pmatrix},$$

giving the fundamental solution of the equation $\nabla_2 \psi_2 = 0$ with initial initial condition $\psi_2(z_0) = 1$, defined for z in a simply connected neighbourhood of z_0. In particular $\psi_2 \psi_1^{-1}$ gives an analytic isomorphism from ∇_1 to ∇_2 on such a neighbourhood. Truncating the series expansion for f after N steps gives a rational function f_N that converges to the component $\psi_2^{12} = f(z_0^{-1} - z^{-1})$ in a neighbourhood of z_0 as $N \to \infty$. These approximations are illustrated in Figure 3.3.

The procedure we have described produces solutions of the irregular singular equation from divergent formal power series in a way that is purely local: the kth term in the Taylor expansion of the solutions on the groupoid relies only on the Taylor coefficients of degree $\leq k$ for the Lie algebroid connection. We close our discussion of Lie theory on curves with this tantalizing suggestion that the groupoid might allow for a geometric description of the theory of multi-summation.

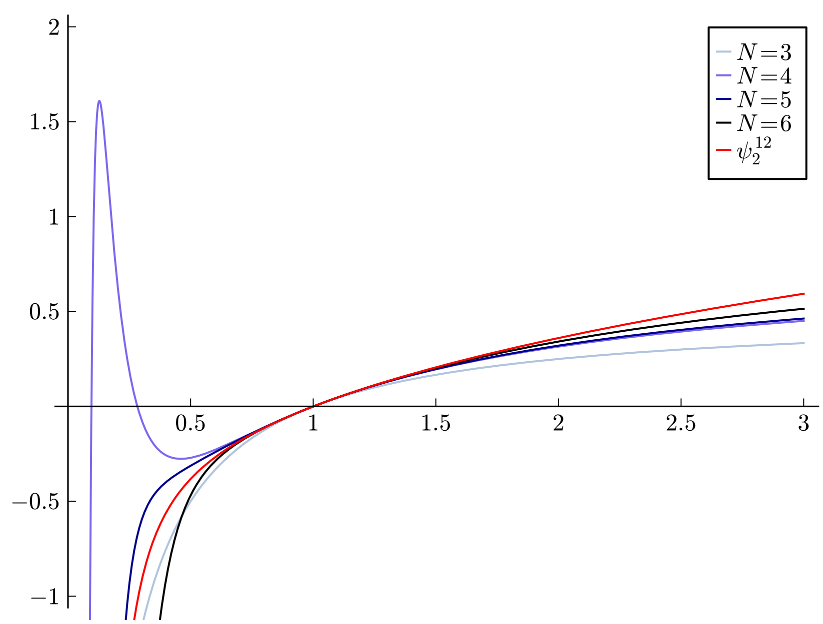


Figure 3.3: Approximations of the component ψ_2^{12} of ψ_2. Truncating the series expansion on the groupoid in degree N gives an approximation by rational functions that converge to the solution as $N \to \infty$.

Chapter 4

Poisson structures in complex geometry

In this chapter, we review some basic aspects of Poisson geometry in the complex analytic setting, in the spirit of [117]. Our goal throughout is to treat Poisson structures on singular spaces on equal footing with their smooth counterparts. In particular, we recall the basic definitions of multiderivations, Poisson structures and Poisson subspaces and the associated Lie algebroids in a manner that is appropriate for general analytic spaces.

4.1 Multiderivations

One of the subtle issues that arise when dealing with singular spaces is that the identification between multivector fields and multiderivations familiar from the case of smooth manifolds fails in general. In this section, we give a brief review of multiderivations in preparation for their use in Poisson geometry; we refer the reader to [93, Chapter 3] for a more detailed treatment.

Let X be a complex manifold or analytic space. The ***sheaf of k-derivations on X*** is the sheaf of $\mathbb{C}$-multilinear maps

$$\underbrace{\mathcal{O}_{\mathsf{X}} \times \cdots \times \mathcal{O}_{\mathsf{X}}}_{k \text{ times}} \to \mathcal{O}_{\mathsf{X}}$$

that are totally skew-symmetric and are derivations in each argument. In particular, $\mathscr{X}^1_{\mathsf{X}} = \mathcal{T}_{\mathsf{X}} = (\Omega^1_{\mathsf{X}})^\vee$ is just the tangent sheaf of X. More generally, the pairing

$$\Omega^k_{\mathsf{X}} \otimes \mathscr{X}^k_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}$$

defined on generators by

$$(df_1 \wedge \cdots \wedge df_k) \otimes \xi \mapsto \xi(f_1, \ldots, f_k)$$

gives an isomorphism $\mathscr{X}^k_{\mathsf{X}} \cong (\Omega^k_{\mathsf{X}})^\vee = \mathcal{H}om\big(\Omega^k_{\mathsf{X}}, \mathcal{O}_{\mathsf{X}}\big)$. When X is smooth, we also have the duality $\Omega^k_{\mathsf{X}} = (\mathscr{X}^k_{\mathsf{X}})^\vee$, but the latter isomorphism fails, in general, when X is singular.

Using the usual formula for the product of alternating linear maps (the shuffle product), we obtain a wedge product

$$\wedge : \mathscr{X}^k_{\mathsf{X}} \times \mathscr{X}^l_{\mathsf{X}} \to \mathscr{X}^{k+l}_{\mathsf{X}},$$

that is $\mathcal{O}_{\mathsf{X}}$-linear, making $\mathscr{X}^\bullet_{\mathsf{X}} = \bigoplus_{k\geq 0} \mathscr{X}^k_{\mathsf{X}}$ into a graded-commutative algebra. Notice that there are obvious contraction maps

$$\Omega^k_{\mathsf{X}} \otimes \mathscr{X}^l_{\mathsf{X}} \to \mathscr{X}^{l-k}_{\mathsf{X}}$$

for $l \geq k$ and

$$\Omega^k_{\mathsf{X}} \otimes \mathscr{X}^l_{\mathsf{X}} \to \Omega^{k-l}_{\mathsf{X}}$$

for $l \leq k$, satisfying the usual identities familiar from exterior products of vector spaces and their duals. If $\mathcal{E}$ is a vector bundle or $\mathcal{O}_{\mathsf{X}}$-module, we may also consider the sheaf $\mathscr{X}^k_{\mathsf{X}}(\mathcal{E}) = \mathcal{H}om\big(\Omega^k_{\mathsf{X}}, \mathcal{E}\big)$ of ***k-derivations with values in*** $\mathcal{E}$, and we note that $\mathscr{X}^\bullet_{\mathsf{X}}(\mathcal{E})$ is naturally a graded module over $\mathscr{X}^\bullet_{\mathsf{X}}$.

Additionally, there is a bracket

$$[\cdot,\cdot] : \mathscr{X}^k_{\mathsf{X}} \times \mathscr{X}^l_{\mathsf{X}} \to \mathscr{X}^{l+k-1}_{\mathsf{X}}$$

called the ***Schouten bracket***, which extends the Lie bracket on vector fields and makes $\mathscr{X}^\bullet_{\mathsf{X}}$ into a Gerstenhaber algebra. In particular, there is a natural action of $\mathcal{T}_{\mathsf{X}} = \mathscr{X}^1_{\mathsf{X}}$ on $\mathscr{X}^\bullet_{\mathsf{X}}$ by the ***Lie derivative***

$$\mathscr{L}_Z \xi = [Z, \xi]$$

for $Z \in \mathscr{X}^1_{\mathsf{X}}$ and $\xi \in \mathscr{X}^\bullet_{\mathsf{X}}$ that is compatible with wedge products and contractions with differential forms.

Related to the multiderivations are the ***multivector fields*** $\wedge^\bullet \mathcal{T}_{\mathsf{X}}$, which are $\mathcal{O}_{\mathsf{X}}$-linear combinations of wedge products of vector fields. There is a natural algebra homomorphism $\wedge^k \mathcal{T}_{\mathsf{X}} \to \mathscr{X}^k_{\mathsf{X}}$ defined by sending the multivector field $Z_1 \wedge \cdots \wedge Z_k$ with $Z_1, \ldots, Z_l \in \mathscr{X}^1_{\mathsf{X}}$ to the k-derivation

$$\xi(f_1, \ldots, f_k) = \det(Z_i(f_j))_{1\leq i,j,\leq k}.$$

When X is smooth, Φ is an isomorphism, but in general it is neither injective nor surjective. The failure of Φ to be an isomorphism is already visible for the quadric cone $\mathsf{Y} \subset \mathbb{C}^3$ from

The Example, as we shall in Example 4.3.2. This failure shows that in order to do Poisson geometry on singular spaces we must work with $\mathscr{X}^\bullet_{\mathsf{X}}$ rather than $\wedge^\bullet \mathcal{T}_{\mathsf{X}}$. Once this fact is recognized, it is straightforward to adapt many of the techniques from Poisson geometry in the smooth category to the singular case.

4.2 Poisson brackets

Let X be a complex manifold, or, more generally, an analytic space. A ***Poisson structure on*** X is a $\mathbb{C}$-linear Lie bracket $\{\cdot,\cdot\} : \mathcal{O}_{\mathsf{X}} \times \mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}$ on holomorphic functions that is a derivation in both arguments, i.e., it satisfies the Leibniz rule

$$\{f, gh\} = \{f, g\}h + f\{g, h\}$$

for all $f, g, h \in \mathcal{O}_{\mathsf{X}}$. Thus, the Poisson bracket is defined by a global section $\sigma \in \mathsf{H}^0(\mathsf{X}, \mathscr{X}^2_{\mathsf{X}})$ of the sheaf of two-derivations. The Jacobi identity for the bracket is equivalent to the equation

$$[\sigma, \sigma] = 0$$

for the Schouten bracket of σ. From now on, we refer to the pair (X, σ) as a ***Poisson analytic space***; if X is a manifold, then we also call (X, σ) a ***complex Poisson manifold***. If (X, σ) and (Y, η) are Poisson analytic spaces, a ***morphism from*** (X, σ) ***to*** (Y, η) is a holomorphic map $F : \mathsf{X} \to \mathsf{Y}$ such that the pullback of functions $F^* : F^{-1}\mathcal{O}_{\mathsf{Y}} \to \mathcal{O}_{\mathsf{X}}$ is compatible with the brackets, i.e., $\{F^*(f), F^*(g)\} = F^*\{f, g\}$.

Example 4.2.1. If X is two-dimensional, then $\mathscr{X}^3_{\mathsf{X}} = 0$, and hence any two-derivation $\sigma \in \mathsf{H}^0(\mathsf{X}, \mathscr{X}^2_{\mathsf{X}})$ defines a Poisson structure. If X is a compact manifold, then $\mathscr{X}^2_{\mathsf{X}} = \omega^{-1}_{\mathsf{X}}$ is the anticanonical line bundle and hence the Poisson structure is determined up to rescaling by the curve $\mathsf{C} \subset \mathsf{X}$ on which it vanishes, which is an anti-canonical divisor. □

Example 4.2.2. Let X be an even-dimensional complex manifold and let $\beta \in \mathsf{H}^0(\mathsf{X}, \Omega^2_{\mathsf{X}})$ be a holomorphic two-form that is closed an nondegenerate, i.e., a ***holomorphic symplectic form***. The nondegeneracy allows us to form the inverse $\sigma = \beta^{-1} \in \mathsf{H}^0(\mathsf{X}, \mathscr{X}^2_{\mathsf{X}})$, and the condition $d\beta = 0$ ensures that this two-derivation is a Poisson structure. Conversely, a Poisson structure that is nondegenerate can be inverted to obtain a holomorphic symplectic form. □

Example 4.2.3. The Poisson structure on $\mathbb{C}^3$ from The Example is defined by the brackets

$$\{x, y\} = 2y$$
$$\{x, z\} = -2z$$
$$\{y, z\} = x,$$

for the coordinate functions, which generate $\mathcal{O}_{\mathbb{C}^3}$. The corresponding two-derivation is

$$\sigma = x\partial_y \wedge \partial_z + 2y\partial_x \wedge \partial_y - 2z\partial_x \wedge \partial_z,$$

which is a global section of $\mathscr{X}^2_{\mathbb{C}^3}$, making $(\mathbb{C}^3, \sigma)$ a complex Poisson manifold. $\square$

Example 4.2.4. If $\mathfrak{g}$ is a complex Lie algebra, the dual space $\mathfrak{g}^\vee$ is a complex Poisson manifold, as follows: we identify $\mathfrak{g} \subset \mathcal{O}_{\mathfrak{g}^\vee}$ with the space of linear functions on the dual $\mathfrak{g}^\vee$. Thus $\mathfrak{g}$ generates $\mathcal{O}_{\mathfrak{g}^\vee}$ as a commutative algebra and we obtain a Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{O}_{\mathfrak{g}^\vee}$ by requiring that

$$\{x, y\} = [x, y]$$

for all $x, y \in \mathfrak{g}$, where $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket. This Poisson structure is known as the ***Lie–Poisson structure*** or the ***Kirillov–Kostant–Soriau*** Poisson structure. The Example corresponds to the case when $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{C})$. $\square$

Example 4.2.5. Let $\mathfrak{g}$ be a complex Lie algebra. Then $\wedge^\bullet\mathfrak{g}$ inherits a bracket that extends the Lie bracket on $\mathfrak{g}$. An element $\gamma \in \wedge^2\mathfrak{g}$ is called a ***triangular r-matrix*** if it satisfies the ***classical Yang–Baxter equation*** $[\gamma, \gamma] = 0$. Suppose that X is an analytic space and $\rho : \mathfrak{g} \to \mathsf{H}^0(\mathsf{X}, \mathscr{X}^1_{\mathsf{X}})$ is a Lie algebra homomorphism, i.e., and infinitesimal action of $\mathfrak{g}$ on X. If $\gamma \in \wedge^2\mathfrak{g}$ is a triangular r-matrix, then its image $\sigma = \wedge^2\rho(\gamma) \in \mathsf{H}^0(\mathsf{X}, \mathscr{X}^2_{\mathsf{X}})$ is a Poisson structure on X. We will use this approach to construct Poisson structures on projective space associated with linear free divisors in Section 7.5. $\square$

Example 4.2.6. Many moduli spaces in gauge theory and algebraic geometry, including the moduli spaces parametrizing the meromorphic connections that we studied in Chapter 3, come equipped with natural Poisson structures whose geometry is intimately connected with the moduli problem under consideration. See, for example, [14, 15, 23, 24, 25, 26, 57, 100, 105, 118]. $\square$

When (X, σ) is a Poisson analytic space, there is a natural morphism $\sigma^\sharp : \Omega^1_{\mathsf{X}} \to \mathscr{X}^1_{\mathsf{X}}$ defined by

$$\sigma^\sharp(\alpha)(\beta) = \sigma(\alpha \wedge \beta)$$

for $\alpha, \beta \in \Omega^1_{\mathsf{X}}$. This map is the anchor map for a Lie algebroid structure on Ω^1_{X}, which we denote by $\Omega^1_{\mathsf{X},\sigma}$. Notice that this Lie algebroid is a vector bundle if and only if X is smooth. The bracket is given by the ***Koszul bracket*** [91]

$$[\alpha, \beta] = \mathscr{L}_{\sigma^\sharp(\alpha)}\beta - \mathscr{L}_{\sigma^\sharp(\beta)}\alpha - d(\sigma(\alpha \wedge \beta)).$$

The image $\mathcal{F} = \mathcal{I}mg(\sigma^\sharp) \subset \mathscr{X}^1_{\mathsf{X}}$ is involutive and therefore defines a subalgebroid of the tangent sheaf $\mathscr{X}^1_{\mathsf{X}}$. In particular, when X is a manifold, it is partitioned into orbits which are immersed submanifolds $\mathsf{Y} \subset \mathsf{X}$. Moreover, σ restricts to give a nongenerate Poisson structure on Y. Thus the orbits are holomorphic symplectic manifolds, called the ***symplectic leaves***.

If (X, σ) is a Poisson analytic space, then we say that a vector field $Z \in \mathscr{X}^1_{\mathsf{X}}$ is a ***Poisson vector field*** if $\mathscr{L}_Z \sigma = 0$. In other words, Poisson vector fields are the local infinitesimal symmetries of σ. We denote by $\mathcal{P}ois(\sigma) \subset \mathscr{X}^1_{\mathsf{X}}$ the sheaf of Poisson vector fields. Notice that this sheaf is not an $\mathcal{O}_{\mathsf{X}}$-module; it is only a sheaf of complex vector spaces.

For any function $f \in \mathcal{O}_{\mathsf{X}}$, we have the ***Hamiltonian vector field*** $\sigma^\sharp(df)$, which is the derivation $\mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}$ obtained by bracketing with f:

$$\sigma^\sharp(df) = \{f, \cdot\}$$

We denote by $\mathcal{H}am(\sigma) = \mathcal{I}mg\big(\sigma^\sharp \circ d\big) \subset \mathscr{X}^1_{\mathsf{X}}$ the sheaf of Hamiltonian vector fields. It follows from the Jacobi identity that every Hamiltonian vector field is Poisson:

$$\mathcal{H}am(\sigma) \subset \mathcal{P}ois(\sigma)\,,$$

but the inclusion is, in general strict.

Example 4.2.7. In The Example, the Hamiltonian vector fields of x, y and z are the vector fields

$$\begin{aligned} X_x &= \sigma^\sharp(dx) = 2y\partial_y - 2z\partial_z \\ X_y &= \sigma^\sharp(dy) = x\partial_z - 2y\partial_x \\ X_z &= \sigma^\sharp(dz) = 2z\partial_x - x\partial_y \end{aligned}$$

on $\mathbb{C}^3$ that generate the reflexive Lie algebroid $\mathcal{F} = \mathcal{I}mg\big(\sigma^\sharp\big) \subset \mathscr{X}^1_{\mathbb{C}^3}$. □

Example 4.2.8. Let $\mathsf{X} = \mathbb{C}^2$ with linear coordinates x and y, and let $\sigma = x\partial_x \wedge \partial_y$. Then the vector field ∂_y is Poisson. Notice that every Hamiltonian vector field vanishes on the locus where $x = 0$, and hence ∂_y, which is non-vanishing, cannot be Hamiltonian. □

We remark that Poisson vector fields, which are local symmetries, need not extend to global vector fields: not every local section of $\mathcal{P}ois(\sigma)$ comes from a global one.

Example 4.2.9. Let $\mathsf{X} = \mathbb{P}^2$. Since $\mathscr{X}^2_{\mathbb{P}^2} = \omega^{-1}_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(3)$, a Poisson structure σ on $\mathbb{P}^2$ is determined up to rescaling by the cubic curve $\mathsf{Y} \subset \mathbb{P}^2$ on which it vanishes. Suppose that Y is smooth (hence, an elliptic curve) and consider an affine chart $\mathbb{C}^2 \subset \mathbb{P}^2$ with coordinates x, y. Then the Poisson structure has the form

$$\sigma|_{\mathbb{C}^2} = f\partial_x \wedge \partial_y$$

where f is a defining equation for Y in this chart. Since the Poisson structure has a factor of f, the vector field

$$Z = \sigma^\sharp(df)/f$$

is holomorphic on all of $\mathbb{C}^2$. One easily verifies that $Z \in \mathcal{P}ois(\sigma)$ and that it preserves the curve $\mathsf{Y} \cap \mathbb{C}^2$. However, it cannot extended to a Poisson vector field on $\mathbb{P}^2$: indeed, if it were to extend, then by continuity, the extension would have to preserve Y; i.e., it would be a section of $\mathcal{T}_{\mathbb{P}^2}(-\log \mathsf{Y})$. But a straightforward calculation in sheaf cohomology shows that $\mathcal{T}_{\mathbb{P}^2}(-\log \mathsf{Y})$ has no global sections. □

4.3 Poisson subspaces

If (X, σ) is a Poisson analytic space, a ***Poisson subspace of*** (X, σ) is an analytic subspace Y equipped with a Poisson structure $\eta \in \mathsf{H}^0(\mathsf{Y}, \mathscr{X}^2_{\mathsf{Y}})$ such that the inclusion $i : \mathsf{Y} \to \mathsf{X}$ is a morphism of Poisson analytic spaces, i.e., it is compatible with the brackets. Notice that if such an η exists, then it is necessarily unique since the map $i^* : i^{-1}\mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{Y}}$ is surjective at every point of Y.

If $\mathsf{Y} \subset \mathsf{X}$ is open, then it inherits in an obvious way the structure of a Poisson subspace since the bracket is defined on all open sets. However, if $\mathsf{Y} \subset \mathsf{X}$ is closed the condition of being a Poisson subspace is rather special:

Proposition 4.3.1. *Let (X, σ) be a Poisson analytic space and let $\mathsf{Y} \subset \mathsf{X}$ be a closed subspace with ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$. Then the following statements are equivalent*

1. *Y admits the structure of a Poisson subspace.*
2. *$\mathcal{I}$ is a sheaf of Poisson ideals, i.e., $\{\mathcal{I}, \mathcal{O}_{\mathsf{X}}\} \subset \mathcal{I}$.*
3. *$\mathcal{I}$ is invariant under the action of any Hamiltonian vector field.*
4. *Y is an invariant subspace for the Lie algebroid $\Omega^1_{\mathsf{X},\sigma}$ in the sense of Definition 2.2.6; that is, $\sigma^\sharp(\alpha)$ preserves $\mathcal{I}$ for all $\alpha \in \Omega^1_{\mathsf{X}}$.*

Proof. If $i : \mathsf{Y} \to \mathsf{X}$ is the inclusion, then the map $i^* : i^{-1}\mathcal{O}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{Y}}$ is a surjective, bracket preserving morphism with kernel $\mathcal{I}$ and so the equivalence of 1 and 2 is immediate. The equivalence of 2 and 3 follows from the relationship between Hamiltonian vector fields and the bracket. The equivalence of 3 and 4 is obtained by noting that the Lie algebroid $\Omega^1_{\mathsf{X},\sigma}$ is generated by exact one-forms and hence the image of the anchor map is generated by Hamiltonian vector fields. □

A function $f \in \mathcal{O}_{\mathsf{X}}$ is ***Casimir*** if its Hamiltonian vector field is zero. In other words, f is Casimir if $\{f, g\} = 0$ for all $g \in \mathcal{O}_{\mathsf{X}}$. In this case, the ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ generated by f is a Poisson ideal: every element of $\mathcal{I}$ has the form hf for some $h \in \mathcal{O}_{\mathsf{X}}$, and

$$\{g, hf\} = \{g, h\}f + h\{g, f\} = \{g, h\}f \in \mathcal{I}$$

for all $g \in \mathcal{O}_{\mathsf{X}}$. Thus $\mathcal{I}$ is closed under bracketing with arbitrary holomorphic functions.

Example 4.3.2. Let us describe the (reduced) closed Poisson subspaces in The Example, where

$$\sigma = x\partial_y \wedge \partial_z + 2y\partial_x \wedge \partial_y - 2z\partial_x \wedge \partial_z$$

on $\mathbb{C}^3$. The function $f = x^2 + 4yz - c$ is a Casimir for all $c \in \mathbb{C}$. The zero set of $f - c$ is exactly the c-level set of f and is reduced for all $c \in \mathbb{C}$. The level sets for $c \neq 0$ are smooth symplectic submanifolds, while the level set for $c = 0$ is the singular cone $\mathsf{Y} \subset \mathbb{C}^3$. Thus this singular space is a Poisson subspace. The only other reduced closed Poisson subspace is the origin $\{0\}$, which is the singular locus of Y.

We claim that the induced Poisson structure $\sigma|_{\mathsf{Y}} \in \mathscr{X}^2_{\mathsf{Y}}$ on Y cannot be represented by any bivector field on Y. Indeed, one can verify that the Hamiltonian vector fields $X_x = \sigma^\sharp(dx), X_y = \sigma^\sharp(dy)$ and $X_z = \sigma^\sharp(dz)$ generate $\mathcal{T}_{\mathsf{Y}}$, and hence $X_x \wedge X_Y$, $X_x \wedge X_z$ and $X_y \wedge X_z$ generate $\wedge^2\mathcal{T}_{\mathsf{Y}}$. But the pairing of any of these bivector fields with a two-form $\mu \in \Omega^2_{\mathsf{Y}}$ results in a function that vanishes to order at least two at the origin, while the pairing of σ with $dx \wedge dy$, $dx \wedge dz$ and $dy \wedge dz$ vanishes only to first order. Hence $\sigma|_{\mathsf{Y}}$ cannot be described as a bivector field on Y, even though it is the restriction of a bivector field on $\mathbb{C}^3$. □

It will be very useful for us to consider a special class of Poisson subspaces that are invariant under all local symmetries:

Definition 4.3.3. Let (X, σ) be a Poisson analytic space. A closed subspace $\mathsf{Y} \subset \mathsf{X}$ is a ***strong Poisson subspace*** if it is invariant under all of the local infinitesimal symmetries of σ. In other words, we require that the ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$ is preserved by $\mathscr{L}_Z$ for every Poisson vector field $Z \in \mathcal{P}ois(\sigma)$.

Notice an important distinction: for a subspace to be strong, we require it to be preserved by Poisson vector fields defined on arbitrary open sets in X. It is frequently the case that such a vector field is not the restriction of any global vector field on X. Thus, asking Y to be invariant under globally defined vector fields is a strictly weaker condition in general.

As with the case of invariant subspaces for Lie algebroids, we may directly apply Theorem 2.1.10 with $\mathcal{F} = \mathcal{H}am(\sigma)$ or $\mathcal{F} = \mathcal{P}ois(\sigma)$ to conclude that the irreducible components, the singular locus and the reduced subspace of a given Poisson subspace (resp., strong Poisson subspace) are themselves Poisson (resp., strong Poisson), and similarly for unions and intersections.

Since Hamiltonian vector fields are always Poisson, it follows from Proposition 4.3.1 that strong Poisson subspaces are Poisson subspaces. However, the converse fails, in general:

Example 4.3.4. Let $\mathsf{X} = \mathbb{C}^3$ with coordinates x, y and z, and consider the Poisson structure $\sigma = \partial_x \wedge \partial_y$. The function z is a Casimir function. Its level sets are the Poisson subspaces defined by the symplectic leaves, which give a parallel family of planes. The vector field

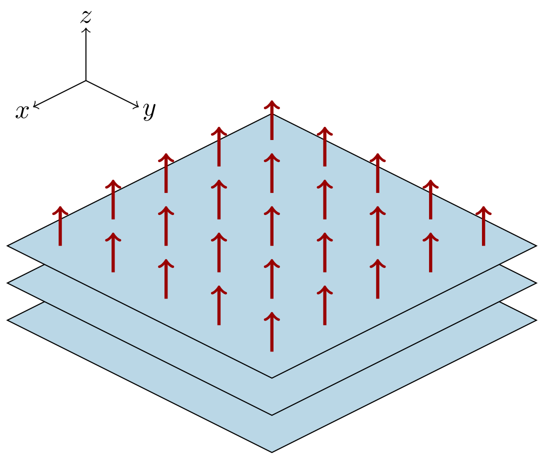


Figure 4.1: The Poisson vector field ∂_z is transverse to the symplectic leaves of $\partial_x \wedge \partial_y$, and hence they are not strong Poisson subspaces.

$Z = \partial_z$ is clearly Poisson but it does not preserve any of these subspaces, and hence none of them are strong. This situation is illustrated in Figure 4.1. □

More generally, strong Poisson subspaces are concentrated where the rank of σ drops:

Proposition 4.3.5. *Let X be a connected manifold, and let σ be a Poisson structure on X which has constant rank, i.e., the rank of the anchor map $\sigma^\sharp : \Omega^1_{\mathsf{X}} \to \mathscr{X}^1_{\mathsf{X}}$ is the same at every point of X. Then the only strong Poisson subspace of X is X itself.*

Proof. By Weinstein's Splitting Theorem [142, Theorem 2.1], we may pick local coordinates $x_1, \ldots, x_k$, $p_1, \ldots, p_k$ and $y_1, \ldots, y_r$ near any point in X so that the Poisson structure has the Darboux form

$$\sigma = \partial_{x_1} \wedge \partial_{p_2} + \cdots + \partial_{x_k} \wedge \partial_{p_k}.$$

Then all of the coordinate vector fields are Poisson and hence the sheaf $\mathcal{P}ois(\sigma)$ of Poisson vector fields spans $\mathscr{X}^1_{\mathsf{X}}$. Since X is a connected manifold, the only subspace of X that is invariant under all of $\mathscr{X}^1_{\mathsf{X}}$ is X itself. □

Remark 4.3.6. The local normal form used in the proof is obtained by analytic means. For this reason, the proof is not applicable in the algebraic category and it is conceivable that the proposition fails if we replace the sheaf of analytic Poisson vector fields with the sheaf of algebraic Poisson vector fields in the Zariski topology. It would be interesting to study this issue in more detail. □

We will be particularly interested in studying the degeneracy loci of Poisson structures, where the rank drops. When we defined the degeneracy loci of a Lie algebroid $\mathcal{A}$, we used the exterior power $\wedge^{k+1} a : \wedge^{k+} \mathcal{A} \to \wedge^{k+1} \mathcal{T}_{\mathsf{X}}$ to define the kth degeneracy locus. However, if we try to use this definition with the Poisson algebroid $\mathcal{A} = \Omega^1_{\mathsf{X},\sigma}$ we find that it is a poor choice:

Example 4.3.7. Consider the Poisson structure $\sigma = x\partial_x \wedge \partial_y$ on $\mathbb{C}^2$. Clearly, this Poisson structure vanishes transversally and its zero locus should be defined by the equation $x = 0$. However, if we write the anchor map $\sigma^\sharp$ as a matrix in the coordinate basis, we have

$$\sigma^\sharp = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$$

and the determinant is x^2, which does not give a reduced equation for the zero locus. □

The problem is that $\sigma^\sharp$ is skew-symmetric, and hence it is not a "generic" map $\Omega^1_{\mathsf{X}} \to \mathscr{X}^1_{\mathsf{X}}$. In particular, if we write it locally as a matrix of functions, the determinants of the minors never give reduced equations for the degeneracy loci. The reason that the determinant of a skew-symmetric matrix is the square of its Pfaffian. It is therefore much better for us to work with the Pfaffians, which are modelled by the exterior powers

$$\sigma^{k+1} = \underbrace{\sigma \wedge \cdots \wedge \sigma}_{(k+1) \text{ times}}.$$

This multiderivation gives a natural map $\Omega^{2k+2}_{\mathsf{X}} \to \mathcal{O}_{\mathsf{X}}$ and the $2k^{th}$ ***degeneracy locus of*** σ is the closed analytic subspace $\mathsf{Dgn}_{2k}(\sigma)$ defined by this ideal. The geometry of these subspaces will be the focus of Chapter 6.

Polishchuk [117] proved that the degeneracy loci are Poisson subspaces and noted that they are preserved by all Poisson vector fields. We thus have

Proposition 4.3.8. *Let* (X, σ) *be a Poisson analytic space and* $k \geq 0$*. Then the degeneracy locus* $\mathsf{Dgn}_{2k}(\sigma)$ *is a strong Poisson subspace of* X*.*

Proof. The proof is similar to Proposition 2.2.9 for Lie algebroids. We simply use the fact that if $\mu \in \Omega^{2k+2}_{\mathsf{X}}$, then

$$\mathscr{L}_Z(\sigma^{k+1}(\mu)) = (\mathscr{L}_Z\sigma^{k+1})(\mu) + \sigma^{k+1}(\mathscr{L}_Z\mu).$$

If Z is any Poisson vector field, then $\mathscr{L}_Z\sigma^{k+1} = 0$, and it follows that the image of σ^{k+1} is closed under the action of Z, as required. □

Example 4.3.9. In The Example, with the Poisson structure

$$x\partial_y \wedge \partial_z + 2y\partial_x \wedge \partial_y - 2z\partial_x \wedge \partial_z$$

on $\mathbb{C}^3$, the zero locus $\mathsf{Dgn}_0(\sigma)$ is the locus where the bracket $\{f, g\}$ vanishes for all $f, g \in \mathcal{O}_{\mathbb{C}^3}$. The corresponding ideal is generated by the elementary brackets $\{x, y\} = 2y$, $\{x, z\} = -2z$

and $\{y, z\} = x$ which defines the origin $\{0\} \subset \mathbb{C}^3$ as an analytic subspace with its reduced structure. Hence, the origin is a strong Poisson subspace. $\square$

In what follows, it will be useful for us to consider the singular locus of the space X and the degeneracy loci together, since they are the pieces of X where the nontrivial behaviour of the Poisson structure occurs. We therefore make the following

Definition 4.3.10. Let (X, σ) be a Poisson analytic space and suppose that the maximum rank of σ on X is $2k$. The ***Poisson singular locus of*** (X, σ) is the strong Poisson subspace

$$\mathsf{Sing}(\mathsf{X}, \sigma) = \mathsf{X}_{sing} \cup \mathsf{Dgn}_{2k-2}(\sigma)\,,$$

which is the union of the singular locus of X as an analytic space, and the locus where the rank of σ drops. We say that (X, σ) is ***regular*** if X is smooth and σ has constant rank, i.e., $\mathsf{Sing}(\mathsf{X}, \sigma) = \varnothing$.

4.4 Poisson hypersurfaces and log symplectic structures

Let (X, σ) be a complex manifold (or more generally, a normal analytic space) a ***Poisson hypersurface*** is a codimension one analytic space $\mathsf{D} \subset \mathsf{X}$ that is a Poisson subspace. A ***Poisson divisor*** [117] is an element of the free abelian group generated by the reduced and irreducible Poisson hypersurfaces.

Recall from Section 2.3 that if $\mathsf{D} \subset \mathsf{X}$ is a codimension one analytic space, then the sheaf of logarithmic vector fields for D is the subsheaf $\mathscr{X}^1_{\mathsf{X}}(-\log \mathsf{D}) \subset \mathscr{X}^1_{\mathsf{X}}$ consisting of vector fields that preserve D, and that the corresponding de Rham complex is the complex $\Omega^\bullet_{\mathsf{X}}(\log \mathsf{D})$ of forms with logarithmic singularities along D. We may also speak of logarithmic multiderivations as being dual to logarithmic forms: $\mathscr{X}^\bullet_{\mathsf{X}}(-\log \mathsf{D}) = (\Omega^\bullet_{\mathsf{X}}(\log \mathsf{D}))^\vee$. When D is a free divisor, so that the logarithmic vector fields form a vector bundle, we have $\mathscr{X}^\bullet_{\mathsf{X}}(-\log \mathsf{D}) = \wedge^\bullet \mathscr{X}^1_{\mathsf{X}}(-\log \mathsf{D})$ but this isomorphism may fail if D is not free.

Proposition 4.4.1. *Suppose that $\mathsf{D} \subset \mathsf{X}$ is a Poisson hypersurface. Then σ lies in the subsheaf $\mathscr{X}^2_{\mathsf{X}}(-\log \mathsf{D}) \subset \mathscr{X}^\bullet_{\mathsf{X}}$.*

Proof. Since $\mathscr{X}^\bullet_{\mathsf{X}}(-\log \mathsf{D}) = \mathscr{X}^\bullet_{\mathsf{X}}(-\log \mathsf{D}_{red})$, we may assume that D is reduced. Furthermore, since $\mathscr{X}^2_{\mathsf{X}}(-\log \mathsf{D})$ is reflexive and the singular locus of D has codimension at least two in X, it is enough to verify the claim on the open set where D is smooth.

Let $f \in \mathcal{O}_{\mathsf{X}}$ be a local equation for D in the neighbourhood of a smooth point $x \in \mathsf{D}$. Near x, the sheaf $\Omega^2_{\mathsf{X}}(\log \mathsf{D})$ is generated as an $\mathcal{O}_{\mathsf{X}}$-module by $\frac{df}{f} \wedge \Omega^1_{\mathsf{X}}$ and Ω^2_{X}. It is therefore sufficient to check that the function $\sigma(df/f, dg) = f^{-1}\{f, g\}$ is holomorphic for all $g \in \mathcal{O}_{\mathsf{X}}$. But D is a Poisson subspace, and hence the ideal generated by f is Poisson. It follows that $\{f, g\} = hf$ for some $h \in \mathcal{O}_{\mathsf{X}}$, so that $f^{-1}\{f, g\}$ is holomorphic. $\square$

It follows that when D is a Poisson hypersurface, the anchor map $\sigma^\sharp$ induces a skew-symmetric Lie algebroid morphism $\Omega^1_{\mathsf{X}}(\log \mathsf{D}) \to \mathscr{X}^1_{\mathsf{X}}(-\log \mathsf{D})$, which we also denote by $\sigma^\sharp$. As a result, the triple $(\Omega^1_{\mathsf{X}}(\log \mathsf{D}), \mathscr{X}^1_{\mathsf{X}}(-\log \mathsf{D}), \sigma)$ is an example of a ***triangular Lie bialgebroid*** in the sense of Mackenzie and Xu [104]. Notice that we do not require these Lie algebroids to be vector bundles, but in order to make sense of the duality condition in the definition of a bialgebroid, we do need the sheaves to be reflexive.

If X is a complex manifold (or more generally, a normal analytic space) of dimension $2n$ with $n > 0$, we can consider Poisson structures for which the anchor map $\sigma^\sharp : \Omega^1_{\mathsf{X}} \to \mathscr{X}^1_{\mathsf{X}}$ is generically an isomorphism. In this case, the degeneracy locus $\mathsf{Dgn}_{2n-2}(\sigma)$ is a Poisson hypersurface, defined as the zero locus of the section $\sigma^n \in \mathsf{H}^0\big(\mathsf{X}, \omega_{\mathsf{X}}^{-1}\big)$. Here, as always, $\omega_{\mathsf{X}}^{-1} = \mathscr{X}^{2n}_{\mathsf{X}} = \det \mathscr{X}^1_{\mathsf{X}}$ is the anticanonical line bundle. In this case, the complement $\mathsf{U} = \mathsf{X} \setminus \mathsf{Dgn}_{2n-2}(\sigma)$ is a holomorphic symplectic manifold. For this reason, we say that σ is ***generically symplectic***.

Proposition 4.4.2. *Suppose that (X, σ) is a generically symplectic Poisson manifold, and let D be a Poisson hypersurface. Then the logarithmic anchor map*

$$\sigma^\sharp : \Omega^1_{\mathsf{X}}(\log \mathsf{D}) \to \mathscr{X}^1_{\mathsf{X}}(-\log \mathsf{D})$$

is an isomorphism if and only if $\mathsf{D}_{red} = \mathsf{Dgn}_{2n-2}(\sigma)$, where $\mathsf{D}_{red} \subset \mathsf{D}$ is the reduced subspace.

Proof. Since D is a Poison subspace, then the Poisson structure must have rank less than $2n-1$ on D because of the dimension, and hence we always have $\mathsf{D}_{red} \subset \mathsf{Dgn}_{2n-2}(\sigma)$. It remains to prove that the reverse inclusion holds only when $\mathsf{D}_{red} = \mathsf{Dgn}_{2n-2}(\sigma)$.

Let $\mathcal{A} = \mathscr{X}^1_{\mathsf{X}}(-\log \mathsf{D})$. Since $\mathcal{A}$ and $\mathcal{A}^\vee = \Omega^1_{\mathsf{X}}(\log \mathsf{D})$ are reflexive, we only need to consider the open set where they are locally free (see Section 2.1.2). On this open set the anchor map $\sigma^\sharp : \mathcal{A}^\vee \to \mathcal{A}$ is an isomorphism if and only if $\sigma^n \in \det \mathcal{A}$ is nonvanishing. But $\mathcal{A}$ is also equal to $\mathscr{X}^1_{\mathsf{X}}(-\log \mathsf{D}_{red})$. Therefore, by Saito's criterion (Theorem 2.3.3), we have $\det \mathcal{A} = \omega_{\mathsf{X}}^{-1}(-\mathsf{D}_{red})$. Thus, the locus where $\sigma^\sharp$ fails to be an isomorphism is precisely the locus where the section $\sigma^n \in \omega_{\mathsf{X}}^{-1}(-\mathsf{D}_{red})$ vanishes. But this locus is precisely the divisor $\mathsf{Dgn}_{2n-2}(\sigma) - \mathsf{D}_{red}$, and hence σ is an isomorphism if and only if $\mathsf{Dgn}_{2n-2}(\sigma) = \mathsf{D}_{red}$. □

In the case when $\mathsf{D} = \mathsf{Dgn}_{2n-2}(\sigma)$ is reduced, the inverse of the isomorphism $\sigma^\sharp : \Omega^1_{\mathsf{X}}(\log \mathsf{D}) \to \mathscr{X}^1_{\mathsf{X}}(-\log \mathsf{D})$ defines a two-form $\omega \in \mathcal{H}om\big(\mathscr{X}^2_{\mathsf{X}}(-\log \mathsf{D}), \mathcal{O}_{\mathsf{X}}\big)$. When D is a free divisor, the latter sheaf is simply $\Omega^2_{\mathsf{X}}(\log \mathsf{D})$. For this reason, we say that a Poisson manifold (X, σ) is ***log symplectic*** if σ is generically symplectic and the degeneracy divisor $\mathsf{Dgn}_{2n-2}(\sigma)$ is reduced. (The same definition applies more generally when X is a normal analytic space.)

Log symplectic Poisson structures were introduced by R. Goto in [64], where they are used to construct Rozansky–Witten invariants associated with the moduli space of reduced

$\mathsf{SU}(2)$-monopoles of charge k. One finds some discussion of the cohomology and prequantization of log symplectic structures in [45, 46]. Recently, they have also been studied in the smooth category [33, 71, 109], where they are also referred to as b-Poisson [74, 127] structures, and give higher-dimensional analogues of Radko's topologically stable Poisson structures on surfaces [120]. In the smooth category, it is assumed that the hypersurface $\mathsf{D} = \mathsf{Dgn}_{2n-2}(\sigma)$ is a manifold. We emphasize that this condition is not enforced in this thesis. Indeed, one of our main results in Chapter 6, namely Theorem 6.6.1, will be concerned with the structure of the singular locus of D.

4.5 Lie algebroids and symplectic leaves

Let (X, σ) be a connected complex Poisson manifold. Associated to σ are a number of Lie algebroids: among these are the Poisson Lie algebroid $\Omega^1_{\mathsf{X},\sigma}$ and the image $\mathcal{I}mg\left(\sigma^\sharp\right) \subset \mathscr{X}^1_{\mathsf{X}}$ of the anchor map, which is the tangent sheaf to the symplectic leaves. One of the troublesome features of the latter algebroid is that is not a vector bundle. In fact, it will often not even be reflexive. However, its double dual $\mathcal{I}mg\left(\sigma^\sharp\right)^{\vee\vee} \subset \mathscr{X}^1_{\mathsf{X}}$ is reflexive and has the following description: suppose that $\dim \mathsf{X} = n$ and that the rank of σ is generically equal to $2k < n$. Consider the map

$$\sigma^k \wedge : \mathscr{X}^1_{\mathsf{X}} \to \mathscr{X}^{2k+1}_{\mathsf{X}}$$

defined by taking the wedge product with σ^k. It is easy to see that the kernel of this map is an involutive subsheaf $\mathcal{F} \subset \mathscr{X}^1_{\mathsf{X}}$, hence a Lie algebroid. The sections of $\mathcal{F}$ are precisely those vector fields that are tangent to all of the top-dimensional symplectic leaves, as we saw in The Example for the case when X is the dual of $\mathfrak{sl}(2, \mathbb{C})$.

Notice that, since $\mathcal{F}$ is presented as the kernel of a vector bundle map, it must be reflexive. Moreover, it coincides with the image of $\sigma^\sharp$ away from the degeneracy locus $\mathsf{Dgn}_{2k-2}(\sigma)$. Therefore, if the codimension of $\mathsf{Dgn}_{2k-2}(\sigma)$ is at least two, $\mathcal{F}$ must be the double dual $\mathcal{F}^{\vee\vee}$.

Since $\mathcal{F}$ serves as a replacement for the tangent bundle to the top-dimensional leaves, it inherits a symplectic structure:

Proposition 4.5.1. *If the codimension of $\mathsf{Dgn}_{2k-2}(\sigma)$ is at least two, then $\mathcal{F}$ is a symplectic Lie algebroid, in the sense that it carries a non-degenerate closed two-form $\beta \in \mathsf{H}^0\left(\mathsf{X}, \Omega^2_{\mathcal{F}}\right)$, where $\Omega^2_{\mathcal{F}}$ is the dual of $\wedge^2 \mathcal{F}$.*

Proof. Away from $\mathsf{Dgn}_{2k-2}(\sigma)$, $\mathcal{F}$ is the tangent bundle to the $2k$-dimensional leaves, and hence it carries a symplectic form. Since $\Omega^2_{\mathcal{F}}$ is reflexive and $\mathsf{Dgn}_{2k-2}(\sigma)$ has codimension at least two, this form extends to a nondegenerate closed form on all of X. $\square$

As was the case with Poisson structures tangent to a hypersurface, the map $\sigma^\sharp : \mathcal{F}^\vee \to \mathcal{F}$ makes $(\mathcal{F}, \mathcal{F}^\vee, \sigma)$ into a reflexive triangular Lie bialgebroid.

Notice that the zero locus of the morphism $\sigma^k\wedge$ is precisely the degeneracy locus $\mathsf{Y} = \mathsf{Dgn}_{2k-2}(\sigma)$ *as an analytic space.* In the case when $\dim \mathsf{X} = 2k+1$, we therefore have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathscr{X}^1_{\mathsf{X}} \xrightarrow{\sigma^k\wedge} \omega_{\mathsf{X}}^{-1} \longrightarrow \omega_{\mathsf{X}}^{-1}|_{\mathsf{Dgn}_{2k-2}(\sigma)} \longrightarrow 0 \tag{4.1}$$

As a result, there is a close relationship between the structure of $\mathcal{F}$ and that of $\mathsf{Dgn}_{2k-2}(\sigma)$ (see also [40, 61]):

Proposition 4.5.2. *Suppose that (X,σ) is a connected Poisson manifold of dimension $2k+1$ and that the rank of σ is generically equal to $2k$. If the associated Lie algebroid $\mathcal{F} = \mathcal{K}er(\sigma^k\wedge) \subset \mathscr{X}^1_{\mathsf{X}}$ is a vector bundle, then every component of $\mathsf{Dgn}_{2k-2}(\sigma)$ has codimension at most two in X. Conversely, if every component of $\mathsf{Dgn}_{2k-2}(\sigma)$ is smooth of codimension two (or, more generally, if $\mathsf{Dgn}_{2k-2}(\sigma)$ is Cohen–Macaulay of codimension two), then $\mathcal{F}$ is a vector bundle.*

Proof. If $\mathcal{F}$ is a vector bundle, the exact sequence (4.1) shows that the projective dimension of $\mathcal{O}_{\mathsf{Dgn}_{2k-2}(\sigma)}$ is at most two. But the projective dimension of $\mathcal{O}_{\mathsf{Dgn}_{2k-2}(\sigma)}$ is an upper bound for the codimension of every component.

Conversely, if $\mathsf{Dgn}_{2k-2}(\sigma)$ is Cohen–Macaulay of codimension two, then the projective dimension of $\mathcal{O}_{\mathsf{Dgn}_{2k-2}(\sigma)}$ is equal to two, and hence $\mathcal{F}$ is a vector bundle. ☐

Example 4.5.3. Consider the Poisson structure on $\mathbb{C}^3$ defined by

$$\sigma = (x\partial_x + y\partial_y) \wedge \partial_z.$$

The zero locus of σ is the z-axis, defined by the equations $x = y = 0$. This locus is smooth of codimension two and hence $\mathcal{F}$ is a vector bundle. Indeed, the vector fields $x\partial_x + y\partial_y$ and ∂_z give a basis for $\mathcal{F}$: every vector field tangent to the two-dimension symplectic leaves can be written uniquely as a linear combination of these two. Notice, though, that ∂_z does not lie in the image of $\sigma^\sharp$, and hence $\mathcal{F}$ cannot be generated by Hamiltonian vector fields. The symplectic form on $\mathcal{F}$ can be represented by the meromorphic two-form $(\frac{dx}{x} + \frac{dy}{y}) \wedge dz$ ☐

Example 4.5.4. We return to The Example of the linear Poisson structure

$$\sigma = x\partial_y \wedge \partial_z + 2y\partial_x \wedge \partial_y - 2z\partial_x \wedge \partial_z$$

on $\mathbb{C}^3$. This Poisson structure vanishes only at the origin, which has codimension three. Correspondingly, $\mathcal{F}$ fails to be locally free there. Instead, it has projective dimension equal to one: the sequence

$$0 \longrightarrow \Omega^3_{\mathbb{C}^3} \xrightarrow{\iota_\sigma} \Omega^1_{\mathbb{C}^3} \xrightarrow{\sigma^\sharp} \mathcal{F} \longrightarrow 0$$

is exact, giving a length-one resolution of $\mathcal{F}$ by vector bundles. In this case, $\mathcal{F}$ is generated by Hamiltonian vector fields. $\square$

Chapter 5

Geometry of Poisson modules

In this section, we recall the notion of a Poisson module, which serves as the replacement for a vector bundle with flat connection in Poisson geometry. We then develop several aspects of their geometry that will be useful in our study of the degeneracy loci of Poisson structures.

5.1 Poisson modules

Let (X, σ) be a complex Poisson manifold or analytic space, and let $\mathcal{E}$ be a holomorphic vector bundle, or more generally a sheaf of $\mathcal{O}_{\mathsf{X}}$-modules. A ***Poisson connection*** on $\mathcal{E}$ is a $\mathbb{C}$-linear morphism of sheaves $\nabla : \mathcal{E} \to \mathscr{X}^1_{\mathsf{X}}(\mathcal{E})$ satisfying the Leibniz rule

$$\nabla(fs) = -\sigma^\sharp(df) \otimes s + f\nabla s$$

for all $f \in \mathcal{O}_{\mathsf{X}}$ and $s \in \mathcal{E}$. Recall that $\mathscr{X}^1_{\mathsf{X}}(\mathcal{E}) = \mathcal{H}om\big(\Omega^1_{\mathsf{X}}, \mathcal{E}\big)$, so if X is a manifold or $\mathcal{E}$ is a vector bundle, we have $\mathscr{X}^1_{\mathsf{X}}(\mathcal{E}) \cong \mathscr{X}^1_{\mathsf{X}} \otimes \mathcal{E}$. Thus, a Poisson connection is simply a connection for the Poisson Lie algebroid $\Omega^1_{\mathsf{X},\sigma}$ (see Section 2.4). When this connection is flat, we say that $\mathcal{E}$ is a ***Poisson module***.

Using a Poisson connection, we may differentiate a section of $\mathcal{E}$ along a one-form: if $\alpha \in \Omega^1_{\mathsf{X}}$, we set

$$\nabla_\alpha s = (\nabla s)(\alpha).$$

If $f \in \mathcal{O}_{\mathsf{X}}$ is a function, we write

$$\{f, s\} = \nabla_{df} s.$$

This operation satisfies the Leibniz rules

$$\{f, gs\} = \{f, g\}s + g\{f, s\}$$
$$\{fg, s\} = f\{g, s\} + g\{f, s\}$$

for $f, g \in \mathcal{O}_{\mathsf{X}}$ and $s \in \mathcal{E}$. The bracket $\{\cdot,\cdot\} : \mathcal{O}_{\mathsf{X}} \times \mathcal{E} \to \mathcal{E}$ completely determines the connection ∇ because Ω^1_{X} is generated by exact forms. Flatness of the connection is equivalent to the identity

$$\{\{f, g\}, s\} = \{f, \{g, s\}\} - \{g, \{f, s\}\}$$

for all $f, g \in \mathcal{O}_{\mathsf{X}}$ and $s \in \mathcal{E}$. Thus a Poisson module can alternatively be viewed as a sheaf that carries an action of the Poisson algebra of functions.

Remark 5.1.1. Let $\mathcal{J}^1\mathcal{E}$ be the sheaf of one-jets of sections of $\mathcal{E}$. Then a Poisson connection is an $\mathcal{O}_{\mathsf{X}}$-linear map $\nabla : \mathcal{J}^1\mathcal{E} \to \mathscr{X}^1_{\mathsf{X}}(\mathcal{E})$ making the following diagram commute:

$$\begin{tikzcd}
\Omega^1_{\mathsf{X}} \otimes \mathcal{E} \arrow[r] \arrow[d, "-\sigma^\sharp \otimes \mathrm{id}_{\mathcal{E}}"'] & \mathcal{J}^1\mathcal{E} \arrow[d, "\nabla"] \\
\mathcal{T}_{\mathsf{X}} \otimes \mathcal{E} \arrow[r] & \mathscr{X}^1_{\mathsf{X}}(\mathcal{E})
\end{tikzcd}$$

In other words, a Poisson connection is a first-order differential operator on $\mathcal{E}$ whose symbol is $-\sigma^\sharp \otimes \mathrm{id}_{\mathcal{E}}$. □

Recall from Section 2.4 that every locally free Lie algebroid $\mathcal{A}$ on a manifold X comes equipped with two natural modules: the trivial bundle $\mathcal{O}_{\mathsf{X}}$, and the canonical module $\omega_{\mathcal{A}} = \det \mathcal{A} \otimes \omega_{\mathsf{X}}$. For Poisson structures, these modules have the following descriptions:

The natural Poisson module structure on $\mathcal{O}_{\mathsf{X}}$ is given by the composite morphism

$$-\sigma^\sharp d : \mathcal{O}_{\mathsf{X}} \to \mathscr{X}^1_{\mathsf{X}} = \mathscr{X}^1_{\mathsf{X}}(\mathcal{O}_{\mathsf{X}})$$

which takes a function to minus its Hamiltonian vector field. Meanwhile, for $\mathcal{A} = \Omega^1_{\mathsf{X},\sigma}$ we have $\det \mathcal{A} = \omega_{\mathsf{X}}$, the canonical bundle of X. Hence $\omega_{\mathcal{A}} = \omega^2_{\mathsf{X}}$. It follows that ω_{X} itself is a Poisson module, and the module structure is given by the formula

$$\nabla_\alpha \mu = -\alpha \wedge d\iota_\sigma \mu, \tag{5.1}$$

for $\alpha \in \Omega^1_{\mathsf{X}}$ and $\mu \in \omega_{\mathsf{X}}$, where ι_σ denotes interior contraction by σ. For a function $f \in \mathcal{O}_{\mathsf{X}}$, we have the identity

$$\{f, \mu\} = -\mathscr{L}_{\sigma^\sharp(df)}\mu.$$

we will refer to this Poisson module as the ***canonical Poisson module*** [53, 117]. As we shall see in Chapter 6, the geometry of the canonical module is intimately connected with the structure of the degeneracy loci of σ.

If $\mathcal{L}$ is a line bundle (an invertible sheaf) equipped with a Poisson connection ∇, and if $s \in \mathcal{L}$ is a local trivialization, we obtain a unique vector field $Z \in \mathscr{X}^1_{\mathsf{X}}$ such that

$$\nabla s = Z \otimes s.$$

This vector field is called the ***connection vector field*** for the trivialization s. Then ∇ is flat if and only if Z is a Poisson vector field.

Definition 5.1.2. Let (X, σ) be a complex Poisson manifold. The connection vector field for a local trivialization $\phi \in \omega_{\mathsf{X}}$ of the canonical module is called the ***modular vector field*** associated to ϕ.

5.2 Pushforward of Poisson modules

One useful property of Poisson modules is that they can be pushed forward along Poisson maps. We now describe this procedure and use it to produce some nontrivial Poisson modules.

Suppose that $\phi : \mathsf{X} \to \mathsf{Y}$ is a holomorphic map and that $\mathcal{E}$ is a vector bundle or coherent sheaf on X. We may then form the pushforward $\phi_*\mathcal{E}$, which is a sheaf on Y. If $\mathsf{U} \subset \mathsf{Y}$ is an open set, then the sections of $\phi_*\mathcal{E}$ over U are given by the sections of $\mathcal{E}$ on $\phi^{-1}(\mathsf{U}) \subset \mathsf{X}$. Since we have a pull back map $\mathcal{O}_{\mathsf{Y}}(\mathsf{U}) \to \mathcal{O}_{\mathsf{X}}(\phi^{-1}(\mathsf{U}))$, the sheaf $\phi_*\mathcal{E}$ inherits in a natural way an action of $\mathcal{O}_{\mathsf{Y}}$, making it into an $\mathcal{O}_{\mathsf{Y}}$-module. If $\phi : \mathsf{X} \to \mathsf{Y}$ is a k-to-one branched covering of complex manifolds of equal dimension, then the pushforward of a rank-r vector bundle on X is a rank-kr vector bundle on Y.

Now suppose that X and Y are equipped with Poisson structures, and that $\mathcal{E}$ is a Poisson module on X. If ϕ is a Poisson map, then the pullback of functions is compatible with Poisson brackets. The composition $\mathcal{O}_{\mathsf{Y}} \times \phi_*\mathcal{E} \to \phi_*\mathcal{O}_{\mathsf{X}} \times \phi_*\mathcal{E}$ with the action of $\mathcal{O}_{\mathsf{X}}$ on $\mathcal{E}$ makes $\mathcal{E}$ into a Poisson module.

Example 5.2.1. Consider $\mathsf{X} = \mathbb{C}^2$ with coordinates x, y. The Poisson structure

$$\{x, y\} = x - y$$

is invariant under the $\mathbb{Z}_2$-action $(x, y) \mapsto (y, x)$, and hence the quotient $\mathsf{Y} = \mathbb{C}^2/\mathbb{Z}_2$ inherits a Poisson structure. The map $\phi : (x, y) \mapsto (x + y, xy)$ gives an isomorphism $\mathsf{Y} \cong \mathbb{C}^2$. The quotient map $\mathsf{X} \to \mathsf{Y}$ is a two-to-one covering branched over the diagonal, where $x = y$ and the map is one-to-one.

Using coordinates $u = x + y, v = xy$ on the quotient, the Poisson structure is given by

$$\{u, v\} = u^2 - 4v.$$

Let us describe the pushforward of the canonical Poisson module ω_{X} to Y. Let $\mu = dx \wedge dy$ be a non-vanishing section of ω_{X}. Then the $\mathcal{O}_{\mathsf{Y}}$-module $\mathcal{E} = \phi_*\omega_{\mathsf{X}}$ is a rank-two vector bundle, with basis given by the sections $s_1 = \phi_*\mu$ and $s_2 = \phi_*((x-y)\mu)$. Using (5.1), we readily compute that $\{x,\mu\} = \{y,\mu\} = \mu$. To determine the action of v on s_1, we write

$$\begin{aligned}\{v, s_1\} &= \phi_*\{\phi^* v, \mu\} \\ &= \phi_*\{xy, \mu\} \\ &= \phi_* \left(x\{y,\mu\} + y\{x,\mu\}\right) \\ &= \phi_*((x+y)\mu) \\ &= \phi_*((\phi^* u)\cdot\mu) \\ &= u s_1.\end{aligned}$$

Proceeding in the same manner to compute the rest of the action, we find that the Poisson connection is given by

$$\nabla = -\sigma^\sharp \circ d + \partial_u \otimes \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} + \partial_v \otimes \begin{pmatrix} u & 0 \\ 0 & u-1 \end{pmatrix}$$

in the basis t_1, t_2 for $\mathcal{E}$.

5.3 Restriction to subspaces: Higgs fields and adaptedness

When the Poisson structure on X is zero, the Leibniz rule in the definition of a Poisson module reduces to the requirement of $\mathcal{O}_{\mathsf{X}}$-linearity. Thus, a Poisson module structure on a vector bundle $\mathcal{E}$ is determined by a global section

$$\Phi \in \mathsf{H}^0\big(\mathsf{X}, \mathscr{X}^1_{\mathsf{X}} \otimes \mathcal{E}nd(\mathcal{E})\big)$$

satisfying the equation $\Phi \wedge \Phi = 0 \in \mathscr{X}^2_{\mathsf{X}} \otimes \mathcal{E}nd(\mathcal{E})$. Such a section is called a ***Higgs field*** with values in the tangent bundle, and the pair $(\mathcal{E}, \Phi)$ is a ***co-Higgs bundle***. (In contrast, a Higgs bundle has a Higgs field takes values in the cotangent bundle.) See [121] for a detailed study of co-Higgs bundles on $\mathbb{P}^1$ and $\mathbb{P}^2$.

When the Poisson structure is nonzero, a Poisson module structure is a genuine differential operator, and is therefore not given by a Higgs field. Nevertheless, as we now explain, there are still some similar tensors present and they measure some important properties of the Poisson module.

The first Higgs field arises when we attempt to restrict a Poisson module to a Poisson subspace. If the subspace is open, there is no problem, but if the subspace is closed, we encounter some difficulty. Indeed, if $\mathsf{X} \subset \mathsf{Y}$ is a closed Poisson subspace corresponding to a Poisson ideal $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$, then Y is an invariant subspace for the Lie algebroid $\Omega^1_{\mathsf{X},\sigma}$, and hence $\mathcal{E}|_{\mathsf{Y}}$ is a $\Omega^1_{\mathsf{X},\sigma}|_{\mathsf{Y}}$-module. We wish to know when $\mathcal{E}|_{\mathsf{Y}}$ defines a Poisson module on Y. To this end, consider the exact sequence

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega^1_{\mathsf{X}}|_{\mathsf{Y}} \longrightarrow \Omega^1_{\mathsf{Y}} \longrightarrow 0,$$

where $\mathcal{I}/\mathcal{I}^2$ is the conormal sheaf of Y in X. We then have an exact sequence

$$0 \longrightarrow \mathscr{X}^1_{\mathsf{Y}}(\mathcal{E}) \longrightarrow \mathscr{X}^1_{\mathsf{X}}(\mathcal{E})|_{\mathsf{Y}} \longrightarrow \mathcal{H}om\left(\mathcal{I}/\mathcal{I}^2, \mathcal{E}|_{\mathsf{Y}}\right),$$

for any coherent sheaf $\mathcal{E}$. If ∇ is a Poisson connection on $\mathcal{E}$, then the composition

$$\mathcal{E} \xrightarrow{\nabla} \mathscr{X}^1_{\mathsf{X}}(\mathcal{E}) \longrightarrow \mathcal{H}om_{\mathcal{O}_{\mathsf{Y}}}\left(\mathcal{I}/\mathcal{I}^2, \mathcal{E}|_{\mathsf{Y}}\right)$$

is $\mathcal{O}_{\mathsf{X}}$-linear, giving a section

$$\Phi^{\mathsf{Y}}_{\nabla} \in \mathsf{H}^0\left(\mathsf{Y}, \mathcal{H}om\left(\mathcal{E} \otimes \mathcal{I}/\mathcal{I}^2, \mathcal{E}|_{\mathsf{Y}}\right)\right).$$

This section vanishes identically if and only if $(\mathcal{E}|_{\mathsf{Y}}, \nabla|_{\mathsf{Y}})$ defines a map $\mathcal{E}|_{\mathsf{Y}} \to \mathscr{X}^1_{\mathsf{Y}}(\mathcal{E})$, i.e., a Poisson connection for $\sigma|_{\mathsf{Y}}$.

Notice that $\mathcal{I}/\mathcal{I}^2$ inherits the structure of an $\mathcal{O}_{\mathsf{Y}}$-linear Lie algebra from the Poisson bracket. When ∇ is flat, $\Phi^{\mathsf{Y}}_{\nabla}$ defines an action $\mathcal{I}/\mathcal{I}^2 \times \mathcal{E}|_{\mathsf{Y}} \to \mathcal{E}|_{\mathsf{Y}}$ of this Lie algebra on $\mathcal{E}|_{\mathsf{Y}}$. When Y is a submanifold with normal bundle $\mathcal{N}$, we have $\mathcal{N} = (\mathcal{I}/\mathcal{I}^2)^\vee$ and the action is defined by a tensor

$$\Phi^{\mathsf{Y}}_{\nabla} \in \mathsf{H}^0(\mathsf{Y}, \mathcal{N}_{\mathsf{Y}} \otimes \mathcal{E}nd(\mathcal{E}|_{\mathsf{Y}})).$$

If the conormal Lie algebra is abelian, this tensor satisfies the equation $\Phi \wedge \Phi = 0$, but in general a correction is required that accounts for the Lie bracket $B : \wedge^2\mathcal{N}^\vee \to \mathcal{N}^\vee$; we have

$$\Phi \wedge \Phi - \Phi \circ B = 0.$$

Nevertheless, we refer to $\Phi^{\mathsf{Y}}_{\nabla}$ as the ***normal Higgs field of*** ∇ ***along*** Y.

Recall from Definition 4.3.3 that a strong Poisson subspace is one that is preserved by all Poisson vector fields. One of the useful aspects of strong Poisson subspaces is that they behave well with respect to Poisson modules:

Lemma 5.3.1. *Let (X, σ) be a Poisson analytic space and let $(\mathcal{L}, \nabla)$ be an invertible Poisson module (a Poisson line bundle). If the closed analytic subspace $\mathsf{Y} \subset \mathsf{X}$ is a strong Poisson subspace of X, then the restriction $(\mathcal{L}|_{\mathsf{Y}}, \nabla|_{\mathsf{Y}})$ is a Poisson module on Y with respect to the* *induced Poisson structure.*

Proof. Choosing a local trivialization s of $\mathcal{L}$, we have

$$\nabla s = Z \otimes s,$$

where Z is a Poisson vector field. Since Y is a strong Poisson subspace, we have $Z(\mathcal{I}_{\mathsf{Y}}) \subset \mathcal{I}_{\mathsf{Y}}$. Hence the image of $Z \otimes s$ in $\mathcal{H}om\big(\mathcal{I}/\mathcal{I}^2, \mathcal{L}|_{\mathsf{Y}}\big)$ is zero and $Z \otimes s \in \mathscr{X}^1_{\mathsf{Y}}(\mathcal{L})$, as required. □

A second Higgs field arises when we ask the related question of whether a Poisson connection comes from a connection along the symplectic leaves. Let $\mathcal{F} = \mathcal{I}mg\big(\sigma^\sharp\big) \subset \mathscr{X}^1_{\mathsf{X}}$ be the image of the anchor map. We wish to know when the Poisson connection is induced by an action $\mathcal{F} \times \mathcal{E} \to \mathcal{E}$ by way of the anchor map. In general, $\mathcal{F}$ will not be a vector bundle, but we can still form the quotient sheaf $\mathcal{N} = \mathscr{X}^1_{\mathsf{X}}/\mathcal{F}$, and consider the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathscr{X}^1_{\mathsf{X}} \longrightarrow \mathcal{N} \longrightarrow 0.$$

If $\mathcal{E}$ is a vector bundle, then $\mathscr{X}^1_{\mathsf{X}}(\mathcal{E}) \cong \mathscr{X}^1_{\mathsf{X}} \otimes \mathcal{E}$, and the sequence

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{E} \longrightarrow \mathscr{X}^1_{\mathsf{X}}(\mathcal{E}) \longrightarrow \mathcal{N} \otimes \mathcal{E} \longrightarrow 0$$

is also exact. Given a Poisson connection ∇ on $\mathcal{E}$, the composite morphism

$$\mathcal{E} \xrightarrow{\nabla} \mathscr{X}^1_{\mathsf{X}}(\mathcal{E}) \longrightarrow \mathcal{N} \otimes \mathcal{E}$$

is actually $\mathcal{O}_{\mathsf{X}}$-linear because the extra term in the Leibniz rule lies in the image of $\sigma^\sharp$. Therefore this morphism defines a ***normal Higgs field***

$$\Phi_\nabla \in \mathsf{H}^0(\mathsf{X}, \mathcal{N} \otimes \mathcal{E}nd(\mathcal{E}))\,,$$

which measures the failure of the connection vector fields for ∇ to be tangent to the symplectic leaves of σ.

Definition 5.3.2. A Poisson connection ∇ on a vector bundle $\mathcal{E}$ is ***adapted*** if its normal Higgs field Φ_∇ vanishes. In other words, ∇ is adapted exactly when it is induced by a morphism $\mathcal{E} \to \mathcal{F} \otimes \mathcal{E} \subset \mathscr{X}^1_{\mathsf{X}}(\mathcal{E})$.

The usefulness of adapted Poisson modules comes from the fact that they restrict well:

Proposition 5.3.3. *An adapted Poisson module restricts to a Poisson module on every closed Poisson subspace.*

Proof. If ∇ is adapted, it is defined by a map $\mathcal{E} \to \mathcal{F} \otimes \mathcal{E}$. If $\mathsf{Y} \subset \mathsf{X}$ is a closed Poisson subspace, then we have $\mathcal{F}(\mathcal{I}) \subset \mathcal{I}$, and hence the action $\mathcal{O}_\mathsf{X} \times \mathcal{E} \to \mathcal{E}$ descends to an action of $\mathcal{O}_\mathsf{Y} = \mathcal{O}_\mathsf{X}/\mathcal{I}$ on $\mathcal{E}|_\mathsf{Y}$. □

In particular, an adapted Poisson module restricts to a Poisson module on any symplectic leaf $\mathsf{Y} \subset \mathsf{X}$, which is nothing but a flat connection along Y.

Example 5.3.4. Consider the Poisson structure $\sigma = \partial_x \wedge \partial_y$ on $\mathsf{X} = \mathbb{C}^3$ with coordinates x, y, z, previously discussed in Example 4.3.4 and Figure 4.1. Any vector field Z induces a Poisson connection on $\mathcal{O}_\mathsf{X}$ by the formula

$$\nabla f = -\sigma^\sharp(df) + fZ \in \mathscr{X}^1_\mathsf{X}.$$

Notice that $\mathcal{F} = \mathcal{I}mg\big(\sigma^\sharp\big)$ is generated by ∂_x and ∂_y, and the normal Higgs field Φ_∇ is just the class of Z in the normal bundle $\mathcal{N} = \mathscr{X}^1_\mathsf{X}/\mathcal{F}$ to the symplectic leaves. Hence the connection is adapted if and only if Z is an $\mathcal{O}_\mathsf{X}$-linear combination of ∂_x and ∂_y. In particular, the connection associated to the Poisson vector field $Z = \partial_z$ is not adapted, even though it is flat. □

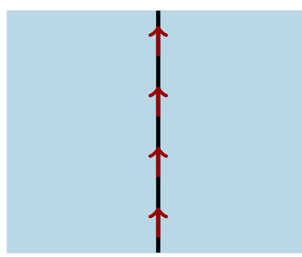


Figure 5.1: The normal Higgs field for the canonical module of the Poisson structure $x\partial_x \wedge \partial_y$ is the restriction of the vector field ∂_y to the y-axis.

Example 5.3.5. Consider the Poisson structure $\sigma = x\partial_x \wedge \partial_y$ on $\mathbb{C}^2$, which vanishes on the y-axis $\mathsf{Y} \subset \mathbb{C}^2$. Thus $\mathcal{F}$ is generated by $x\partial_x$ and $x\partial_y$, and the quotient $\mathscr{X}^1_\mathsf{X}/\mathcal{F} = \mathcal{N}$ is a torsion sheaf; it is precisely the restriction $\mathscr{X}^1_\mathsf{X}|_\mathsf{Y}$ of the tangent bundle of X to a vector bundle on Y. Using (5.1), the Poisson module structure on $\omega_{\mathbb{C}^2}$ is readily computed. We have

$$\nabla(dx \wedge dy) = \partial_y \otimes (dx \wedge dy),$$

and hence the normal Higgs field for the canonical module is

$$\Phi_\nabla = \partial_y|_\mathsf{Y} \neq 0,$$

as illustrated in Figure 5.1. We therefore see that $\omega_{\mathbb{C}^2}$ is not an adapted module for this

Poisson structure. However, its restriction to the open dense set $\mathsf{U} = \mathbb{C}^2 \setminus \mathsf{Y}$ is adapted. Indeed, on U, the Poisson structure is symplectic and its inverse $\omega = \sigma^{-1}$ gives a trivialization of ω_{U}, giving a flat connection along this symplectic leaf.

Furthermore, we notice that Y, being the zero locus of σ, is a strong Poisson subspace. Correspondingly, the normal Higgs field Φ_∇ is tangent to Y, defining a Poisson module structure on the restriction $\omega_{\mathbb{C}^2}|_{\mathsf{Y}}$.

5.4 Lie algebroids associated with Poisson modules

Suppose that (X, σ) is a Poisson analytic space and that $\mathcal{L}$ is a line bundle. Let $\mathcal{A}_{\mathcal{L}} = \mathcal{H}om\left(\mathcal{J}^1\mathcal{L}, \mathcal{L}\right)$ be the sheaf of first-order differential operators on $\mathcal{L}$ (the Atiyah algebroid; see Example 2.2.5). More generally, we denote by $\mathcal{A}^k_{\mathcal{L}}$ the sheaf of totally skew maps

$$\underbrace{\mathcal{L} \times \cdots \times \mathcal{L}}_{k \text{ times}} \to \mathcal{L}^k$$

that are first-order differential operators in each argument. Just as with multiderivations, $\mathcal{A}^\bullet_{\mathcal{L}}$ inherits a wedge product and a Schouten bracket.

A flat Poisson connection ∇ determines a section $\sigma_\nabla \in \mathcal{A}^2_{\mathcal{L}}$ as follows: if $s \in \mathcal{L}$ is a section giving a local trivialization, then we may write the connection as $\nabla s = Z \otimes s$ for a Poisson vector field $Z \in \mathscr{X}^1_{\mathsf{X}}$. The operator is then defined by

$$\sigma_\nabla(fs, gs) = (\{f, g\} + Z(f)g - Z(g)f)s^2.$$

where $\{\cdot, \cdot\}$ is the Poisson bracket on X. This expression is independent of the choice of trivialization. Viewing $\mathcal{L}$ as linear functions on the dual $\mathcal{L}^\vee$, this bracket can be extended to a Poisson structure on the total space of $\mathcal{L}^\vee$ that is invariant under the $\mathbb{C}^*$-action; see, e.g. [87, 117]. Notice that the image of σ_∇ under the natural map $\mathcal{A}^2_{\mathcal{L}} \to \mathscr{X}^2_{\mathsf{X}}$ is the Poisson structure σ.

The section σ_∇ satisfies $[\sigma_\nabla, \sigma_\nabla] = 0$, making the pair $(\mathcal{A}_{\mathcal{L}}, \sigma_\nabla)$ a triangular Lie bialgebroid in the sense of Mackenzie and Xu [104]. In particular, the dual sheaf $\mathcal{A}^\vee_{\mathcal{L}}$ inherits a Lie bracket, which, together with the composition

$$a^* : \mathcal{A}^\vee_{\mathcal{L}} \xrightarrow{\sigma^\sharp_\nabla} \mathcal{A}_{\mathcal{L}} \xrightarrow{a} \mathscr{X}^1_{\mathsf{X}} ,$$

gives $\mathcal{A}^\vee_{\mathcal{L}}$ the structure of a Lie algebroid. We stress that this construction works in significant generality: as long as X is reduced and irreducible, $\mathcal{A}_{\mathcal{L}}$ will be reflexive and hence the usual discussion of Lie bialgebroids on vector bundles carries through no matter how singular X is.

Since $\mathcal{A}_{\mathcal{L}}^{\vee}$ is a Lie algebroid, its orbits define a singular foliation of X. Since the diagram

$$\begin{tikzcd} \Omega^1_{\mathcal{A}_{\mathcal{L}}} \arrow[r, "\sigma^\sharp_\nabla"] & \mathcal{A}_{\mathcal{L}} \arrow[d] \\ \Omega^1_{\mathsf{X}} \arrow[u] \arrow[r, "\sigma^\sharp"] & \mathscr{X}^1_{\mathsf{X}} \end{tikzcd}$$

commutes, it follows that $a^*(\mathcal{A}_{\mathcal{L}})$ contains $\mathcal{I}mg(\sigma^\sharp)$, and thus the leaves of this foliation contain the symplectic leaves of σ.

Geometrically, the orbits of $\mathcal{A}_{\mathcal{L}}^{\vee}$ are the projections to X of the symplectic leaves on the total space of $\mathcal{L}^{\vee}$. In general, these orbits may be strictly larger than the symplectic leaves. Indeed, the foliation is generated locally by a connection vector field $Z \in \mathscr{X}^1_{\mathsf{X}}$ together with the Hamiltonian vector fields $\mathcal{H}am(\sigma)$. Thus, the two foliations agree exactly when the module is adapted in the sense of Definition 5.3.2.

In particular, we note that since every Poisson manifold carries two Poisson line bundles ($\mathcal{O}_{\mathsf{X}}$ and ω_{X}), it follows that every Poisson manifold comes equipped with *two* natural foliations: the usual foliation by symplectic leaves, and the orbits of $\mathcal{A}_{\omega_{\mathsf{X}}}^{\vee}$. We call the latter the ***modular foliation***. It would be interesting to study this secondary foliation in greater detail.

5.5 Residues of Poisson line bundles

In this section, we introduce the notion of residues for Poisson line bundles. These residues are multiderivations supported on the degeneracy loci of the Poisson structure, and they encode features of the connection as well as the degeneracy loci themselves.

The basic idea is simple: suppose that (X, σ) is a Poisson manifold and $\mathcal{L}$ is a Poisson line bundle. In a local trivialization $s \in \mathcal{L}$, the Poisson connection is determined by a Poisson vector field Z. When we change the trivialization, Z changes by a Hamiltonian vector field, which lies in the image of $\sigma^\sharp$. Therefore, if $x \in \mathsf{X}$ is a point at which σ has rank $2k$ or less, the wedge product $Z \wedge \sigma^k|_x$ is actually independent of the choice of trivialization, and so $Z \wedge \sigma^k|_{\mathsf{Dgn}_{2k}(\sigma)}$ gives a well-defined tensor supported on the degeneracy locus. Moreover, since $\mathsf{Dgn}_{2k}(\sigma)$ is a strong Poisson subspace and Z is a Poisson vector field, the resulting tensor is actually tangent to $\mathsf{Dgn}_{2k}(\sigma)$: in other words, if $\mathsf{Dgn}_{2k}(\sigma)$ is a submanifold, then we have a section of $\wedge^{2k+1}\mathcal{T}_{\mathsf{Dgn}_{2k}(\sigma)}$, which we refer to as the residue of $\mathcal{L}$.

Since the degeneracy loci of a Poisson structure are typically singular, we need a definition of the residue that is appropriate for general analytic spaces. We formalize it as follows:

Proposition 5.5.1. *Let (X, σ) be a Poisson analytic space and let $(\mathcal{L}, \nabla)$ be a Poisson line* *bundle. Then the morphism defined by the composition*

$$\mathcal{J}^1\mathcal{L} \xrightarrow{\nabla} \mathscr{X}^1_{\mathsf{X}}(\mathcal{L}) \xrightarrow{\sigma^k} \mathscr{X}^{2k+1}_{\mathsf{X}}(\mathcal{L})$$

descends, upon restriction to $\mathsf{Dgn}_{2k}(\sigma)$*, to a morphism* $\mathcal{L} \to \mathscr{X}^{2k+1}_{\mathsf{Dgn}_{2k}(\sigma)}(\mathcal{L})$*, defining a multiderivation*

$$\mathrm{Res}^k(\nabla) \in \mathsf{H}^0\Big(\mathsf{Dgn}_{2k}(\sigma)\,,\mathscr{X}^{2k+1}_{\mathsf{Dgn}_{2k}(\sigma)}\Big)\,.$$

Proof. Let $\mathsf{Y} = \mathsf{Dgn}_{2k}(\sigma)$. We abuse notation and denote the restricted Poisson structure by $\sigma \in \mathscr{X}^2_{\mathsf{Y}}$. Since Y is a strong Poisson subspace, Lemma 5.3.1 guarantees that $\mathcal{L}|_{\mathsf{Y}}$ is a Poisson module, and so the connection on $\mathcal{L}$ restricts to a morphism $\nabla|_{\mathsf{Y}} : \mathcal{J}^1_{\mathsf{Y}}\mathcal{L} \to \mathscr{X}^1_{\mathsf{Y}}(\mathcal{L})$. Consider the commutative diagram

$$\begin{tikzcd}
\Omega^1_{\mathsf{Y}} \otimes \mathcal{L}|_{\mathsf{Y}} \arrow[d] \arrow[dr, "-\sigma^\sharp\otimes \mathrm{id}_{\mathcal{L}}"] & & \\
\mathcal{J}^1_{\mathsf{Y}}\mathcal{L} \arrow[r, "\nabla|_{\mathsf{Y}}"] \arrow[d] & \mathscr{X}^1_{\mathsf{Y}}(\mathcal{L}) \arrow[r, "\sigma^k"] & \mathscr{X}^{2k+1}_{\mathsf{Y}}(\mathcal{L}) \\
\mathcal{L}|_{\mathsf{Y}} \arrow[urr, dashed] & &
\end{tikzcd}$$

By exactness of the jet sequence, the composition $\sigma^k \circ \nabla|_{\mathsf{Y}}$ will descend to give the dashed arrow we seek provided that $\sigma^k \wedge \sigma^\sharp(\xi) = 0$ for all $\xi \in \Omega^1_{\mathsf{Y}}$. But using the contraction $i : \Omega^1_{\mathsf{Y}} \otimes \mathscr{X}^\bullet_{\mathsf{Y}} \to \mathscr{X}^{\bullet-1}_{\mathsf{Y}}$, we compute

$$\sigma^\sharp(\xi) \wedge \sigma^k = (i_\xi\sigma) \wedge \sigma^k = \tfrac{1}{k+1} i_\xi(\sigma^{k+1}),$$

which vanishes identically on Y since $\sigma^{k+1}|_{\mathsf{Y}} = 0$. □

Definition 5.5.2. The section

$$\mathrm{Res}^k(\nabla) \in \mathsf{H}^0\Big(\mathsf{Dgn}_{2k}(\sigma)\,,\mathscr{X}^{2k+1}_{\mathsf{Dgn}_{2k}(\sigma)}\Big)$$

defined by an invertible Poisson module $(\mathcal{L}, \nabla)$ is called the kth ***residue of*** ∇. The kth residue of the canonical module ω_{X} is called the kth ***modular residue of*** σ, and denoted by

$$\mathrm{Res}^k_{mod}(\sigma) \in \mathsf{H}^0\Big(\mathsf{Dgn}_{2k}(\sigma)\,,\mathscr{X}^{2k+1}_{\mathsf{Dgn}_{2k}(\sigma)}\Big)\,.$$

Remark 5.5.3. The modular residue is so named because it locally has the form $Z \wedge \sigma^k$, where Z is the modular vector field of σ with respect to a local trivialization. In Section 7.6, we will see a family of examples where the modular residues are all non-vanishing, giving trivializations of $\mathscr{X}^{2k+1}_{\mathsf{Dgn}_{2k}(\sigma)}$. □

Example 5.5.4. Let $\mathsf{X} = \mathbb{C}^4$ with coordinates w, x, y and z. Consider the commuting vector fields $Z_w = w\partial_w$, $Z_x = x\partial_x$, $Z_y = y\partial_y$ and $Z_z = z\partial_z$, and the generically symplectic Poisson structure $Z_w \wedge Z_x + Z_y \wedge Z_z$. The modular vector field with respect to the volume form $dz \wedge dx \wedge dy \wedge dz$ is

$$Z = Z_x + Z_z - Z_w - Z_y,$$

and the degeneracy locus $\mathsf{Dgn}_2(\sigma)$ is the zero locus of

$$\sigma^2 = 2wxyz\partial_w \wedge \partial_x \wedge \partial_y \wedge \partial_z,$$

which is the union of the four coordinate hyperplanes. The first modular residue is the restriction of $Z \wedge \sigma$ to $\mathsf{Dgn}_2(\sigma)$. One readily checks that $Z \wedge \sigma$ is tangent to this locus: for example, on the $w = 0$ hyperplane $\mathsf{W} \subset \mathbb{C}^4$, we have

$$Z \wedge \sigma|_{\mathsf{W}} = Z_x \wedge Z_y \wedge Z_z|_{\mathsf{W}}$$

which is tangent to W. It defines multiderivation on $\mathsf{Dgn}_2(\sigma)$ that is independent of the choice of volume form μ

Similarly, the zero locus is the union of the pairwise intersections of the coordinate hyperplanes, and the zeroth residue is the restriction of Z to this locus. The $w = 0$ slice of this situation is illustrated in Figure 5.2. $\square$

In the previous example the residues were obtained by restricting global multiderivations to the degeneracy loci. In general, though, the residue will not be the restriction of a global Poisson vector field on X:

Example 5.5.5. Let $\mathsf{X} = \mathbb{P}^2$, and recall that a Poisson structure on X is determined up to rescaling by the cubic curve $\mathsf{Y} = \mathsf{Dgn}_0(\sigma) \subset \mathbb{P}^2$ on which it vanishes. Since Y is a strong Poisson subspace, any global Poisson vector field on $\mathbb{P}^2$ must preserve Y. If Y is smooth, a computation in sheaf cohomology shows that the only such vector field is 0. Nevertheless the residue is non-trivial: in fact, Y is an elliptic curve and the residue of any nontrivial Poisson line bundle on $\mathbb{P}^2$ gives a nonvanishing vector field on Y. If the curve is singular, then the modular residue, thought of as locally-defined a vector field on $\mathbb{P}^2$, will vanish at the singular points, but still the residue will define a trivialization of the line bundle $\mathscr{X}^1_{\mathsf{Y}}$. Some residues on $\mathbb{P}^2$ are illustrated in Figure 5.3. $\square$

Remark 5.5.6. The residue of ∇ can also be described in terms of the normal Higgs field $\Phi_\nabla \in \mathsf{H}^0(\mathsf{X}, \mathcal{N})$: since the rank of σ on $\mathsf{Dgn}_{2k}(\sigma)$ is $\leq 2k$, the exterior product of Φ_∇ and σ^k gives a well-defined section of $\mathscr{X}^{2k+1}_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)}$, and we have

$$i_* \mathrm{Res}^k(\nabla) = \Phi_\nabla|_{\mathsf{Dgn}_{2k}(\sigma)} \wedge \sigma|^k_{\mathsf{Dgn}_{2k}(\sigma)} \in \mathscr{X}^{2k+1}_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)},$$

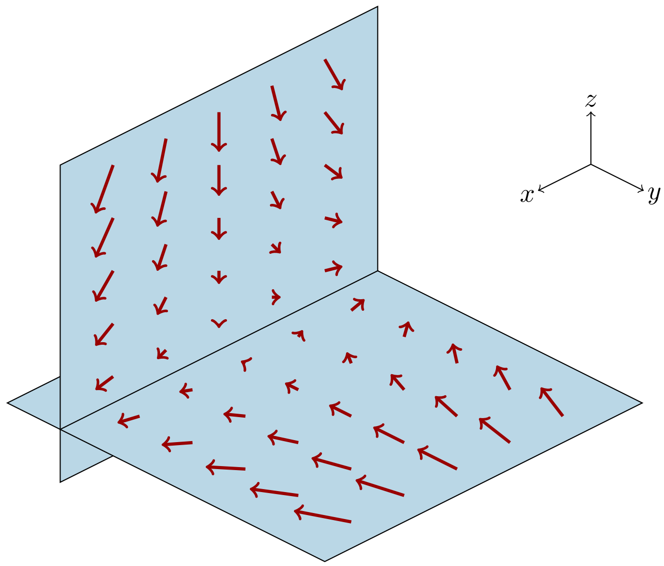


Figure 5.2: Modular residues for the Poisson structure $wx\partial_w \wedge \partial_x + yz\partial_y \wedge \partial_z$, shown in the three-dimensional hyperplane $w = 0$. The first modular residue is a top multivector field that is non-vanishing away from the coordinate planes. The planes $w = y = 0$ and $w = z = 0$ are components of the zero locus of σ, and the zeroth modular residue is a vector field on their union, shown in red.

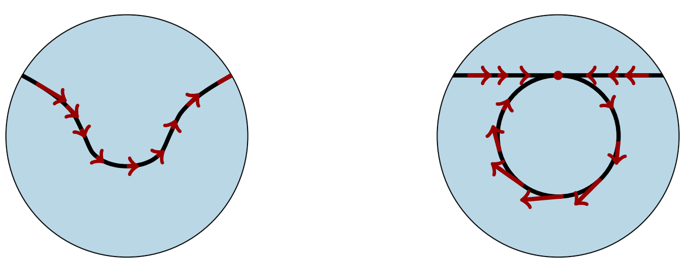


Figure 5.3: The zero locus of a Poisson structure on $\mathbb{P}^2$ is a cubic curve. The red arrows represent the modular residue $\mathrm{Res}^0_{mod}(\sigma)$.

where $i : \mathsf{Dgn}_{2k}(\sigma) \to \mathsf{X}$ is the inclusion. If $Z \in \mathscr{X}^1_{\mathsf{X}}$ is the connection vector field associated to a local trivialization of the module, we have

$$i_* \mathrm{Res}^k(\nabla) = (Z \wedge \sigma^k)|_{\mathsf{Dgn}_{2k}(\sigma)}$$

on the domain of Z. □

5.6 Modular residues

In this section, we give an explicit and simple formula for the modular residues of a Poisson structure. To do so, we recall that if σ is a Poisson structure, we can form the Poisson homology operator [27, 91]

$$\delta = \iota_\sigma d - d\iota_\sigma : \Omega^\bullet_{\mathsf{X}} \to \Omega^{\bullet-1}_{\mathsf{X}}$$

with $\delta^2 = 0$, giving a differential complex. Under the identification $\Omega^\bullet_{\mathsf{X}} \cong \mathscr{X}^{\dim \mathsf{X} - \bullet}_{\mathsf{X}} \otimes \omega_{\mathsf{X}}$, this complex is simply the de Rham complex of the canonical Poisson module. In particular, if $\mu \in \omega_{\mathsf{X}}$ is a volume form, we have $\delta\mu = \iota_Z \mu$, where $Z \in \mathscr{X}^1_{\mathsf{X}}$ is the modular vector field with respect to μ.

This operator satisfies the useful identities

$$[\delta, d] = \delta d + d\delta = 0$$

and

$$[\delta, \iota_\sigma] = \delta\iota_\sigma - \iota_\sigma\delta = 0,$$

from which one readily computes that

$$d \circ \iota_\sigma^{k+1} = \iota_\sigma^{k+1} \circ d - (k+1)\iota_\sigma^k \delta. \tag{5.2}$$

Using these formulae we can compute the residues in terms of the derivative of the Poisson structure:

Theorem 5.6.1. *For $k \geq 0$, denote by*

$$D\sigma^{k+1} = j^1(\sigma^{k+1})|_{\mathsf{Dgn}_{2k}(\sigma)} \in \Omega^1_{\mathsf{X}} \otimes \mathscr{X}^{2k+1}_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)}$$

the derivative of σ^{k+1} along its zero set, and denote by

$$\mathrm{Tr} : \Omega^1_{\mathsf{X}} \otimes \mathscr{X}^{2k+2}_{\mathsf{X}} \to \mathscr{X}^{2k+1}_{\mathsf{X}}$$

the contraction. Then the k^{th} modular residue is given by the formula

$$i_*\mathrm{Res}^k_{mod}(\sigma) = \frac{-1}{k+1}\mathrm{Tr}(D\sigma^{k+1}) \in \mathscr{X}^{2k+1}_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)},$$

where $i : \mathsf{Dgn}_{2k}(\sigma) \to \mathsf{X}$ is the inclusion.

Proof. Let $\mathsf{Y} = \mathsf{Dgn}_{2k}(\sigma)$. The question is local, so we may pick a trivialization $\mu \in \omega_{\mathsf{X}}$. Let Z be the connection vector field with respect to this trivialization, so that $\iota_Z\mu = \delta\mu$.

The residues is given by the restriction of $Z \wedge \sigma^k$, so it suffices to show that

$$(Z \wedge \sigma^k)|_{\mathsf{Y}} = \frac{-1}{k+1}\mathrm{Tr}(D\sigma^{k+1}).$$

We compute

$$\begin{aligned}\iota_{Z\wedge\sigma^k}\mu &= \iota^k_\sigma\iota_Z\mu \\ &= \iota^k_\sigma\delta\mu \\ &= -\frac{1}{k+1}d\iota^{k+1}_\sigma\mu\end{aligned}$$

by (5.2) and the identity $d\mu = 0$. But $\iota^{k+1}_\sigma\mu$ vanishes on $\mathsf{Dgn}_{2k}(\sigma)$, and hence its one-jet restricts to the derivative

$$D(\iota^{k+1}_\sigma\mu) \in \Omega^1_{\mathsf{X}} \otimes \Omega^{n-2k-2}_{\mathsf{X}}|_{\mathsf{Y}},$$

where $n = \dim \mathsf{X}$. Since the symbol of the exterior derivative is the exterior product ϵ, we have that

$$\iota_{Z\wedge\sigma^k}\mu|_{\mathsf{Y}} = \frac{-1}{k+1}\epsilon(D(\iota^{k+1}_\sigma\mu)) \in \Omega^{n-2k-1}_{\mathsf{X}}|_{\mathsf{Y}}.$$

Now, the Hodge isomorphism $\mathscr{X}^\bullet_{\mathsf{X}} \to \Omega^{n-\bullet}_{\mathsf{X}}$ defined by μ intertwines interior contraction and exterior product, so this formula shows that

$$(Z \wedge \sigma^k)|_{\mathsf{Y}} = \frac{-1}{k+1}\mathrm{Tr}(D\sigma^{k+1}),$$

as desired. □

Example 5.6.2. Suppose that X is a log symplectic manifold for which the degeneracy divisor $\mathsf{Y} = \mathsf{Dgn}_{2n-2}(\sigma)$ is smooth. Inverting σ^n gives a meromorphic volume form $\mu = (\sigma^n)^{-1}$ with poles on Y, which has a Poincaré residue $\mathrm{Res}(\mu) \in \mathsf{H}^0(\mathsf{Y}, \omega_{\mathsf{Y}})$. We have the identity

$$\mathrm{Res}^{n-1}_{mod}(\sigma) = -n\mathrm{Res}(\mu)^{-1} \in \mathsf{H}^0\left(\mathsf{Y}, \omega^{-1}_{\mathsf{Y}}\right),$$

so that the modular residues of top degree are residues in the usual sense. □

Example 5.6.3. For the Poisson structure

$$\sigma = (x\partial_x + y\partial_y) \wedge \partial_z$$

on $\mathbb{C}^3$ considered in Example 6.5.3, the zero locus is the z-axis $\mathsf{Z} \subset \mathbb{C}^3$. To compute the modular residue, we calculate the derivative

$$D\sigma = (dx \otimes (\partial_x \wedge \partial_z) + dy \otimes (\partial_y \wedge \partial_z))\,|_{\mathsf{Z}},$$

and contract the one-forms into the bivectors to find

$$\mathrm{Res}^0_{mod}(\sigma) = -\mathrm{Tr}(D\sigma) = -2\partial_z|_{\mathsf{Z}},$$

giving a trivialization of $\mathscr{X}^1_{\mathsf{Z}}$. □

Chapter 6

Degeneracy loci

In this chapter, we undertake a detailed study of the degeneracy loci of Poisson structures and Lie algebroids, highlighting the similarities and differences to the classical methods used to study degeneracy loci in algebraic geometry. One of the main themes in this chapter is the interplay between the properties of the degeneracy loci of a Poisson structure and geometry of the canonical Poisson module. We use these ideas to give new evidence for Bonal's conjecture regarding Poisson structure on Fano varieties.

6.1 Motivation: Bondal's conjecture

Recall that a line bundle $\mathcal{L}$ on a compact complex manifold X is ***ample*** if a sufficiently large power of $\mathcal{L}$ can be used to embed X in a projective space. X is Fano if the anti-canonical line bundle $\det \mathcal{T}_{\mathsf{X}}$ is ample. Examples of Fano manifolds include projective spaces $\mathbb{P}^n$, flag varieties, and smooth hypersurfaces of degree $\leq n$ in $\mathbb{P}^n$.

After observing some examples—particularly the Feigin–Odesskii–Sklyanin Poisson structures on projective space, which we will discuss in Section 7.6—Bondal made the following conjecture:

Conjecture 6.1.1 (Bondal [19])**.** *Let* (X, σ) *be a connected Poisson Fano manifold, and suppose that* $2k < \dim \mathsf{X}$*. Then the degeneracy locus* $\mathsf{Dgn}_{2k}(\sigma)$ *has a component of dimension at least* $2k+1$*.*

This conjecture implies, for example, that the zero locus of a Poisson structure on any Fano manifold must contain a curve. In dimension three, this conjecture is already a departure from the generic situation: $\mathscr{X}^2_{\mathsf{X}}$ has rank three, so a generic bivector field should only vanish at isolated points. The effect is even more dramatic in dimension four, where $\mathscr{X}^2_{\mathsf{X}}$ has rank six, and so we expect that the generic section is non-vanishing. We will see that the situation for other degeneracy loci is similar. Of course, Poisson structures are far from generic sections; they satisfy the nonlinear partial differential equation $[\sigma, \sigma] = 0$, and it is this "integrability" which leads to larger degeneracy loci.

One of the interesting aspects of this conjecture is that it is definitely of a global nature: consider the Poisson structure

$$x\partial_y \wedge \partial_z + 2y\partial_x \wedge \partial_y - 2z\partial_x \wedge \partial_z$$

from The Example. This Poisson structure on $\mathbb{C}^3$ vanishes at a single isolated point, so the zero locus does not contain a curve. Hence the estimate of the dimension of $\mathsf{Dgn}_0(\sigma)$ in the conjecture does not apply in this case. The Fano condition—or something like it—seems to be an essential part of the phenomenon.

Nevertheless, in 1997, Polishchuk provided some evidence in favour of Bondal's conjecture:

Theorem 6.1.2 ([117, Corollary 9.2])**.** *Let (X, σ) be a connected Fano Poisson manifold. If the Poisson structure generically has rank $2k$, then the degeneracy locus $\mathsf{Dgn}_{2k-2}(\sigma)$ is non-empty and has a component of dimension $\geq 2k - 1$.*

In fact, Polishchuk gave the proof in the odd-dimensional "non-degenerate" case in which the dimension of X is $2k+1$, but as Beauville observes in [10], his proof extends easily to the more general case stated here. It follows immediately that Bondal's conjecture holds for Poisson structures whose rank is ≤ 2 everywhere, and, in particular, for Fano varieties of dimension ≤ 3.

The basic tool used in Polishchuk's proof is Bott's vanishing theorem for the characteristic classes of the normal bundle to a nonsingular foliation [22]. Polishchuk applies this theorem to the anti-canonical line bundle ω_{X}^{-1} of X in order to conclude that an appropriate power of its first Chern class vanishes on $\mathsf{X} \setminus \mathsf{Dgn}_{2k-2}(\sigma)$. This vanishing would contradict the ampleness of ω_{X}^{-1} unless $\mathsf{Dgn}_{2k-2}(\sigma)$ had sufficiently large dimension. The crucial observation that allows the application of Bott's theorem is that ω_{X}^{-1} is the determinant of the normal bundle to the $2k$-dimensional symplectic leaves of σ. One might like to prove the full conjecture now by repeating the argument on $\mathsf{Dgn}_{2k-2}(\sigma)$. However, ω_{X}^{-1} can no longer be identified with the appropriate normal bundle, and so the theorem does not apply. Moreover, $\mathsf{Dgn}_{2k-2}(\sigma)$ will not, in general, be Fano, so one cannot apply an inductive argument, either.

In this chapter, we provide further evidence for the conjecture in the even-dimensional case:

Theorem 6.8.3. *Let (X, σ) be a connected Fano Poisson manifold of dimension $2n$. Then either $\mathsf{Dgn}_{2n-2}(\sigma) = \mathsf{X}$ or $\mathsf{Dgn}_{2n-2}(\sigma)$ is a hypersurface in X. Moreover, $\mathsf{Dgn}_{2n-4}(\sigma)$ is non-empty and has at least one component of dimension $\geq 2n - 3$.*

In particular, this result shows that Conjecture 6.1.1 holds for Fano varieties of dimension four.

The statement about $\mathsf{Dgn}_{2n-2}(\sigma)$ is straightforward: this locus is the zero set of the section

$$\sigma^n \in \mathsf{H}^0(\mathsf{X}, \det \mathcal{T}_\mathsf{X}) = \mathsf{H}^0\big(\mathsf{X}, \omega_\mathsf{X}^{-1}\big)\,,$$

so it is an anticanonical hypersurface in X unless $\sigma^n = 0$, in which case $\mathsf{Dgn}_{2n-2}(\sigma)$ is all of X. If $\mathsf{Dgn}_{2n-2}(\sigma)$ is an irreducible hypersurface, it is a Calabi-Yau variety which turns out to be highly singular. We expect similar behaviour from the lower-rank degeneracy loci.

Showing that $\mathsf{Dgn}_{2n-4}(\sigma)$ has a component of dimension $\geq 2n-3$ is the difficult part of the theorem. The main idea of the proof is to exploit the fact that, while the restriction of ω_X^{-1} to the degeneracy locus $\mathsf{Dgn}_{2n-2}(\sigma)$ is not the normal bundle to the symplectic leaves, it is a Poisson module—that is, it carries a flat "Poisson connection". While Bott's theorem does not apply directly to Poisson modules, we use the ideas developed in the previous chapter—particularly the notion of adaptendess—to show that the theorem *can* be applied on the singular locus of $\mathsf{Dgn}_{2n-2}(\sigma)$. By bounding the dimension of the latter space, we are able to bound the dimension of $\mathsf{Dgn}_{2n-4}(\sigma)$, as in Polishchuk's proof.

Before giving the proof Theorem 6.8.3, we will collect several facts about degeneracy loci in general. We begin by recalling some results from the standard theory of degeneracy loci of vector bundle maps in algebraic geometry. We then explain that for Lie algebroids and Poisson structures, the theory is still useful, but it is nevertheless deficient in several ways. The basic problem is that the integrability conditions imposed on these structures mean that the degeneracy loci in question are not the "generic" ones for which the standard theory is suited. Thus, to prove the theorem, we combine the standard theory with the extra structure present on Poisson manifolds—particularly the canonical module.

6.2 Degeneracy loci in algebraic geometry

The degeneracy loci of vector bundle maps have been well studied in algebraic geometry. We give a brief overview here and refer the reader to Chapter 14 in Fulton's book [58] for a detailed discussion and several references.

Let X be a complex manifold or smooth algebraic variety, and suppose that $\mathcal{E}$ and $\mathcal{F}$ are vector bundles on X. Given a morphism $\rho : \mathcal{E} \to \mathcal{F}$, we are interested in studying the locus $\mathsf{Dgn}_k(\rho)$ in X where the rank of ρ is $\leq k$. This locus is described by the zero set of the section $\Lambda^{k+1}\rho$. The corresponding ideal is the image of the induced map

$$\Lambda^{k+1}\mathcal{E} \otimes \Lambda^{k+1}\mathcal{F}^\vee \to \mathcal{O}_\mathsf{X}.$$

Thus, $\mathsf{Dgn}_k(\rho)$ is defined locally by the vanishing of the $(k+1)\times(k+1)$ minors of a local matrix representation for ρ. There are obviously inclusions

$$\cdots \subset \mathsf{Dgn}_{k-1}(\rho) \subset \mathsf{Dgn}_k(\rho) \subset \mathsf{Dgn}_{k+1}(\rho) \subset \cdots$$

We claim that if $\mathsf{Dgn}_k(\rho)$ is a strict subset of X, then it is necessarily singular along $\mathsf{Dgn}_{k-1}(\rho)$: indeed, since the question is local, we may pick connections on $\mathcal{E}$ and $\mathcal{F}$ and compute

$$\nabla(\Lambda^{k+1}\rho) = (k+1)(\nabla\rho) \wedge \Lambda^k \rho,$$

using the Leibniz rule. We therefore see that the derivative of $\Lambda^{k+1}\rho$ vanishes along $\mathsf{Dgn}_{k-1}(\rho)$. It follows that the Zariski tangent space to $\mathsf{Dgn}_k(\rho)$ at a point $x \in \mathsf{Dgn}_{k-1}(\rho)$ is the entire tangent space of X, and hence $\mathsf{Dgn}_k(\rho)$ must be singular. Notice, though, that the singular locus might be strictly larger.

If $\mathcal{E} = \mathcal{F}^\vee$ and ρ is skew-symmetric, defining a section of $\Lambda^2\mathcal{F}$, then the definition of the degeneracy loci above is not appropriate because it does not lead to reduced spaces: the basic point is that the determinant of a skew-symmetric matrix is the square of its Pfaffian and hence the determinant does not generate a radical ideal. We therefore work instead with the exterior powers $\rho^k \in \Lambda^{2k}\mathcal{F}$ and the corresponding map

$$\Lambda^{2k}\mathcal{F}^\vee \to \mathcal{O}_{\mathsf{X}}.$$

Since the rank is always even, we concern ourselves only with the degeneracy loci $\mathsf{Dgn}_{2k}(\rho)$ for $k \geq 0$. As above, we have $\mathsf{Dgn}_{2k-2}(\rho) \subset \mathsf{Dgn}_{2k}(\rho)_{sing}$.

Every degeneracy locus has an "expected" codimension in X that depends only on the ranks of the bundles involved and the integer k. For general maps $\rho : \mathcal{E} \to \mathcal{F}$ the expected codimension of $\mathsf{Dgn}_k(\rho)$ is $c = (e-k)(f-k)$, where e and f are the ranks of $\mathcal{E}$ and $\mathcal{F}$. For skew-symmetric tensors $\rho \in \Lambda^2\mathcal{F}$, the expected codimension of $\mathsf{Dgn}_{2k}(\rho)$ is $c = \binom{f-2k}{2}$.

If $\mathsf{X} = \mathbb{C}^n$ and ρ is suitably generic, then the expected codimensions give the actual codimensions of the degeneracy loci of ρ, and these loci will be reduced. In general, though, the expected codimension only gives an upper bound on the actual codimension of $\mathsf{Dgn}_k(\rho)$ in X. This bound is really a result in commutative algebra: celebrated work of Eagon and Northcott [48] bounds the depth of the ideal defined by the $(k+1) \times (k+1)$ minors of a matrix with entries in a local ring, which in turn bounds the codimension. (One is essentially counting the number of independent constraints imposed by the minors.) For the skew-symmetric case, one deals instead with the Pfaffians; this case was described by Józefiak and Pragacz [86]. We remark that the connection between the properties of the ideal and the codimension fails when we work over the real numbers because they are not algebraically closed.

In addition to giving a tight bound on the codimension of the degeneracy loci, one can define a natural class $\widehat{\mathsf{Dgn}_k(\rho)} \in \mathsf{H}_{2d}(\mathsf{Dgn}_k(\rho), \mathbb{Z})$ in the Borel-Moore homology of

$\mathsf{Dgn}_k(\rho)$, where $d = \dim \mathsf{X} - c$ is the expected dimension of the degeneracy locus. When $\dim \mathsf{Dgn}_k(\rho) = d$, this class is just the fundamental class. Moreover, there is a natural cohomology class $P_k \in \mathsf{H}^{2c}(\mathsf{X}, \mathbb{Z})$ such that

$$i_*\widehat{\mathsf{Dgn}_k(\rho)} = P_k \cap [\mathsf{X}] \in \mathsf{H}_{2d}(\mathsf{X}, \mathbb{Z})\,.$$

The class P_k is a polynomial in the Chern classes of the bundles involved. This polynomial depends on the ranks of the bundles and on k, but is independent of the section ρ. One can therefore conclude that $\mathsf{Dgn}_k(\rho)$ is non-empty merely by computing with Chern classes. The existence of such a polynomial was known to Thom but explicit formulae were given by Porteous [119] for general bundle maps. Formulae in the skew-symmetric case were given by Harris and Tu [76] and Józefiak, Lascoux and Pragacz [85].

6.3 Degeneracy loci of Lie algebroids

Although we shall focus primarily on the degeneracy loci of Poisson structures, we begin with a few remarks about general Lie algebroids. We give some results and examples showing that the degeneracy loci of Lie algebroids are much more complicated than the standard theory of degeneracy loci discussed in the previous section might suggest.

Suppose that X is a complex manifold of dimension n and that $\mathcal{A}$ is a Lie algebroid on X that is a rank-r vector bundle. As in Section 2.2, we denote by $\mathsf{Dgn}_k(\mathcal{A})$ the kth degeneracy locus of the anchor map $\mathcal{A} \to \mathcal{T}_\mathsf{X}$.

Proposition 6.3.1. *Suppose that $\mathsf{Dgn}_k(\mathcal{A})$ is non-empty. Then every reduced component of $\mathsf{Dgn}_k(\mathcal{A})$ has dimension $\geq k$.*

Proof. Suppose that Y is a reduced component of $\mathsf{Dgn}_k(\mathcal{A})$. Then Y is smooth on an open dense set $\mathsf{U} \subset \mathsf{Y}$. Since $\mathsf{Dgn}_k(\mathcal{A})$ is singular along $\mathsf{Dgn}_{k-1}(\mathcal{A})$, it follows that $\mathsf{U} \cap \mathsf{Dgn}_{k-1}(\mathcal{A}) = \varnothing$, and hence the anchor map has rank k on all of U. But Y is an $\mathcal{A}$-invariant subspace, and hence the image of the anchor $\mathcal{A}_\mathsf{U} \to \mathcal{T}_\mathsf{X}|_\mathsf{U}$ lies in the subbundle $\mathcal{T}_\mathsf{U}$. Therefore $\mathcal{T}_\mathsf{U}$ has rank $\geq k$ and the result follows. □

This proposition shows that when the degeneracy loci are reduced, their dimension is a linear function of k. In contrast, the standard theory of degeneracy loci would tell us to expect the dimension of $\mathsf{Dgn}_k(\mathcal{A})$ to be $n - (n-k)(r-k)$, which is a quadratic function of k.

There is a close relationship between the degeneracy loci of a Lie algebroid and the properties of the subsheaf of $\mathcal{T}_\mathsf{X}$ defined by the image of the anchor.

Proposition 6.3.2. *Let $\mathcal{A}$ be a Lie algebroid on X (not necessarily a vector bundle) and let $\mathcal{F} \subset \mathcal{T}_\mathsf{X}$ be the image of the anchor map. Suppose that $\mathcal{F}$ has rank k and that its double-*

dual $\mathcal{F}^{\vee\vee} \subset \mathcal{T}_{\mathsf{X}}$ *is locally free (a vector bundle). Then every component of* $\mathsf{Dgn}_{k-1}(\mathcal{A})$ *has dimension at least* $k-1$*.*

Proof. The vector bundle $\mathcal{F}^{\vee\vee}$ is naturally a Lie algebroid on X. Considering the composition $\mathcal{A} \to \mathcal{F}^{\vee\vee} \to \mathcal{T}_{\mathsf{X}}$ of the anchors, we see that $\mathsf{Dgn}_{k-1}(\mathcal{F}^{\vee\vee}) \subset \mathsf{Dgn}_{k-1}(\mathcal{A})$. But the expected dimension of $\mathsf{Dgn}_{k-1}(\mathcal{F}^{\vee\vee})$ is $n-(n-(k-1))(k-(k-1)) = k-1$, and hence every component of $\mathsf{Dgn}_{k-1}(\mathcal{F}^{\vee\vee})$ has dimension at least $k-1$. $\square$

Similarly, some conclusions can be made in the case when the ranks of $\mathcal{A}$ and $\mathcal{T}_{\mathsf{X}}$ agree:

Proposition 6.3.3. *Suppose that* $\operatorname{rank}(\mathcal{A}) = \dim \mathsf{X} = n$*, so that the expected dimension of* $\mathsf{Dgn}_{n-1}(\mathcal{A})$ *is* $n-1$*. Suppose further that* $\mathsf{Dgn}_{n-1}(\mathcal{A})$ *is a reduced hypersurface. If* $\mathsf{Dgn}_{n-2}(\mathcal{A})$ *is non-empty, then each of its components has dimension* $n-2$*.*

Proof. Let $\mathsf{D} = \mathsf{Dgn}_{n-1}(\mathcal{A})$ be the hypersurface, which is defined by the determinant of the anchor map. Since D is reduced, Saito's criterion (Theorem 2.3.3), shows that D is a free divisor and the anchor map gives a canonical identification $\mathcal{A} \cong \mathcal{T}_{\mathsf{X}}(-\log \mathsf{D})$. Therefore, by Theorem 2.3.8, every component of the singular locus D_{sing} has dimension $n-2$. Since D_{sing} is $\mathcal{A}$-invariant, the rank of $\mathcal{A}$ along D_{sing} must be $\leq n-2$, and hence every point of D_{sing} lies in $\mathsf{Dgn}_{n-2}(\mathcal{A})$. Conversely, every point in $\mathsf{Dgn}_{n-2}(\mathcal{A})$ is a singular point of D. Hence $\mathsf{D}_{sing} = \mathsf{Dgn}_{n-2}(\mathcal{A})$ as sets and the conclusion follows. $\square$

Remark 6.3.4. Notice that according to the standard theory of degeneracy loci, the expected dimension of $\mathsf{Dgn}_{2n-2}(\sigma)$ in Proposition 6.3.3 is $n-4$. Hence, the standard theory predicts the wrong dimension in this case.

From these results, one is tempted to conjecture that $\dim \mathsf{Dgn}_k(\mathcal{A}) \geq k$ always, giving a replacement for the expected dimension in the standard theory of degeneracy loci. However, this proposal fails even for simple Lie-theoretic examples:

Example 6.3.5. Consider the adjoint action of $\mathsf{G} = \mathsf{SL}(3,\mathbb{C})$ on its Lie algebra $\mathfrak{g}$. The only orbit of dimension ≤ 2 is the origin $0 \in \mathfrak{g}$. Hence for the action algebroid $\mathfrak{g} \ltimes \mathfrak{g}$, we have $\dim \mathsf{Dgn}_2(\mathfrak{g} \ltimes \mathfrak{g}) = 0 < 2$. We note, however, that $\mathsf{Dgn}_2(\mathfrak{g} \ltimes \mathfrak{g})$ is not reduced since it is defined by a collection of cubic polynomials (the coefficients of the map $\Lambda^3\mathrm{ad} : \Lambda^3\mathfrak{g} \to \Lambda^3\mathfrak{g}$ of trivial bundles over $\mathfrak{g}$). Hence there is no contradiction with Proposition 6.3.2. $\square$

6.4 Degeneracy loci of Poisson structures

For the rest of the chapter, we restrict our attention to the degeneracy loci of Poisson structures on complex manifolds, although many of the results we discuss should admit generalizations to other Lie algebroids as well.

Arguing exactly as in Proposition 6.3.2 for Lie algebroids, we have the following property of reduced degeneracy loci for Poisson strucutres:

Proposition 6.4.1. *Suppose that* (X, σ) *is a complex Poisson manifold. Then every reduced component of* $\mathsf{Dgn}_{2k}(\sigma)$ *has dimension at least* $2k$*.*

In light of this result, it is natural to wonder if every component of $\mathsf{Dgn}_{2k}(\sigma)$ has dimension at least $2k$ whenever it is nonempty, but as in Example 6.3.5, a counterexample is given by the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$: its coadjoint orbits are the symplectic leaves of the natural Poisson structure on the dual, and so the only leaf of dimension less than four is the origin.

In order to deal effectively with the degeneracy loci, it is useful to have some control over their singularities. To this end, we now describe the Zariski tangent space to a degeneracy locus of a Poisson structure.

At a point $x \in \mathsf{X}$ where σ has rank $2k$, the kernel of the anchor map $\mathsf{N}^*_x \subset \mathsf{T}^*_x\mathsf{X}$ is a subspace of codimension $2k$ and inherits a natural Lie algebra structure as the isotropy algebra of the Poisson Lie algebroid. The bracket has the following simple description: if $f, g \in \mathcal{O}_\mathsf{X}$ are functions such that $\sigma^\sharp(df)$ and $\sigma^\sharp(dg)$ vanish at x, then $df|_x, dg|_x \in \mathsf{N}^*_x$ and their Lie bracket is given by

$$[df|_x, dg|_x] = d\{f, g\}|_x \in \mathsf{N}^*_x.$$

We have the following result, generalizing [117, Lemma 2.5] for the case of the zero locus:

Proposition 6.4.2. *Let* (X, σ) *be a Poisson manifold. Then the Zariski tangent space to* $\mathsf{Dgn}_{2k}(\sigma)$ *at a point* $x \in \mathsf{Dgn}_{2k}(\sigma) \setminus \mathsf{Dgn}_{2k-2}(\sigma)$ *is given by*

$$\mathsf{T}_x\mathsf{Dgn}_{2k}(\sigma) = (\mathsf{T}^*_x\mathsf{X}/[\mathsf{N}^*_x, \mathsf{N}^*_x])^\vee$$

In particular, $\mathsf{Dgn}_{2k}(\sigma)$ *is smooth at* x *if and only its codimension at* x *is equal to the dimension of* $[\mathsf{N}^*_x, \mathsf{N}^*_x]$*.*

Proof. Choose functions $g_1, \ldots, g_r$ whose derivative form a basis for N^*_x and choose more functions $f_1, \ldots, f_{2k}$ whose derivatives extend the given basis of N^*_X to a basis for all of $\mathsf{T}^*_x\mathsf{X}$. The conormal space of $\mathsf{Dgn}_{2k}(\sigma)$ is therefore spanned by expressions of the form

$$d\left(\sigma^{k+1}(dg_{i_1} \wedge \cdots \wedge dg_{i_m} \wedge df_{j_1} \wedge \cdots \wedge df_{j_n})\right)|_x,$$

where $m + n = 2k + 2$. Expanding the derivative using the Leibniz rule, one finds that such an expression can only be nonzero if each of the functions $f_1, \ldots, f_{2k}$ appears exactly once, and in this case the expression reduces to a nonzero constant multiple of $d\{g_{i_1}, g_{i_2}\}|_x$, which is the Lie bracket on N^*_x. Hence the conormal space is spanned by the commutator subalgebra $[\mathsf{N}^*_x, \mathsf{N}^*_x]$ and the result follows. □

This result allows us to constrain the structure of the Higgs field associated with the conormal sheaf:

Proposition 6.4.3. *Let* (X,σ) *be a complex Poisson manifold, and let* Y *be a smooth irreducible component of* $\mathsf{Dgn}_{2k}(\sigma)$*, defined by the ideal* $\mathcal{I} \subset \mathcal{O}_{\mathsf{X}}$*. Then its conormal sheaf* $\mathcal{C} = \mathcal{I}/\mathcal{I}^2$ *is a Poisson module and the corresponding Higgs field*

$$\Phi(\mathcal{C}) \in \mathsf{Hom}_{\mathsf{Dgn}_{2k}(\sigma)}(\mathcal{C}, \mathcal{E}nd(\mathcal{C}))$$

is traceless.

Proof. Notice that $\mathcal{C}$ is a bundle of Lie algebras and the Higgs field is simply the adjoint action of $\mathcal{C}$ on itself, and we may identify the fibre of $\mathcal{C}$ at a point $x \in \mathsf{Y}$ with the commutator subalgebra $[\mathsf{N}^*_x, \mathsf{N}^*_x]$. Since commutator subalgebras are always unimodular, the adjoint action of $\mathcal{C}$ on itself is traceless, as desired. $\square$

This description of the Zariski tangent spaces can be extended to a locally free presentation of the cotangent sheaf of the degeneracy locus. This presentation is an exact sequence

$$\Omega^{2k+2}_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)} \xrightarrow{\phi_k(\sigma)} \Omega^1_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)} \longrightarrow \Omega^1_{\mathsf{Dgn}_{2k}(\sigma)} \longrightarrow 0,$$

presenting the forms on $\mathsf{Dgn}_{2k}(\sigma)$ as the cokernel of a vector bundle map $\phi_k(\sigma)$. This bundle map is just the dual of the one-jet of σ^{k+1} along the degeneracy locus, but it also has an interpretation in terms of the canonical Poisson module ω_{X}, as we now explain.

Recall from Section 5.6 the Poisson homology operator

$$\delta = \iota_\sigma d - d\iota_\sigma : \Omega^\bullet_{\mathsf{X}} \to \Omega^{\bullet-1}_{\mathsf{X}}$$

and the identity

$$d \circ \iota^{k+1}_\sigma = \iota^{k+1}_\sigma \circ d - (k+1)\iota^k_\sigma \delta.$$

By definition, the conormal sheaf of $\mathsf{Dgn}_{2k}(\sigma)$ is generated by the restriction of forms in the image of $d \circ \iota^{k+1}_\sigma : \Omega^{2k+2}_{\mathsf{X}} \to \Omega^1_{\mathsf{X}}$. Since σ^{k+1} vanishes on $\mathsf{Dgn}_{2k}(\sigma)$, the formula above shows that it is also generated by the image of $\iota^k_\sigma\delta$. One can easily check that the restriction of this operator to $\mathsf{Dgn}_{2k}(\sigma)$ is $\mathcal{O}_{\mathsf{X}}$-linear, so that we have the

Theorem 6.4.4. *Let* δ *be the Poisson homology operator on a complex Poisson manifold* (X,σ)*. Then the operator* $\iota^k_\sigma\delta : \Omega^\bullet_{\mathsf{X}} \to \Omega^{\bullet-2k-1}_{\mathsf{X}}$ *restricts to a vector bundle morphism on the degeneracy locus* $\mathsf{Dgn}_{2k}(\sigma)$*, giving an exact sequence*

$$\Omega^{2k+2}_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)} \xrightarrow{\iota^k_\sigma\delta} \Omega^1_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)} \longrightarrow \Omega^1_{\mathsf{Dgn}_{2k}(\sigma)} \longrightarrow 0$$

of coherent sheaves.

6.5 Degeneracy loci of Poisson modules

In this section, we assume that (X, σ) is a complex Poisson manifold. Recall from Section 5.4 that a Poisson line bundle $(\mathcal{L}, \nabla)$ gives rise to a canonical section $\sigma_\nabla \in \mathsf{H}^0\big(\mathsf{X}, \mathcal{A}^2_{\mathcal{L}}\big)$, that is, a skew-symmetric bidifferential operator $\sigma_\nabla : \mathcal{L} \times \mathcal{L} \to \mathcal{L}^2$. We now study the ***degeneracy loci*** $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ of the Poisson module—the loci where the exterior powers σ_∇^{k+1} vanish. These loci provide a new family of Poisson subspaces with the property that the Poisson module is flat along their $2k$-dimensional symplectic leaves.

Remark 6.5.1. One could also think of σ_∇ as the Poisson structure induced on the total space of $\mathcal{L}^\vee$. If Y is the principal $\mathbb{C}^*$-bundle obtained by removing the zero section of $\mathcal{L}^\vee$. Set-theoretically, $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ is the image of the $2k^{th}$ degeneracy locus of the Poisson structure on Y under the projection. □

The degeneracy loci of σ_∇ are closely related with those of the Poisson structure, and with the notion of adaptedness introduced in Definition 5.3.2:

Proposition 6.5.2. *The degeneracy locus $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ of an invertible Poisson module $(\mathcal{L}, \nabla)$ is a Poisson subspace and is equal to the zero locus of the section*

$$i_*\mathrm{Res}^k(\nabla) \in \mathsf{H}^0\big(\mathsf{Dgn}_{2k}(\sigma)\,, \mathscr{X}^{2k+1}_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)}\big)\,,$$

where $i : \mathsf{Dgn}_{2k}(\sigma) \to \mathsf{X}$ is the embedding. In particular, we have inclusions

$$\mathsf{Dgn}_{2k-2}(\sigma) \subset \mathsf{Dgn}_{2k}(\sigma_\nabla) \subset \mathsf{Dgn}_{2k}(\sigma)\,.$$

Moreover, ∇ restricts to a Poisson module on $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ that is adapted on the open set $\mathsf{Dgn}_{2k}(\sigma_\nabla) \setminus \mathsf{Dgn}_{2k-2}(\sigma)$.

Proof. Taking exterior powers of the exact sequence

$$0 \to \mathscr{O}_{\mathsf{X}} \to \mathcal{A}_{\mathcal{L}} \to \mathscr{X}^1_{\mathsf{X}} \to 0,$$

we obtain the exact sequence

$$0 \to \mathscr{X}^{2k+1}_{\mathsf{X}} \to \mathcal{A}^{2k+2}_{\mathcal{L}} \to \mathscr{X}^{2k+2}_{\mathsf{X}} \to 0.$$

The image of σ_∇^{k+1} in $\mathscr{X}^{2k+2}_{\mathsf{X}}$ is simply σ^{k+1}. Hence the zero locus of σ_∇^{k+1} is contained in $\mathsf{Dgn}_{2k}(\sigma)$, and is given by the vanishing of the section

$$s = \sigma_\nabla^{k+1}|_{\mathsf{Dgn}_{2k}(\sigma)} \in \mathscr{X}^{2k+1}_{\mathsf{X}}|_{\mathsf{Dgn}_{2k}(\sigma)}.$$

One readily checks using the definition of σ_∇ that s has the form

$$s = Z \wedge \sigma^k|_{\mathsf{Dgn}_{2k}(\sigma)}, \tag{6.1}$$

where Z is a connection vector field for ∇ with respect to any local trivialization. But then $s = i_*\mathrm{Res}^k(\nabla)$ by Remark 5.5.6, and so $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ is the zero locus of $i_*\mathrm{Res}^k(\nabla)$, as claimed. Since s has a factor σ^k, we have $\mathsf{Dgn}_{2k-2}(\sigma) \subset \mathsf{Dgn}_{2k}(\sigma_\nabla)$.

We wish to show that the zero locus of s is a Poisson subspace. Given a function $f \in \mathcal{O}_\mathsf{X}$ and using the identity $\mathscr{L}_Z\sigma = 0$, we have

$$\begin{aligned} \mathscr{L}_{\sigma^\sharp(df)}(\sigma^k \wedge Z) &= (\mathscr{L}_{\sigma^\sharp(df)}\sigma^k) \wedge Z + \sigma^k \wedge (\mathscr{L}_{\sigma^\sharp(df)}Z) \\ &= -\sigma^k \wedge \mathscr{L}_Z\sigma^\sharp(df) \\ &= -\sigma^k \wedge \sigma^\sharp(\mathscr{L}_Z df) \end{aligned}$$

which vanishes on $\mathsf{Dgn}_{2k}(\sigma)$. It follows that the ideal defining $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ is preserved by all Hamiltonian vector fields, and hence $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ is a Poisson subspace.

Finally, to see that the module is adapted on $\mathsf{Dgn}_{2k}(\sigma_\nabla) \setminus \mathsf{Dgn}_{2k-2}(\sigma)$, simply notice that if at some point $x \in \mathsf{Dgn}_{2k}(\sigma)$ we have $\sigma_x^k \neq 0$, then Z_x lies in the image of $\sigma_x^\sharp$ if and only if $Z_x \wedge \sigma_x^k = 0$, i.e., $x \in \mathsf{Dgn}_{2k}(\sigma_\nabla)$. □

Example 6.5.3. Consider the Poisson structure $\sigma = (x\partial_x + y\partial_y) \wedge \partial_z$ on the affine space $\mathbb{C}^3$ with coordinates x, y and z. Denote by $\mathsf{Z} \subset \mathbb{C}^3$ the z-axis, given by $x = y = 0$. The degeneracy loci of the Poisson structure are $\mathsf{Dgn}_2(\sigma) = \mathbb{C}^3$ and $\mathsf{Dgn}_0(\sigma) = \mathsf{Z}$. Every plane W containing Z is a Poisson subspace, and $\mathsf{W} \setminus \mathsf{Z}$ is a symplectic leaf.

The vector field

$$U = x\partial_x$$

is Poisson, and hence it defines a Poisson module structure on $\mathcal{O}_\mathsf{X}$ by the formula

$$\nabla f = -\sigma^\sharp(df) \otimes 1 + U \otimes f.$$

The degeneracy locus $\mathsf{Dgn}_2(\sigma_\nabla)$ is given by the vanishing of the tensor

$$U \wedge \sigma = xy\partial_x \wedge \partial_y \wedge \partial_z,$$

and hence $\mathsf{Dgn}_2(\sigma_\nabla)$ is the union of the $x = 0$ and $y = 0$ planes, which are indeed Poisson subspaces. We therefore see that the inclusions

$$\mathsf{Dgn}_0(\sigma) \subset \mathsf{Dgn}_2(\sigma_\nabla) \subset \mathsf{Dgn}_2(\sigma)$$

are strict in this case. □

6.6 Structural results in small codimension

6.6.1 Codimension one: log symplectic singularities

Let (X, σ) a log symplectic Poisson manifold of dimension $2n$ in the sense of Section 4.4. Let $\mathsf{D} = \mathsf{Dgn}_{2n-2}(\sigma)$ be the degeneracy divisor of σ which is reduced by definition, but need not be smooth.

Recall from Theorem 2.3.8 that if D is singular, it will be a free divisor if and only if its singular locus is Cohen–Macaulay of codimension two in X. We note that this condition need not be satisfied for general log symplectic manifolds: for example, for the log symplectic Poisson structures of Feigin and Odesskii [56, 57], illustrated in Figure 6.1 and discussed in Section 7.6, the degeneracy hypersurface is smooth away from a subset of codimension three.

Remarkably, the codimension of the singular locus can never be more than three and thus D is quite close to being free when compared to a general singular hypersurface. Indeed, in this section we will prove the following

Theorem 6.6.1. *Let (X, σ) be a log symplectic Poisson manifold of dimension $2n$ with degeneracy locus $\mathsf{D} = \mathsf{Dgn}_{2n-2}(\sigma)$, and let $\mathsf{W} = \mathsf{D}_{sing}$ be the singular locus of D. Then every component of W has codimension ≤ 3 in X. Moreover, if every component has codimension three, then the following statements hold:*

1. *The fundamental class $[\mathsf{W}] \in \mathsf{H}^{2n-6}(\mathsf{X}, \mathbb{Z})$ in the cohomology of X is given by*

$$[\mathsf{W}] = c_1c_2 - c_3,$$

   where c_1, c_2, c_3 are the Chern classes of X.

2. *The complex*

$$0 \longrightarrow \omega_{\mathsf{X}}^2 \xrightarrow{\sigma_\triangledown^n} \mathcal{A}_{\omega_{\mathsf{X}}}^\vee \otimes \omega_{\mathsf{X}} \xrightarrow{\sigma_\triangledown^\sharp \otimes 1} \mathcal{A}_{\omega_{\mathsf{X}}} \otimes \omega_{\mathsf{X}} \xrightarrow{\sigma_\triangledown^n} \mathcal{O}_{\mathsf{X}} \longrightarrow \mathcal{O}_{\mathsf{W}} \longrightarrow 0$$

   gives a locally free resolution of $\mathcal{O}_{\mathsf{W}}$.

3. *W is Gorenstein with dualizing sheaf $\omega_{\mathsf{X}}^{-1}|_{\mathsf{W}}$.*

The proof, which we delay for a moment, is a direct consequence of the following

Proposition 6.6.2. *Let (X, σ) be a log symplectic Poisson manifold of dimension $2n$ with degeneracy locus $\mathsf{D} = \mathsf{Dgn}_{2n-2}(\sigma)$, and let $\mathsf{W} = \mathsf{D}_{sing}$ be the singular locus of D. Then $\mathsf{W} = \mathsf{Dgn}_{2n-2}(\sigma_\triangledown)$ is a degeneracy locus of the canonical Poisson module.*

Proof. We apply the description of the one-forms on D from Theorem 6.4.4. In this case, it amounts to an exact sequence

$$\omega_{\mathsf{X}}|_{\mathsf{D}} \xrightarrow{\iota_\sigma^{n-1}\delta} \Omega^1_{\mathsf{X}}|_{\mathsf{D}} \longrightarrow \Omega^1_{\mathsf{D}} \longrightarrow 0$$

The singular locus is therefore canonically identified with the zero locus of the section

$$\iota_\sigma^{n-1}\delta|_{\mathsf{D}} \in \omega_{\mathsf{X}}^{-1} \otimes \Omega^1_{\mathsf{X}}|_{\mathsf{D}} \cong \mathscr{X}^{2n-1}_{\mathsf{X}}|_{\mathsf{D}}$$

Since $\delta : \omega_{\mathsf{X}} \to \Omega^{2n-1}_{\mathsf{X}} \cong \mathscr{X}^1_{\mathsf{X}} \otimes \omega_{\mathsf{X}}$ is simply the Poisson connection on ω_{X}, the tensor in question is given locally by $\sigma^{n-1} \wedge Z$ where Z is the modular vector field. The singular locus is therefore defined by the simultaneous vanishing of $\sigma^{n-1} \wedge Z$ and σ^n, and hence it coincides with the degeneracy locus of σ_∇ (see Proposition 6.5.2). $\square$

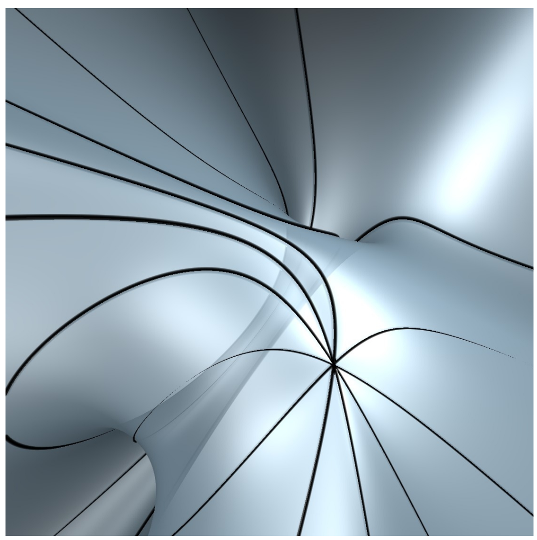


Figure 6.1: A three-dimensional cross-section of one of Feigin and Odesskii's elliptic Poisson structures on $\mathbb{P}^4$. The blue surface represents the degeneracy hypersurface D, which is the secant variety to an elliptic normal curve. The hypersurface D is foliated by two-dimensional symplectic leaves, represented here by the black curves. The leaves intersect along the singular locus of D, which has codimension three in the ambient space.

Proof of Theorem 6.6.1. The proof is now a straightforward application of the standard theory of degeneracy loci. In this case, the vector bundle is the Atiyah algebroid $\mathcal{A} = \mathcal{A}_{\omega_{\mathsf{X}}}$, which has rank $2n+1$. Since $\mathcal{A}$ is an extension

$$0 \longrightarrow \mathscr{O}_{\mathsf{X}} \longrightarrow \mathcal{A} \longrightarrow \mathscr{X}^1_{\mathsf{X}} \longrightarrow 0,$$

we have $c_j(\mathcal{A}_{\mathcal{L}}) = c_j(\mathsf{X})$ for all j. The statements about the codimension and fundamental class then follow from the general theory of skew-symmetric degeneracy loci as in [76, 85, 86].

Locally, we are interested in the vanishing of the submaximal Pfaffians of a $(2n+1) \times (2n+1)$ matrix of functions. The fact that such degeneracy loci are Gorenstein when the codimension is three, and that the free resolutions have the form in question, was proven by Buchsbaum and Eisenbud [28]. The global version of the resolution for skew-symmetric bundle maps is described by Okonek in [113]; we simply apply the formula therein to the tensor $\sigma_\nabla \in \Lambda^2\mathcal{A}$, noting that for the Atiyah algebroid $\mathcal{A}$, we have $\det \mathcal{A} \cong \omega_{\mathsf{X}}^{-1}$. $\square$

Notice that that every free divisor D is a degeneracy locus of the Lie algebroid $\mathcal{T}_{\mathsf{X}}(-\log \mathsf{D})$. Moreover, Buchsbaum and Eisenbud [28] showed that every codimension three Gorenstein scheme is the degeneracy locus of a skew form. These observations lead the author to wonder if there might be a sort of converse for Theorem 6.6.1:

Question 6.6.3. *Let $(\mathsf{D}, 0)$ be the germ of a reduced hypersurface in $\mathbb{C}^{2n}$ and suppose that the singular locus of D is Gorenstein of pure codimension three. Does there exist a germ of a log symplectic Poisson structure that has D as its degeneracy divisor?*

6.6.2 Codimension two: degeneracy in odd dimension

We now treat the case when X is odd-dimensional, say $\dim \mathsf{X} = 2k+1$ and the Poisson structure generically has rank $2k$.

We have the following straightforward generalization of [117, Theorem 13.1], which deals with the case when $\dim \mathsf{X} = 3$.

Proposition 6.6.4. *Suppose that (X, σ) is a Poisson manifold of dimension $2k+1$ and that Y is a connected component of the degeneracy locus $\mathsf{Dgn}_{2k-2}(\sigma)$ that is smooth of codimension two. Then the conormal sheaf $\mathcal{C} \subset \Omega^1_{\mathsf{X}}|_{\mathsf{Y}}$ is a sheaf of abelian Lie algebras, and $\omega_{\mathsf{Y}}^2 \cong \mathcal{O}_{\mathsf{Y}}$.*

Proof. The first statement is immediate from Proposition 6.4.3 and the fact that any two-dimensional unimodular Lie algebra is abelian. Let $\mathcal{N}^\vee \subset \Omega^1_{\mathsf{X}}|_{\mathsf{Y}}$ be the kernel of $\sigma^\sharp|_{\mathsf{Y}}$. Since $\mathcal{C}$ is abelian, the bracket on $\mathcal{N}$ descends to an action

$$\mathcal{N}^\vee/\mathcal{C} \otimes \mathcal{C} \to \mathcal{C}.$$

Since $\mathcal{N}^\vee/\mathcal{C}$ is isomorphic to the kernel of the anchor map $\Omega^1_{\mathsf{Y}} \to \mathscr{X}^1_{\mathsf{Y}}$, it is a line bundle isomorphic to ω_{Y}. We therefore obtain an isomorphism

$$\omega_{\mathsf{Y}} \otimes \mathcal{C} \cong \mathcal{C}$$

and the result follows from computing the determinants. $\square$

6.6.3 Codimension three: submaximal degeneracy in even dimension

Let (X, σ) be a Poisson manifold of dimension $2n$. Then the conormal Lie algebra N_x^* at a point $x \in \mathsf{X}$ where the rank of σ is equal to $2n-4$ is a four-dimensional Lie algebra. Using this fact, we can constrain the structure of the normal Higgs field at a smooth point of $\mathsf{Dgn}_{2n-4}(\sigma)$:

Proposition 6.6.5. *Let (X, σ) be a Poisson manifold of dimension $2n$, let $x \in \mathsf{Dgn}_{2n-4}(\sigma)$ be a smooth point. Then $\mathsf{Dgn}_{2n-4}(\sigma)$ has codimension at most three in X. Let $\mathsf{C}_x \subset \mathsf{T}_x^*\mathsf{X}$ be the conormal space. If the codimension of $\mathsf{Dgn}_{2n-4}(\sigma)$ is less than three, then C_x is abelian. If the codimension is equal to three, then one of the following three statements holds:*

1. *C_x is abelian.*
2. *$\dim([\mathsf{C}_x, \mathsf{C}_x]) = 1$ and C_x is isomorphic to the Heisenberg algebra $\langle x, y, z \,|\, [x, y] = z \rangle$.*
3. *$\dim([\mathsf{C}_x, \mathsf{C}_x]) = 3$ and C_x is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.*

Proof. Since $\mathsf{Dgn}_{2n-4}(\sigma)$ is smooth at x, our description of the Zariski tangent space in Proposition 6.4.2 implies that $\mathsf{C}_x = [\mathsf{N}_x^*, \mathsf{N}_x^*]$ is the commutator subalgebra of a four dimensional Lie algebra. The result now follows from the classification of four-dimensional Lie algebras [110] (see also [1]). □

Corollary 6.6.6. *Let (X, σ) be a complex Poisson manifold of dimension $2n$. Suppose that Y is an irreducible component of $\mathsf{Dgn}_{2n-4}(\sigma)$ that is reduced. Then the codimension of Y is at most three.*

6.7 Non-emptiness via Chern classes

Our goal in this section is to explain how Chern classes can often be used to verify that degeneracy loci are non-empty and bound their dimensions. The methods here differ substantially from the intersection-theoretic approach typical in the study of degeneracy problems. Indeed, by examining the different Poisson structures on $\mathbb{P}^3$ (see Chapter 8), it is apparent that there cannot be a universal intersection-theoretic formula for the fundamental classes of degeneracy loci in the Poisson setting: a generic Poisson structure on $\mathbb{P}^3$ vanishes on the union of a curve and a finite collection of points, but the degree of the curve and the number of isolated points depend on the particular Poisson structure that is chosen, and so they cannot be invariants of the underlying manifold.

As a result, new techniques are required. Instead of relying on intersection theory, we make use of the characteristic classes of Poisson modules. In particular, we give a vanishing theorem for the characteristic classes of adapted modules, and explain how it may be used to construct homology classes in the Poisson singular locus. To do so, we will need to use some basic results regarding the Borel–Moore homology of complex analytic spaces, which we presently recall. We refer the reader to [58, Chapter 19] for more details.

If X is a complex analytic space, we denote by $\mathsf{H}^\bullet(\mathsf{X}) = \mathsf{H}^\bullet(\mathsf{X}, \mathbb{C})$ its singular cohomology. A vector bundle $\mathcal{E}$ then has Chern classes

$$c_p(\mathcal{E}) \in \mathsf{H}^{2p}(\mathsf{X}).$$

We denote by $\mathsf{Chern}(\mathcal{E})$ the subring generated by the Chern classes.

Similarly we denote by $\mathsf{H}_\bullet(\mathsf{X})$ the Borel-Moore homology groups of X with complex coefficients. These groups satisfy the following properties:

1. If $\mathsf{Y} \subset \mathsf{X}$ is a closed subspace with complement $\mathsf{U} = \mathsf{X} \setminus \mathsf{Y}$, then there is a long exact sequence

$$\cdots \longrightarrow \mathsf{H}_j(\mathsf{Y}) \longrightarrow \mathsf{H}_j(\mathsf{X}) \longrightarrow \mathsf{H}_j(\mathsf{U}) \longrightarrow \mathsf{H}_{j-1}(\mathsf{Y}) \longrightarrow \cdots \tag{6.2}$$

2. If $\dim \mathsf{X} = n$, then $\mathsf{H}_j(\mathsf{X}) = 0$ for $j > 2n$, and $\mathsf{H}_{2n}(\mathsf{X})$ is the vector space freely generated by the n-dimensional irreducible components of X. Moreover, there is a fundamental class $[\mathsf{X}] \in \mathsf{H}_{2n}(\mathsf{X})$, which is a linear combination of these generators with coefficients given by the multiplicities of the components.

3. There are cap products $\mathsf{H}_j(\mathsf{X}) \otimes \mathsf{H}^k(\mathsf{X}) \to \mathsf{H}_{j-k}(\mathsf{X})$ satisfying the usual compatibilities.

Here, as always, $\dim \mathsf{X}$ denotes the complex dimension of X.

We have the following analogue of Bott's vanishing theorem in the case of adapted Poisson modules:

Theorem 6.7.1. *Let (X, σ) be a regular Poisson manifold of dimension n, and suppose that σ has constant rank $2k$. If $\mathcal{E}$ is an locally free, adapted Poisson module, then its Chern ring vanishes in degree $> 2(n-2k)$:*

$$\mathsf{Chern}^p(\mathcal{E}) = 0 \subset \mathsf{H}^p(\mathsf{X})$$

if $p > 2(n-2k)$.

Proof. Since X is smooth and σ has constant rank, the image of σ defines an involutive subbundle $\mathcal{F} \subset \mathcal{T}_\mathsf{X}$. Since $\mathcal{E}$ is an adapted Poisson module, the Poisson connection descends to a flat partial connection $\nabla : \mathcal{E} \to \mathcal{F}^\vee \otimes \mathcal{E}$ along $\mathcal{F}$. Therefore the proof of Bott's vanishing theorem [22] applies (substituting $\mathcal{E}$ for the normal bundle $\mathcal{T}_\mathsf{X}/\mathcal{F}$) and establishes the theorem. $\square$

Baum and Bott [9] applied Bott's vanishing theorem to a singular foliation, obtaining "residues" in the homology of the singular set that are Poincaré dual to the Chern classes of the normal sheaf of the foliation. In the same way, we apply Theorem 6.7.1 to a locus Y on which a Poisson module $\mathcal{L}$ is generically adapted, obtaining residues in the homology of the Poisson singular locus of Y that are dual to the Chern classes of $\mathcal{L}$. Notice that we do not require Y to be a manifold:

Corollary 6.7.2. *Let (Y, σ) be an irreducible Poisson analytic space of dimension n, and let $(\mathcal{L}, \nabla)$ be a Poisson line bundle. Suppose that the rank of σ is generically equal to $2k$ and the module is adapted away from the Poisson singular locus $\mathsf{Z} = \mathsf{Sing}(\mathsf{Y}, \sigma) = \mathsf{Y}_{sing} \cup \mathsf{Dgn}_{2k-2}(\sigma)$.*

If $p > n - 2k$, then there is a class $R_p \in \mathsf{H}_{2n-2p}(\mathsf{Z})$ such that

$$i_*(R_p) = c_1(\mathcal{L})^p \cap [\mathsf{Y}] \in \mathsf{H}_{2n-2p}(\mathsf{Y}),$$

where $i : \mathsf{Z} \to \mathsf{Y}$ is the inclusion. In particular, if

$$c_1(\mathcal{L})^{n-2k+1} \cap [\mathsf{Y}] \neq 0$$

then Z has a component of dimension $\geq 2k + 1$.

Proof. Let $\mathsf{Y}^\circ = \mathsf{Y} \setminus \mathsf{Z}$ be the regular locus. Since $\nabla|_{\mathsf{Y}^\circ}$ is adapted, the previous theorem informs us that $c_1(\mathcal{L}|_{\mathsf{Y}^\circ})^p = 0$, and hence

$$c_1(\mathcal{L}|_{\mathsf{Y}^\circ})^p \cap [\mathsf{Y}^\circ] = 0 \in \mathsf{H}_{2n-2p}(\mathsf{Y}^\circ).$$

Appealing to the long exact sequence (6.2), we find the desired class R_p. If

$$c_1(\mathcal{L})^p \cap [\mathsf{Y}] \neq 0,$$

it follows that $R_p \neq 0$, and so the statement regarding the dimension of Z now follows from the vanishing $\mathsf{H}_j(\mathsf{Z}) = 0$ for $j > 2 \dim \mathsf{Z}$. □

Remark 6.7.3. Under certain non-degeneracy conditions, Baum and Bott relate their homology classes to the local behaviour of the foliation near its singular set. One therefore wonders whether there may be a connection between the residue multiderivations defined in Section 5.5 and the homology classes in the Poisson singular locus. It would be interesting to clarify this issue. □

We say that a Poisson line bundle $(\mathcal{L}, \nabla)$ is ***ample*** if the underlying line bundle $\mathcal{L}$ is ample. We are particularly interested in applying Corollary 6.7.2 to ample Poisson modules, because the positivity of their Chern class implies the nonvanishing of the homology classes R_p described above, placing lower bounds on the dimensions of certain degeneracy loci.

Lemma 6.7.4. *Let* (X, σ) *be a projective Poisson variety and let* $(\mathcal{L}, \nabla)$ *be an ample Poisson line bundle. Suppose that* $k > 0$*, and that* Y *is a closed, strong Poisson subspace of* $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ *that is not contained in* $\mathsf{Dgn}_{2k-2}(\sigma)$*. Then the Poisson singular locus* $\mathsf{Sing}(\mathsf{Y}, \sigma|_\mathsf{Y})$ *is non-empty and has a component of dimension* $\geq 2k-1$*.*

Proof. Assume without loss of generality that Y is reduced an irreducible. Since Y is a strong Poisson subspace of $\mathsf{Dgn}_{2k}(\sigma_\nabla)$, the module ∇ restricts to a Poisson module on Y, and we have $\mathsf{Dgn}_{2k}(\sigma_{\nabla|_\mathsf{Y}}) = \mathsf{Y}$. Hence the pair $(\mathsf{Y}, \nabla|_\mathsf{Y})$ satisfies the hypotheses of Corollary 6.7.2.

Let d be the dimension of Y. Then $d \geq 2k$ because $\sigma|_\mathsf{Y}$ has generic rank $2k$. Since $k > 0$, we have

$$0 < d - 2k + 1 < d.$$

Since $\mathcal{L}$ is ample, so is $\mathcal{L}|_\mathsf{Y}$, and hence

$$c_1(\mathcal{L}|_\mathsf{Y})^{d-2k+1} \cap [\mathsf{Y}] \neq 0 \in \mathsf{H}_{2(2k-1)}(\mathsf{Y})$$

Therefore

$$\dim \mathsf{Sing}(\mathsf{Y}, \sigma|_\mathsf{Y}) \geq 2k - 1,$$

as required. □

Corollary 6.7.5. *Let* (X, σ) *be a projective Poisson variety and let* $(\mathcal{L}, \nabla)$ *be an ample Poisson line bundle. Suppose that* $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ *is non-empty and has a component of dimension* $\geq 2k-1$*. Then* $\mathsf{Dgn}_{2k-2}(\sigma)$ *is also non-empty and has a component of dimension* $\geq 2k-1$*.*

Proof. Select an irreducible component Y'_0 of $\mathsf{Dgn}_{2k}(\sigma_\nabla)$ of maximal dimension, and let $\mathsf{Y}_0 = (\mathsf{Y}'_0)_{red}$ be the reduced subspace. For $j > 0$, select by induction an irreducible component Y'_j of $\mathsf{Sing}(\mathsf{Y}_{j-1}, \sigma|_{\mathsf{Y}_{j-1}})$ of maximal dimension, and let $\mathsf{Y}_j = (\mathsf{Y}'_j)_{red}$. We therefore obtain a decreasing chain

$$\mathsf{Y}_0 \supset \mathsf{Y}_1 \supset \mathsf{Y}_2 \supset \cdots$$

of strong Poisson subspaces of $\mathsf{Dgn}_{2k}(\sigma_\nabla)$, with each inclusion strict.

Since X is Noetherian with respect to the Zariski topology, the chain must eventually terminate, and so there is a maximal integer J such that $\dim \mathsf{Y}_J \geq 2k-1$. We claim that the rank of $\sigma|_{\mathsf{Y}_J}$ is $\leq 2k-2$. Indeed, if it were not, then Y_J would satisfy the conditions of the previous lemma, and so we would find that Y_{J+1} is non-empty of dimension $\geq 2k-1$, contradicting the maximality of J. □

Corollary 6.7.6. *Suppose that* (X, σ) *is a projective Poisson analytic space which admits an ample Poisson line bundle (e.g., a Fano variety), and suppose that there exists a strong Poisson subspace* $\mathsf{W} \subset \mathsf{X}$ *of dimension* $2k$*. Then* $\mathsf{Dgn}_{2k-2}(\sigma)$ *has a component of dimension* $\geq 2k-1$*.*

Proof. W satisfies the hypotheses of Corollary 6.7.5 for any ample Poisson line bundle. □

6.8 Fano manifolds

Suppose that σ is a generically symplectic Poisson structure on a Fano manifold X with $\dim \mathsf{X} = 2n > 2$, and let $\mathsf{D} = \mathsf{Dgn}_{2n-2}(\sigma)$ be the hypersurface on which it degenerates. We claim that D must be singular: indeed, if D were smooth then the Poisson structure on D would have rank $2n-2$ at every point. In particular, we would have a nonzero section $\sigma^{n-1} \in \mathsf{H}^0\big(\mathsf{D}, \mathscr{X}_{\mathsf{D}}^{2n-2}\big)$, but this situation is impossible:

Proposition 6.8.1. *Suppose that X is a Fano manifold of dimension $n > 2$ and that $\mathsf{Y} \subset \mathsf{X}$ is a smooth anti-canonical divisor, then*

$$\mathsf{H}^0(\mathsf{Y}, \mathscr{X}_{\mathsf{Y}}^q) = 0$$

for $0 < q < n-1$.

Proof. The line bundle ω_{Y} is trivial by the adjunction formula We therefore have

$$\mathscr{X}_{\mathsf{Y}}^q \cong \Omega_{\mathsf{Y}}^{n-1-q}.$$

But if $0 < q < n-1$, then $0 < n-1-q < n-1$, and hence

$$\mathsf{H}^0\Big(\mathsf{Y}, \Omega_{\mathsf{Y}}^{n-1-q}\Big) \cong \mathsf{H}^0\Big(\mathsf{X}, \Omega_{\mathsf{X}}^{n-1-q}\Big) = 0$$

by the Lefschetz hyperplane theorem and the following well-known lemma. □

Lemma 6.8.2. *If X is a Fano manifold then $\mathsf{H}^0(\mathsf{X}, \Omega_{\mathsf{X}}^q) = 0$ for $q > 0$.*

Proof. By the Hodge decomposition theorem, the space in question is the complex conjugate of $\mathsf{H}^q(\mathsf{X}, \mathcal{O}_{\mathsf{X}})$, which is Serre dual to $\mathsf{H}^{n-q}(\mathsf{X}, \omega_{\mathsf{X}})$. But ω_{X} is anti-ample, so the latter space is zero by Kodaira's vanishing theorem. □

Now, using the non-emptiness of the singular locus, we can deduce the non-emptiness of the degeneracy locus:

Theorem 6.8.3. *Let (X, σ) be a connected Fano Poisson manifold of dimension $2n$. Then either $\mathsf{Dgn}_{2n-2}(\sigma) = \mathsf{X}$ or $\mathsf{Dgn}_{2n-2}(\sigma)$ is a hypersurface in X. Moreover, $\mathsf{Dgn}_{2n-4}(\sigma)$ is non-empty and has at least one component of dimension $\geq 2n-3$.*

Proof. If the rank of σ is less than $2n$ everywhere, then $\mathsf{Dgn}_{2n-2}(\sigma) = \mathsf{X}$, and $\mathsf{Dgn}_{2n-4}(\sigma)$ has dimension $\geq 2n-3$ by Theorem 6.1.2. So, we may assume that σ is generically symplectic, i.e., that the section $\sigma^n \in \mathsf{H}^0\big(\mathsf{X}, \omega_{\mathsf{X}}^{-1}\big)$ is non-zero.

Let $\mathsf{D} = \mathsf{Dgn}_{2n-2}(\sigma)$ be the zero locus of σ^n, which is non-empty of dimension $2n-1$ because X is Fano. By our discussion above, D must be singular. By Proposition 6.6.2, every component of its singular locus must have dimension at least $2n-3$. We are therefore in the situation of Corollary 6.7.5, and we conclude that $\mathsf{Dgn}_{2n-4}(\sigma)$ is non-empty with a component of dimension $\geq 2n-3$, as desired. □

Corollary 6.8.4. *Bondal's Conjecture 6.1.1 holds for Fano manifolds of dimension four.*

We can also obtain some evidence for the conjecture on odd-dimensional projective spaces. The following result shows, in particular, that for any Poisson structure on $\mathbb{P}^5$, there is at least one point at which the Poisson structure vanishes.

Theorem 6.8.5. *Let σ be a Poisson structure on $\mathbb{P}^{2k+1}$ with $k \geq 2$. Then $\mathsf{Dgn}_{2k-2}(\sigma)$ has a component of dimension at least $2k-1$, and $\mathsf{Dgn}_{2k-4}(\sigma)$ is non-empty.*

Proof. If the rank of σ is $< 2k$ everywhere, then $\mathsf{Dgn}_{2k-2}(\sigma) = \mathbb{P}^{2k+1}$ and $\mathsf{Dgn}_{2k-4}(\sigma)$ has a component of dimension $\geq 2k-3$ by Polishchuk's result (Theorem 6.1.2). So, we may suppose that the rank of σ is generically equal to $2k$.

By Theorem 6.1.2 again, we know that $\mathsf{Dgn}_{2k-2}(\sigma)$ has a component of dimension at least $\geq 2k-1$. Let Y be the reduced space underlying an irreducible component of $\mathsf{Dgn}_{2k-2}(\sigma)$ of maximal dimension. There are two possibilities: either $\dim \mathsf{Y} = 2k-1$, or Y is a hypersurface. If $\dim \mathsf{Y} = 2k-1$, we are in the situation of Lemma 6.8.6 and the theorem holds. So, let us assume that Y is a hypersurface.

If Y is smooth, then either $\sigma|_{\mathsf{Y}}$ is zero, in which case the theorem is proven, or the degree of Y is < 4 (see Lemma 7.3.5). In the latter case, Y is Fano of dimension $2k$, and hence $\mathsf{Dgn}_{2k-4}(\sigma|_{\mathsf{Y}})$ has a component of dimension $\geq 2k-3$ by Theorem 6.8.3.

If, on the other hand, Y is singular, then its singular locus Y_{sing} is a strong Poisson subspace of dimension $\leq 2k-1$. Once again, we are in the situation of Lemma 6.8.6 below and we conclude that $\mathsf{Dgn}_{2k-4}(\sigma)$ is non-empty. □

Lemma 6.8.6. *Suppose that σ is a Poisson structure on $\mathbb{P}^{2k+1}$ with $k \geq 2$. If there is a strong Poisson subspace $\mathsf{Y} \subset \mathbb{P}^{2k+1}$ of dimension $\leq 2k-1$, then $\mathsf{Dgn}_{2k-4}(\sigma)$ is non-empty.*

Proof. Assume without loss of generality that Y is reduced and irreducible. If the dimension of Y is less than $2k-2$, then the rank of σ on Y is at most $2k-4$, showing that $\mathsf{Dgn}_{2k-4}(\sigma)$ is non-empty. Hence, we may assume that the rank of $\sigma|_{\mathsf{Y}}$ is generically equal to $2k$.

If the dimension of Y is equal to $2k-2$, the module $\omega^{-1}_{\mathbb{P}^{2k+1}}|_{\mathsf{Y}}$ will be adapted on an open dense set. We are therefore in the situation of Corollary 6.7.6, and we again conclude that $\mathsf{Dgn}_{2k-4}(\sigma)$ is non-empty.

Suppose now that $\dim \mathsf{Y} = 2k-1$. If Y is singular, then its singular locus has dimension at most $2k-2$. Replacing Y with its singular locus, we can argue as in the previous two paragraphs to conclude that $\mathsf{Dgn}_{2k-4}(\sigma)$ is non-empty.

It remains to deal with the case when Y is smooth of dimension $2k-1$. In this case, let ∇ be the natural Poisson module structure on $\omega^{-1}_{\mathbb{P}^{2k+1}}|_{\mathsf{Y}}$. Since $\sigma^k|_{\mathsf{Y}} = 0$ and Y is strong, the tensor $\sigma^k_\nabla|_{\mathsf{Y}}$ defines a section $\mu \in \mathsf{H}^0\big(\mathsf{Y}, \mathscr{X}^{2k+1}_{\mathsf{Y}}\big)$ of the anticanonical line bundle of Y. If μ is nonvanishing, then the canonical bundle of Y is trivial and $\sigma^{k-1}|_{\mathsf{Y}}$ can be viewed as a global one-form $\alpha \in \mathsf{H}^0\big(\mathsf{Y}, \Omega^1_{\mathsf{Y}}\big)$. But $h^{1,0}(\mathsf{Y}) = 0$ by the Barth–Lefschetz theorem [8]. Therefore $\sigma^{k-1} = 0$, contradicting the fact that μ contains a factor of σ^{k-1}. We conclude that μ must vanish on a subspace $\mathsf{W} \subset \mathsf{Y}$ of codimension at most one. But then W is a subspace of $\mathsf{Dgn}_{2k-2}(\sigma_\nabla)$ of dimension at least $2k-2$, and so we conclude from Corollary 6.7.5 that $\mathsf{Dgn}_{2k-4}(\sigma)$ is non-empty. $\square$

Chapter 7

Poisson structures on projective spaces

In this chapter, we focus our attention on the geometry of Poisson structures on projective spaces. In particular, we discuss their cohomology and give constructions of some non-trivial examples. We do so for several reasons: first and foremost, projective spaces support a wide variety of interesting examples of Poisson structures, a property that seems to be relatively rare amongst all projective varieties. Indeed, as we shall see, a smooth hypersurface in projective space can only admit a Poisson structure if its degree is at most three. Second, since the algebraic geometry of projective spaces is well understood and fairly simple, we can hope to make some progress in understanding these examples and their classifications. Third, as we shall recall, Poisson structures on projective space are intimately connected with quadratic Poisson structures on vector spaces. The latter are, in some sense, the simplest class of Poisson structures beyond the Poisson structures on the dual of a Lie algebra and therefore are of significant interest in their own right. Finally, the quantizations of Poisson structures on projective spaces is a topic of particular interest in noncommutative algebraic geometry, a theme we shall explore in Chapter 8.

7.1 Review of Poisson cohomology

We begin with a brief review of Poisson cohomology. Let (X,σ) be a complex Poisson manifold. In [98], Lichnerowicz observed that the operation

$$d_\sigma = [\sigma,\cdot] : \mathscr{X}^\bullet_{\mathsf{X}} \to \mathscr{X}^{\bullet+1}_{\mathsf{X}}$$

of taking the Schouten bracket of a multivector field with σ gives rise to a complex

$$0 \longrightarrow \mathcal{O}_{\mathsf{X}} \xrightarrow{d_\sigma} \mathscr{X}^1_{\mathsf{X}} \xrightarrow{d_\sigma} \mathscr{X}^2_{\mathsf{X}} \xrightarrow{d_\sigma} \cdots,$$

called the ***Lichnerowicz–Poisson complex*** associated to σ. Up to some signs, this complex is simply the de Rham complex of the Lie algebroid $(\Omega^1_{\mathsf{X}}, [\cdot,\cdot]_\sigma, \sigma^\sharp)$ associated to σ.

Of particular interest are the low-degree pieces of this complex. A function $f \in \mathcal{O}_{\mathsf{X}}$ is d_σ-closed if and only if it is Casimir ($\{f,g\} = 0$ for all $g \in \mathcal{O}_{\mathsf{X}}$). Meanwhile a vector field $Z \in \mathscr{X}^1_{\mathsf{X}}$ is d_σ-closed if and only if $\mathscr{L}_Z\sigma = 0$, i.e., Z is an infinitesimal symmetry of σ—a Poisson vector field. Meanwhile the exact vector fields are those of the form $\sigma^\sharp(df)$ for a function $f \in \mathcal{O}_{\mathsf{X}}$—the Hamiltonian vector fields.

The ***Poisson cohomology of*** (X, σ) is the hypercohomology

$$\mathsf{H}^\bullet(\sigma) = \mathbb{H}^\bullet(\mathscr{X}^\bullet_{\mathsf{X}}, d_\sigma)$$

of this complex of sheaves; in other words, it is the Lie algebroid cohomology of $\Omega^1_{\mathsf{X},\sigma}$. In low degree, the Poisson cohomology groups have natural interpretations: $\mathsf{H}^0(\sigma)$ is the space of global Casimir functions, $\mathsf{H}^1(\sigma)$ is the tangent space to the Picard group of σ as in Corollary 2.5.6, $\mathsf{H}^2(\sigma)$ is the space of infinitesimal deformations of (X, σ) as a generalized complex manifold [70, Theorem 5.5], and $\mathsf{H}^3(\sigma)$ controls the obstructions to deformations.

When X is compact we have $\mathsf{H}^0(\sigma) = \mathbb{C}$ since every holomorphic function is constant. Let us denote by $\mathsf{Aut}(\sigma)$ the group of holomorphic automorphisms of X that preserve σ, which by [16] is a complex Lie group. Its Lie algebra $\mathfrak{aut}(\sigma)$ is isomorphic to the space of global Poisson vector fields—i.e., the closed global sections of $\mathscr{X}^1_{\mathsf{X}}$. By Corollary 2.5.6, there is a natural embedding

$$\mathfrak{aut}(\sigma) \hookrightarrow \mathsf{H}^1(\sigma),$$

and if the fundamental group of X is finite—for example, if X is a Fano variety—then this embedding is an isomorphism, so that $\mathsf{H}^0(\sigma)$ can be interpreted as the space of infinitesimal symmetries of σ.

7.2 Multivector fields in homogeneous coordinates

In this section, we describe how multivector fields on projective space are related to multivector fields in "homogeneous coordinates", i.e., multivector fields on the corresponding vector space. We begin by recalling the Helmholtz-type decomposition of polynomial multivector fields and quadratic Poisson structures on vector spaces as described in [19, 89, 99], and then explain the connection with projective space, following Bondal [19].

7.2.1 Helmholtz decomposition on vector spaces

Let V be a complex vector space of dimension $n+1$. Denote by $\mathscr{X}^\bullet_{\mathsf{V}}$ the space of global polynomial multivector fields on V. In other words, if we pick linear coordinates $x_0, \ldots, x_n$ on V, the coefficients of a section of $\mathscr{X}^\bullet_{\mathsf{V}}$ in these coordinates is required to be a polynomial function. Identifying polynomials with elements in the symmetric algebra $\mathsf{Sym}^\bullet\mathsf{V}^*$ of the dual, we see that there is a natural isomorphism

$$\mathscr{X}^k_{\mathsf{V}} \cong \bigoplus_{j\geq 0} \mathsf{Sym}^j\mathsf{V}^* \otimes \Lambda^k\mathsf{V}.$$

We denote by

$$\mathscr{X}^{k,l}_{\mathsf{V}} = \mathsf{Sym}^{k+l}\mathsf{V}^* \otimes \Lambda^k\mathsf{V}$$

multivector fields that are homogeneous of weight l for the action of $\mathbb{C}^*$.

Recall that the natural flat connection ∇ on $\Omega^{n+1}_{\mathsf{V}}$ coming from the standard trivialization of TV induces the so-called ***BV operator***

$$\delta : \mathscr{X}^\bullet_{\mathsf{V}} \to \mathscr{X}^{\bullet-1}_{\mathsf{V}}.$$

If $\omega \in \det \mathsf{V}^*$ is a constant volume form on V, and $P : \mathscr{X}^\bullet_{\mathsf{V}} \to \Omega^{\dim \mathsf{V}-\bullet}_{\mathsf{V}}$ the corresponding isomorphism, then

$$\delta = P^{-1} \circ d \circ P$$

where P is the exterior derivative. Thus $\delta^2 = 0$. As shown by Koszul [91], this operator generates the Schouten bracket in the sense that

$$[\xi, \eta] = (-1)^{|\xi|}\delta\xi \wedge \eta - \xi \wedge \delta\eta - (-1)^{|\xi|}\delta(\xi \wedge \eta)$$

for all $\xi \in \mathscr{X}^{|\xi|}_{\mathsf{V}}$ and $\eta \in \mathscr{X}^\bullet_{\mathsf{V}}$. We say that a multivector fields $\xi \in \mathscr{X}^\bullet_{\mathsf{V}}$ is ***divergence-free*** or ***solenoidal*** if $\delta\xi = 0$, and write $\mathscr{X}^\bullet_{\mathsf{V},sol}$ for the space of solenoidal multivector fields.

Let $E \in \mathscr{X}^1_{\mathsf{V}}$ be the Euler vector field on V. Thus $E = \sum_{j=0}^n x_j \partial_{x_j}$. We abuse notation and use the same symbol for the operator

$$E = E\wedge : \mathscr{X}^\bullet_{\mathsf{V}} \to \mathscr{X}^{\bullet+1}_{\mathsf{V}}$$

of wedging with E. We say that a multivector field $\xi \in \mathscr{X}^\bullet_{\mathsf{V}}$ is ***vertical*** if $E\xi = 0$, and denote by $\mathscr{X}^\bullet_{\mathsf{V},vert}$ the space of vertical multivector fields. The reason for this terminology is as follows: let $\mathbb{P} = \mathbb{P}(\mathsf{V})$ be the projective space of lines in V. Then the vertical multivector fields are exactly the ones which are killed by the projection $\mathsf{V} \setminus \{0\} \to \mathbb{P}(\mathsf{V})$.

The following facts are straightforward consequences of the definitions and the basic identities satisfied by the bracket:

Lemma 7.2.1. *We have*

$$[\mathscr{X}^\bullet_{\mathsf{V},sol}, \mathscr{X}^\bullet_{\mathsf{V},sol}] \subset \mathscr{X}^\bullet_{\mathsf{V},sol}$$

and

$$[\mathscr{X}^\bullet_{\mathsf{V},vert}, \mathscr{X}^\bullet_{\mathsf{V},vert}] \subset \mathscr{X}^\bullet_{\mathsf{V},vert}$$

Proposition 7.2.2. *We have the identity*

$$\delta E + E\delta = P^{-1} \circ \mathscr{L}_E \circ P$$

Proof. The operator P intertwines E with the interior product $\iota_E : \Omega^\bullet_{\mathsf{V}} \to \Omega^{\bullet-1}_{\mathsf{V}}$. The result then follows from the Cartan formula $\mathscr{L}_E = \iota_E d + d\iota_E$. ☐

Corollary 7.2.3 ([89])**.** *The space $\mathscr{X}^\bullet_{\mathsf{V}}$ of polynomial multivector fields on V decomposes as a direct sum*

$$\mathscr{X}^\bullet_{\mathsf{V}} = \mathscr{X}^\bullet_{\mathsf{V},sol} \oplus \mathscr{X}^\bullet_{\mathsf{V},vert},$$

Moreover, wedging with the Euler vector field gives an isomorphism

$$E : \mathscr{X}^{k-1}_{\mathsf{V},sol} \to \mathscr{X}^{k}_{\mathsf{V},vert}$$

for $1 \le k \le \dim \mathsf{V}$.

Proof. Consider the subspace $\mathscr{X}^{k,l}_{\mathsf{V}} \subset \mathscr{X}^{k}_{\mathsf{V}}$ with $k, l \ge 0$. Then P sends W to the subspace $P(\mathsf{W}) = \mathsf{Sym}^{k+l}\mathsf{V}^* \otimes \mathsf{\Lambda}^{n+1-k}\mathsf{V}^* \subset \Omega^{n+1-k}_{\mathsf{V}}$. Therefore $P(\mathsf{W})$ is an eigenspace of $\mathscr{L}_E$ with eigenvalue $n+1+l > 0$, and so we have the identity

$$(\delta E + E\delta)|_{\mathsf{W}} = (n+1+l)\mathrm{id}_{\mathsf{W}},$$

from which the result follows easily. ☐

By [117, Lemma 4.4], the modular vector field of a Poisson structure σ with respect to any constant volume form on V is given by $-\delta\sigma$. The following result now follows easily:

Theorem 7.2.4 ([19, 99])**.** *Every homogeneous quadratic Poisson structure $\sigma \in \mathscr{X}^{2,0}_{\mathsf{V}}$ has a unique decomposition*

$$\sigma = \sigma_0 + Z \wedge E$$

where σ_0 is a unimodular Poisson structure and $Z \in \mathscr{X}^{1,0}_{\mathsf{V}}$ is a divergence-free vector field such that $\mathscr{L}_Z \sigma_0 = 0$. In this case, the modular vector field of σ is $(\dim \mathsf{V})Z$.

7.2.2 Multivector fields on projective space

There is a close relationship between the Helmholtz decomposition of multivector fields on a vector space of dimension $n+1$ and the multivector fields on the corresponding projective space $\mathbb{P} = \mathbb{P}(\mathsf{V})$ of dimension n. Notice that a multivector field $\xi \in \mathscr{X}^k_{\mathsf{V}}$ descends to a multivector field on $\mathbb{P}$ if and only if it is invariant under the $\mathbb{C}^*$ action. This is equivalent to requiring that the coefficients of ξ be homogeneous polynomials of degree k, i.e. that $\xi \in \mathscr{X}^{k,0}_{\mathsf{V}} \subset \mathscr{X}^k_{\mathsf{V}}$. If ξ is homogeneous but not invariant, then it descends to multivector field on $\mathbb{P}$ with values in a line bundle. We formalize this correspondence as follows.

Denote by $\mathcal{O}_{\mathbb{P}}(-1)$ the tautological line bundle on $\mathbb{P}$ whose fibre over a point $\mathsf{W} \in \mathbb{P}$ is the line $\mathsf{W} \subset \mathsf{V}$. Its dual is the hyperplane bundle $\mathcal{O}_{\mathbb{P}}(1)$, and we set $\mathcal{O}_{\mathbb{P}}(l) = \mathcal{O}_{\mathbb{P}}(1)^{\otimes l}$ for $l \in \mathbb{Z}$. If $\mathcal{E}$ is a holomorphic vector bundle or a sheaf of $\mathcal{O}_{\mathbb{P}}$-modules, we set $\mathcal{E}(l) = \mathcal{E} \otimes \mathcal{O}_{\mathbb{P}}(l)$ for $l \in \mathbb{Z}$.

Recall that the tangent bundle of $\mathbb{P}$ sits in an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}} \longrightarrow \mathcal{V}(1) \longrightarrow \mathcal{T}_{\mathbb{P}} \longrightarrow 0,$$

where $\mathcal{V}$ is sheaf of sections of the trivial bundle over $\mathbb{P}$ with fibre V. Taking exterior powers of this sequence and tensoring with $\mathcal{O}_{\mathbb{P}}(l)$ we obtain the exact sequence

$$0 \longrightarrow \mathscr{X}^{k-1}_{\mathbb{P}}(l) \longrightarrow \wedge^k \mathcal{V}(l+1) \longrightarrow \mathscr{X}^k_{\mathbb{P}}(l) \longrightarrow 0.$$

The cohomology of these sheaves can be obtained from the Bott formulae [21] (see also [114, p. 8]) using the isomorphism $\omega_{\mathbb{P}} \cong \mathcal{O}_{\mathbb{P}}(-n-1)$:

Theorem 7.2.5. *The Betti numbers of the sheaves of twisted multivector fields on $\mathbb{P}^n$ are given by*

$$h^q(\mathbb{P}^n, \mathscr{X}^k_{\mathbb{P}^n}(l)) = \begin{cases} \binom{k+l+n+1}{l+n+1}\binom{l+n}{n-k} & \text{if } q = 0,\ 0 \le k \le n \text{ and } l > -(k+1) \\ 1 & \text{if } l = -(n+1) \text{ and } 0 \le q = n-k \le n \\ \binom{-(k+l+1)}{-(l+n+1)}\binom{-(l+n+2)}{k} & \text{if } q = n,\ 0 \le k \le n \text{ and } l < -(k+n+1) \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $l > -(n+1)$, the sheaves $\mathscr{X}^\bullet_{\mathbb{P}^n}(l)$ are acyclic, and so we obtain the short exact sequence

$$0 \longrightarrow \mathsf{H}^0\big(\mathbb{P}, \mathscr{X}^{k-1}_{\mathbb{P}}(l)\big) \xrightarrow{\iota} \mathsf{H}^0\big(\mathbb{P}, \wedge^k\mathcal{V}(k+l)\big) \xrightarrow{\pi} \mathsf{H}^0\big(\mathbb{P}, \mathscr{X}^k_{\mathbb{P}}(l)\big) \longrightarrow 0,$$

of global sections. Since the sections of $\mathcal{O}_{\mathbb{P}}(k+l)$ are naturally identified with the space

$\mathsf{Sym}^{k+l}\mathsf{V}^*$ of homogeneous polynomials of degree $k+l$, there is a natural isomorphism

$$\begin{aligned}\mathsf{H}^0\big(\mathbb{P},\Lambda^k\mathcal{V}(k+l)\big) &\cong \mathsf{H}^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(k+l))\otimes\Lambda^k\mathsf{V}\\ &\cong \mathsf{Sym}^{k+l}\mathsf{V}^*\otimes\Lambda^k\mathsf{V}\\ &\cong \mathscr{X}^{k,l}_{\mathsf{V}}\end{aligned}$$

with the space of weight-l multivector fields on V. As a result, we obtain a short exact sequence

$$0 \longrightarrow \mathsf{H}^0\big(\mathbb{P},\mathscr{X}^{k-1}_{\mathbb{P}}(l)\big) \xrightarrow{\ \iota\ } \mathscr{X}^{k,l}_{\mathsf{V}} \xrightarrow{\ \pi\ } \mathsf{H}^0\big(\mathbb{P},\mathscr{X}^{k}_{\mathbb{P}}(l)\big) \longrightarrow 0.$$

for $l > -(n+1)$. As we have observed, the kernel of π consists of the vertical multivector fields $\mathscr{X}^{k,l}_{\mathsf{V},vert}$, and so we obtain natural isomorphisms

$$\mathscr{X}^{k,l}_{\mathsf{V},sol} = \mathsf{H}^0\big(\mathbb{P},\mathscr{X}^{k}_{\mathbb{P}}(l)\big)$$

and

$$\mathscr{X}^{k,l}_{\mathsf{V},vert} = \mathsf{H}^0\big(\mathbb{P},\mathscr{X}^{k-1}_{\mathbb{P}}(l)\big)$$

for $l \geq -n$. We therefore have the

Theorem 7.2.6. *For a vector space* V *of dimension* $n+1$, *there are natural isomorphisms*

$$\mathscr{X}^{\bullet}_{\mathsf{V},vert} \cong \det\mathsf{V}\oplus\bigoplus_{l\geq -n}\mathsf{H}^0\big(\mathbb{P},\mathscr{X}^{\bullet-1}_{\mathbb{P}}(l)\big)$$

and

$$\mathscr{X}^{\bullet}_{\mathsf{V},sol} \cong \bigoplus_{l\geq -n}\mathsf{H}^0(\mathbb{P},\mathscr{X}^{\bullet}_{\mathbb{P}}(l))\,.$$

Proof. The polynomial multivector fields $\mathscr{X}^{k,l}_{\mathsf{V}}$ have $l \geq -k \geq -(n+1)$. Hence the discussion above works for all k and l except the case $k = l = n+1$, which corresponds to the constant covolumes on V. The latter are all vertical and give a one-dimensional vector space isomorphic to $\det\mathsf{V}$. ☐

In the case $k = 2$ and $l = 0$, we see that the decomposition of quadratic Poisson structures in Theorem 7.2.4 is equivalent to the following result of Bondal and Polishchuk

Corollary 7.2.7 ([19],[117])**.** *There is a natural bijective correspondence between the set of quadratic Poisson structures on* V *and the set of pairs* $(\sigma_{\mathbb{P}}, Z)$ *where* $\sigma_{\mathbb{P}}$ *is a Poisson structure on* $\mathbb{P}(\mathsf{V})$ *and* $Z \in \mathfrak{aut}(\sigma_{\mathbb{P}})$ *is a global Poisson vector field. In particular, every Poisson structure on* V *has a unique lift to a unimodular quadratic Poisson structure on* V.

Suppose that $\sigma_{\mathbb{P}}$ is a Poisson structure on $\mathbb{P}(\mathsf{V})$. Since $\mathcal{O}_{\mathbb{P}}(n+1) \cong \omega_{\mathbb{P}}^{-1}$ is the canonical bundle, it is naturally a Poisson module, and hence so is $\mathcal{O}_{\mathbb{P}}(1)$. Any other Poisson module structure on $\mathcal{O}_{\mathbb{P}}(1)$ differs from this one by a Poisson vector field. Hence, as Polishchuk observes [117, Section 12], there is also a natural bijective correspondence between quadratic Poisson structures on V and pairs $(\sigma_{\mathbb{P}}, \nabla)$ of a Poisson structure $\sigma_{\mathbb{P}}$ on $\mathbb{P}(\mathsf{V})$ together with $\sigma_{\mathbb{P}}$-module structure ∇ on $\mathcal{O}_{\mathbb{P}}(1)$.

7.2.3 A comparison theorem for quadratic Poisson structures

We will now explain how this decomposition can be extended to Poisson cohomology. Let $\sigma \in \mathscr{X}_{\mathsf{V}}^{2,0}$ be a quadratic Poisson structure on V. Thus $[\sigma, E] = 0$ because σ is homogeneous. Since every element of $\mathscr{X}_{\mathsf{V},vert}^{k}$ has the form $E \wedge \xi$ for some $\xi \in \mathscr{X}_{\mathsf{V}}^{\bullet}$, it follows from the Leibniz rule that the Poisson differential d_σ preserves $\mathscr{X}_{\mathsf{V},vert}^{\bullet}$. The exact sequence

$$0 \longrightarrow \mathscr{X}_{\mathsf{V},vert}^{\bullet} \longrightarrow \mathscr{X}_{\mathsf{V}}^{\bullet} \longrightarrow \mathscr{X}_{\mathsf{V},sol}^{\bullet} \longrightarrow 0$$

is therefore an exact sequence of complexes. Moreover, if σ is unimodular, so that $\sigma \in \mathscr{X}_{\mathsf{V},sol}^{2}$, this sequence of complexes splits because the solenoidal multivector fields are closed under the Schouten bracket by Lemma 7.2.1.

Corresponding to σ, we have a Poisson structure $\sigma_{\mathbb{P}}$ on $\mathbb{P}(\mathsf{V})$ and a Poisson module structure ∇ on $\mathcal{O}_{\mathbb{P}}(1)$. If $l \geq -n$, then all of the sheaves $\mathscr{X}_{\mathbb{P}(\mathsf{V})}^{k}(l)$ are acyclic and the Poisson cohomology $\mathsf{H}^{\bullet}(\sigma_{\mathbb{P}}, \nabla, \mathcal{O}_{\mathbb{P}}(l))$ reduces to the cohomology of the complex

$$0 \longrightarrow \mathsf{H}^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(l)) \xrightarrow{d_{\sigma_{\mathbb{P}}}^{\nabla}} \mathsf{H}^0\big(\mathbb{P}, \mathscr{X}_{\mathbb{P}}^{1}(l)\big) \xrightarrow{d_{\sigma_{\mathbb{P}}}^{\nabla}} \mathsf{H}^0\big(\mathbb{P}, \mathscr{X}_{\mathbb{P}}^{2}(l)\big) \longrightarrow \cdots$$

of global sections. Moreover, it is identified with the Poisson complexes $(\mathscr{X}_{\mathsf{V},vert}^{\bullet-1,l}, d_\sigma)$ and $(\mathscr{X}_{\mathsf{V},sol}^{\bullet,l}, d_\sigma)$ on V.

We arrive at the following result, comparing the Poisson cohomologies of V and $\mathbb{P}(\mathsf{V})$:

Theorem 7.2.8. *Let σ be a homogeneous quadratic Poisson structure on the vector space V of dimension $n+1$. Let $\sigma_{\mathbb{P}}$ be the corresponding Poisson structure on $\mathbb{P}(\mathsf{V})$ and ∇ the corresponding Poisson module structure on $\mathcal{O}_{\mathbb{P}}(l)$. Then there is a natural long exact* *sequence*

$$\begin{tikzcd}
 & 0 \arrow[r] & \mathsf{H}^0_{poly}(\sigma) \arrow[r] & \bigoplus_{l\geq -n}\mathsf{H}^0(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(l)) \arrow[dll] & \\
 & \bigoplus_{l\geq -n}\mathsf{H}^0(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(l)) \arrow[r] & \mathsf{H}^1_{poly}(\sigma) \arrow[r] & \bigoplus_{l\geq -n}\mathsf{H}^1(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(l)) \arrow[r] & \cdots \\
\cdots \arrow[r] & \bigoplus_{l\geq -n}\mathsf{H}^n(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(l)) \arrow[r] & \mathsf{H}^{n+1}_{poly}(\sigma) \arrow[r] & \det \mathsf{V} \arrow[r] & 0
\end{tikzcd}$$

where $\mathsf{H}^\bullet_{poly}(\sigma)$ *is the polynomial Poisson cohomology (the Poisson cohomology computed using polynomial multivector fields).*

In addition, when σ *is unimodular, the connecting homomorphisms vanish and we obtain isomorphisms*

$$\mathsf{H}^k_{poly}(\sigma) \cong \begin{cases} \bigoplus_{l\geq -n}\mathsf{H}^0(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(l)) & k=0 \\ \bigoplus_{l\geq -n}\mathsf{H}^{k-1}(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(l)) \oplus \bigoplus_{l\geq -n}\mathsf{H}^k(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(l)) & 1 \leq k \leq n \\ \det \mathsf{V} \oplus \bigoplus_{l\geq -n}\mathsf{H}^n(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(l)) & k = n+1. \end{cases}$$

Notice that the maps in the theorem are compatible with the natural gradings on $\bigoplus_{l\geq -n}\mathsf{H}^k\big(\sigma_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}(l)}\big)$ and $\mathsf{H}^k_{poly}(\sigma)$. (The latter grading is the one using the action of $\mathbb{C}^*$ on V by Poisson isomorphisms.)

This theorem may be of use in computing Poisson cohomology. On the one hand, one might wish to compute the cohomology of a Poisson structure $\sigma_{\mathbb{P}}$ on $\mathbb{P}^n$ by lifting it to a unimodular Poisson structure on V and computing the relevant piece of the polynomial Poisson cohomology. This procedure gives a very explicit finite-dimensional linear algebra computation, but the dimensions of the vector spaces involved grow very rapidly with the dimension of V, so a direct assault is difficult without the use of computers.

On the other hand, one might start with a quadratic Poisson structure on V and compute a piece of its cohomology by using the geometry of the corresponding Poisson structure on projective space. For example, we can prove the following:

Theorem 7.2.9. *Let* σ *be a homogeneous quadratic Poisson structure on* $\mathbb{C}^{2d+1}$ *whose rank is generically equal to* $2d$*. Let* f *be the homogeneous polynomial of degree* $2d+1$ *that is the coefficient of* $\sigma^d \wedge E$ *with respect to the standard basis. If* f *is irreducible, then the following statements hold:*

1. *The only linear vector fields preserving* σ *are constant multiples of the Euler field* E*. In particular,* σ *is unimodular.*

2. *Every polynomial Casimir function is a polynomial in* f*, i.e.* $\mathsf{H}^0_{poly}(\sigma) \cong \mathbb{C}[f]$*.*

3. *There is a natural embedding* $\mathbb{C}[f] \cdot E \subset \mathsf{H}^1_{poly}(\sigma)$.

4. *If* $l \geq 0$ *is an integer such that* $2d+1 | p^k l$ *for some prime* p*, then any divergence-free Poisson vector field for* σ *whose coefficients are homogeneous polynomials of degree* $l+1$ *is necessarily Hamiltonian.*

Proof. We consider the corresponding Poisson structure $\sigma_{\mathbb{P}}$ on projective space $\mathbb{P} = \mathbb{P}^{2d}$, which is generically symplectic and degenerates along the hypersurface $\mathsf{D} \subset \mathbb{P}$ defined by f. By Theorem 7.2.8, the first statement reduces to showing that the Poisson structure on $\mathbb{P}$ admits no Poisson vector fields, which was proven in [117, Proposition 15.1]. Since the modular vector field Z of σ must be linear and divergence-free, it follows that $Z = 0$ and hence σ is unimodular.

Furthermore, by Theorem 7.2.8, we may compute the Casimir functions for σ as the direct sum of the spaces of flat sections of $\mathcal{O}_{\mathbb{P}}(l)$ with $l \geq 0$. Since $\sigma_{\mathbb{P}}$ is symplectic on $\mathbb{P} \setminus \mathsf{D}$, any nonzero Poisson-flat section of $\mathcal{O}_{\mathbb{P}}(l)$ must be non-vanishing away from D. Since D is reduced and irreducible, every such section must be a power of the section $\sigma_{\mathbb{P}}^{2d}$ that defines D. But $\sigma_{\mathbb{P}}^{d}$ exactly corresponds to the Casimir function f, giving the second statement. The third statement follows from the second using Theorem 7.2.8 again.

Finally, by Theorem 7.2.8 once again, the fourth statement can be reduced to the computation to the first Poisson cohomology of $\mathcal{O}_{\mathbb{P}}(l)$, where it becomes a special case of the remarkable result [117, Theorems 15.3] of Polishchuk, proved using Galois coverings of the complement $\mathbb{P} \setminus \mathsf{D}$. □

7.3 Projective embedding

Let X be a projective analytic space, and suppose that $\mathcal{L}$ is a very ample line bundle on X. Let $\mathsf{V} = \mathsf{H}^0(\mathsf{X}, \mathcal{L})^\vee$. Thus $\mathcal{L}$ defines an embedding $\mathsf{X} \hookrightarrow \mathbb{P}(\mathsf{V})$. We are interested in the following

Question 7.3.1. *If* X *has a Poisson structure* σ_{X} *and* $\mathcal{L}$ *is a Poisson module, when does the embedding space* $\mathbb{P}(\mathsf{V})$ *inherit a Poisson structure that restricts to the given one on* V*?*

One sufficient—but highly restrictive—criterion is as follows:

Proposition 7.3.2. *Suppose that* X *is connected, and that the multiplication map*

$$\mathsf{Sym}^k \mathsf{V}^* \to \mathsf{H}^0\left(\mathsf{X}, \mathcal{L}^k\right)$$

is an isomorphism for $k = 2$ *and injective for* $k = 3$*. Then there is a unique pair* (σ, ∇) *where* σ *is a Poisson structure on* $\mathbb{P}(\mathsf{V})$ *and* ∇ *is a Poisson module structure on* $\mathcal{O}_{\mathbb{P}(\mathsf{V})}(1)$ *such that* $\sigma|_{\mathsf{X}} = \sigma_{\mathsf{X}}$ *and* $\mathcal{O}_{\mathbb{P}(\mathsf{V})}(1)|_{\mathsf{X}} \cong \mathcal{L}$ *as Poisson modules.*

Proof. A Poisson structure on $\mathbb{P}^n = \mathbb{P}(\mathsf{V})$ together with a Poisson module structure on $\mathcal{O}_{\mathbb{P}^n}(-1)$ is equivalent to the specification of a quadratic Poisson structure on V, i.e., a homogeneous Poisson bracket on the graded ring $\bigoplus_{k\geq 0} \mathsf{H}^0(\mathsf{D}, \mathcal{O}_\mathsf{D}(k)) \cong \mathsf{Sym}^\bullet \mathsf{V}^*$, where $\mathsf{V}^* = \mathsf{H}^0(\mathsf{X}, \mathcal{L})$. Since this ring is generated over $\mathbb{C}$ by elements of degree one, it suffices to define the bracket

$$\{\cdot,\cdot\} : \mathsf{V}^* \times \mathsf{V}^* \to \mathsf{Sym}^2\mathsf{V}^*$$

on elements of degree one and to check that the Jacobiator

$$\begin{array}{cccc} J & : & \mathsf{V}^* \times \mathsf{V}^* \times \mathsf{V}^* & \to & \mathsf{Sym}^3\mathsf{V}^* \\ & & (f,g,h) & \mapsto & \{f\{g,h\}\} + \{g\{h,f\}\} + \{h\{f,g\}\} \end{array}$$

vanishes. But the Poisson module structure on $\mathcal{L}$ induces a homogeneous bracket on the graded ring $\bigoplus_{k\geq 0} \mathsf{H}^0(\mathsf{X}, \mathcal{L}^k)$ and our assumptions allows us to transfer this bracket to obtain the data above on $\mathsf{Sym}^\bullet\mathsf{V}^*$. ☐

It would be interesting to produce some Poisson structure on projective spaces using this method, but it is not obvious how find subspaces of $\mathbb{P}^n$ that satisfy the constraints of the theorem and yet also admit Poisson structures with interesting Poisson modules. Nevertheless, the result does have some potential applications in classifying Poisson structures:

Corollary 7.3.3. *Let $\mathsf{D} \subset \mathbb{P}^n$ be a reduced effective divisor of degree at least four, and let σ and η be Poisson structures on $\mathbb{P}^n$ for which D is a Poisson divisor. Then $\sigma = \eta$ if and only if they have the same linearization along D.*

Proof. Since D is a Poisson divisor, there is unique σ-module structure ∇ on $\mathcal{O}_{\mathbb{P}^n}(\mathsf{D})$ for which the inclusion $\mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(\mathsf{D})$ is a morphism of Poisson modules. The normal bundle $\mathcal{N}_\mathsf{D} = \mathcal{O}_{\mathbb{P}^n}(\mathsf{D})|_\mathsf{D}$ therefore inherits the structure of a Poisson module, and this Poisson module on D is exactly the linearization of σ. Since the degree of D is at least four, the previous proposition applies and hence the map that sends the pair (σ, ∇) to the corresponding Poisson module structure on $\mathcal{N}_\mathsf{D}$ is injective. ☐

Proposition 7.3.4. *Let σ be a Poisson structure on $\mathbb{P}^n$ with $n \geq 4$, and suppose that D is a smooth Poisson divisor of degree $d \geq 4$. Then $\sigma = 0$.*

Proof. The following lemma show that D admits no Poisson structures. Hence a Poisson module structure on $\mathcal{N}_\mathsf{D}$ is specified uniquely by a vector field on D. But the proposition also show that D admits not global vector fields, and hence the linearization of σ is identically zero. By the previous corollary, σ itself must be zero. ☐

Lemma 7.3.5. *Let $\mathsf{D} \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Then*

$$\mathsf{H}^0(\mathsf{D}, \mathscr{X}^k_\mathsf{D}) = 0$$

for $0 < k < d-1$.

Proof. We have $\mathscr{X}^k_{\mathsf{D}} \cong \Omega^{n-1-k}_{\mathsf{D}} \otimes \omega^{-1}_{\mathsf{D}} \cong \Omega^{n-1-k}_{\mathsf{D}}(n+1-d)$ by the adjunction formula. Now apply the following lemma with $q = n-1-k$ and $l = n+1-d$. □

Lemma 7.3.6. *Let* $\mathsf{D} \subset \mathbb{P}^n$ *be a smooth hypersurface of degree* $d > 1$. *Then*

$$\mathsf{H}^0(\mathsf{D}, \Omega^q_{\mathsf{D}}(l)) = 0$$

for $q < n-1$ *and* $l \le q$.

Proof. For $l \in \mathbb{Z}$, consider the restriction exact sequence

$$0 \longrightarrow \Omega^q_{\mathbb{P}^n}(l-d) \longrightarrow \Omega^q_{\mathbb{P}^n}(l) \longrightarrow \Omega^q_{\mathbb{P}^n}(l)|_{\mathsf{D}} \longrightarrow 0.$$

Using the Bott formulae (see [21] or [114, p. 8]) we have $h^p(\mathbb{P}^n, \Omega^q_{\mathbb{P}^n}(l)) = 0$ for $p+q < n$ and $l \le q$. We conclude that

$$\mathsf{H}^p(\mathsf{D}, \Omega^q_{\mathbb{P}^n}(l)|_{\mathsf{D}}) = 0, \text{ for } p+q < n-1 \text{ and } l < q. \tag{7.1}$$

The exact sequence

$$0 \longrightarrow \mathcal{N}^\vee_{\mathsf{D}} \longrightarrow \Omega^1_{\mathbb{P}^n}|_{\mathsf{D}} \longrightarrow \Omega^1_{\mathsf{D}} \longrightarrow 0$$

and the identification $\mathcal{N}^\vee_{\mathsf{D}} \cong \mathcal{O}_{\mathsf{D}}(-d)$ give the exact sequence

$$0 \longrightarrow \Omega^{q-1}_{\mathsf{D}}(l-d) \longrightarrow \Omega^q_{\mathbb{P}^n}(l)|_{\mathsf{D}} \longrightarrow \Omega^q_{\mathsf{D}}(l) \longrightarrow 0$$

for $l \in \mathbb{Z}$. By repeated use of the corresponding long exact sequence and Equation 7.1, we find a sequence of injections

$$\mathsf{H}^0(\mathsf{D}, \Omega^q_{\mathsf{D}}(l)) \hookrightarrow \mathsf{H}^1\Big(\mathsf{D}, \Omega^{q-1}_{\mathsf{D}}(l-d)\Big) \hookrightarrow \cdots \hookrightarrow \mathsf{H}^q(\mathsf{D}, \mathcal{O}_{\mathsf{D}}(l-qd))$$

for $q < n-1$ and $l \le q$. But the rightmost group vanishes for $d > 1$ by the Kodaira–Nakano vanishing theorem, giving the result. □

7.4 Poisson structures on $\mathbb{P}^n$ admitting a normal crossings anticanonical divisor

The structure of anticanonical Poisson divisors on projective space is heavily constrained by the following theorem of Polishchuk:

Theorem 7.4.1 ([117, Theorem 11.1]). *Suppose that σ is a nonzero Poisson structure on $\mathbb{P}^n$ and that $\mathsf{D} \subset \mathbb{P}^n$ is an anti-canonical Poisson divisor that is the union of m smooth components with normal crossings. Then $m \geq n-1$.*

It follows that we must have

$$\mathsf{D} = \mathsf{H} + \mathsf{D}'$$

where $\mathsf{H} = \mathsf{H}_1 + \cdots + \mathsf{H}_{n-3}$ is a union of hyperplanes, and $\mathsf{D}' = \mathsf{D}_1 + \ldots + \mathsf{D}_k$ is a degree-four divisor with at least two components. Hence the only possibilities for the degrees $(d_1, \ldots, d_k)$ of the components of D' are $(1,1,1,1)$, $(1,1,2)$, $(1,3)$ and $(2,2)$.

Definition 7.4.2. A Poisson structure on $\mathbb{P}^n$ has ***normal crossings type*** $(d_1, \ldots, d_k)$ if it admits a normal crossings anti-canonical Poisson divisor that is the union of $n-3$ hyperplanes and $k \geq 2$ smooth components of degrees $d_1 \leq \ldots \leq d_k$.

Remark 7.4.3. For $\mathbb{P}^3$, each of the four normal crossings types gives an irreducible component in the space of Poisson structures; see Chapter 8. □

Notice that if D is a normal crossings anticanonical divisor, then it is free, and Saito's criterion (Theorem 2.3.3) gives $\det \mathcal{T}_{\mathbb{P}^n}(-\log \mathsf{D}) = \omega_{\mathbb{P}^n}^{-1}(-\mathsf{D}) \cong \mathcal{O}_{\mathbb{P}^n}$. We conclude that $\mathscr{X}^2_{\mathbb{P}^n}(-\log \mathsf{D}) \cong \Omega^{n-2}_{\mathbb{P}^n}(\log \mathsf{D})$, and hence, after a choice of trivialization for $\omega^{-1}_{\mathbb{P}^n}(-\mathsf{D})$, a Poisson structure of normal crossings type is uniquely determined by a logarithmic $(n-2)$-form satisfying an appropriate integrability condition. The complete description of such Poisson structures will be the subject of future work by the author; for now, we give some examples of type $(1,1,1,1)$ and $(1,3)$:

Example 7.4.4. If we choose homogeneous coordinates $x_0, \ldots, x_n$, any bracket of the form

$$\{x_i, x_j\} = \lambda_{ij} x_i x_j$$

with $\lambda_{ij} = -\lambda_{ji} \in \mathbb{C}$ for $0 \leq i, j \leq n$ is a Poisson bracket, and induces a Poisson structure of normal crossings type $(1,1,1,1)$ on $\mathbb{P}^n$. Such a bracket is sometimes called ***log canonical*** or ***skew polynomial***. □

Example 7.4.5. Suppose that $x_0, \ldots, x_4$ are linear coordinates on $\mathbb{C}^5$ and f is a homogeneous cubic polynomial. The two-form

$$\widetilde{\alpha} = 3\frac{dx_0}{x_0} \wedge \frac{dx_1}{x_1} - \frac{dx_0}{x_0} \wedge \frac{df}{f} + \frac{dx_1}{x_1} \wedge \frac{df}{f}$$

is invariant under the action of $\mathbb{C}^*$ and the interior product $\iota_E \widetilde{\alpha}$ with the Euler vector field is zero. Hence $\widetilde{\alpha}$ descends to a two-form α on $\mathbb{P}^4$ with logarithmic singularities along the divisor $\mathsf{D} = \mathsf{H}_0 + \mathsf{H}_1 + \mathsf{Y}$, where H_0 and H_1 are the hyperplanes corresponding to x_0 and x_1, and Y is the cubic hypersurface defined by f. The two-form α is closed and has rank two.

It therefore defines a Poisson structure on $\mathbb{P}^4$ by the formula

$$\{g, h\} = \langle dg \wedge dh \wedge \alpha, \mu \rangle$$

where μ is any fixed element of the one-dimensional space of sections of $\omega_{\mathbb{P}^4}^{-1}(-\mathsf{D})$. This Poisson structure generically has rank two. The rational functions $\frac{x_0^3}{f}$ and $\frac{x_1^3}{f}$ on $\mathbb{P}^4$ are Casimir, and hence the symplectic leaves are the surfaces in the net spanned by the triple planes $3\mathsf{H}_0$ and $3\mathsf{H}_1$ and the cubic Y. $\square$

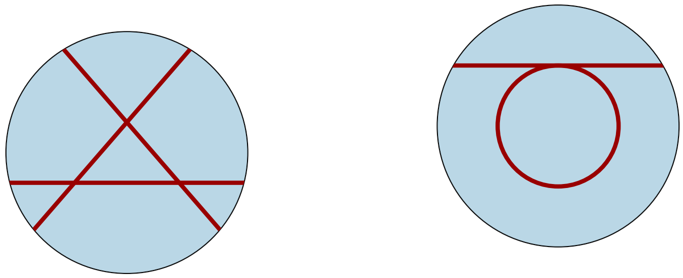
7.5 Poisson structures associated with linear free divisors

Recall from Section 2.3 that a free divisor in a complex manifold X is a hypersurface $\mathsf{Y} \subset \mathsf{X}$ for which the sheaf $\mathscr{X}_{\mathsf{X}}^1(-\log \mathsf{Y})$ of logarithmic vector fields is locally free. When $\mathsf{X} = \mathbb{C}^{d+1}$ there is a special class of free divisors—the ***linear free divisors*** [66]—for which $\mathcal{T}_{\mathbb{C}^{d+1}}(-\log \mathsf{Y})$ has a basis made up out of $d+1$ homogeneous linear vector fields. It follows immediately from Saito's criterion (Theorem 2.3.3) that Y must be homogeneous degree $d+1$, and therefore its projectivization $\mathsf{D} = \mathbb{P}(\mathsf{Y}) \subset \mathbb{P}^d$ is an anticanonical divisor for which $\mathscr{X}_{\mathsf{X}}^1(-\log \mathsf{D})$ is a *globally* trivial vector bundle.

The divergence-free vector linear vector fields tangent to Y form a Lie algebra of dimension d, which is identified with the space $\mathfrak{g} = \mathsf{H}^0\big(\mathbb{P}^d, \mathscr{X}_{\mathsf{X}}^1(-\log \mathsf{D})\big)$ of global logarithmic vector fields on $\mathbb{P}^d$. Therefore, $\mathfrak{g}$ is a Lie algebra of dimension d which acts faithfully on $\mathbb{P}^d$. Hence, for any element $\gamma \in \Lambda^2\mathfrak{g}$ of rank $2k$ with $[\gamma, \gamma] = 0$ (a triangular r-matrix; see Example 4.2.5), we obtain a nonzero Poisson structure on $\mathbb{P}^d$ of generic rank $2k$. In particular, if $d = 2n$ is even and we find an r-matrix of rank $2n$, then we obtain a generically symplectic Poisson structure σ on $\mathbb{P}^{2n}$ whose degeneracy hypersurface is the free divisor: $\mathsf{Dgn}_{2n-2}(\sigma) = \mathsf{D}$.

Linear free divisors in $\mathbb{C}^{d+1}$ for $d \leq 3$ were classified in [66]. For $d = 2$, there are two isomorphism types: the hypersurfaces with equations of the form $xyz = 0$ or $(y^2 + xz)z = 0$. The corresponding generically symplectic Poisson structures on $\mathbb{P}^2$ are the ones for which the cubic curve of zeros is the union of three lines in general position or the union of a smooth conic and a tangent line, illustrated in Figure 7.1.

Work in progress of B. Pike gives a conjectural classification of linear free divisors in $\mathbb{C}^5$. We thank him for sharing his work with us. Using his list, one can construct many examples of log symplectic Poisson structures on $\mathbb{P}^4$. We close this section by exploring one such example, which is similar in spirit to the normal crossings type $(1, 3)$, except that the cubic hypersurface is singular. In contrast to the example of type $(1, 3)$ given in the previous section, this Poisson structure is generically symplectic.



(a) $xyz = 0$ (b) $(y^2 + xz)z = 0$

Figure 7.1: Log symplectic Poisson structures on the projective plane associated with linear free divisors in $\mathbb{C}^3$.

Example 7.5.1. Let $x_1, \ldots, x_5$ be coordinates on $\mathbb{C}^5$. The zero set of the homogeneous quintic polynomial

$$f = x_3x_5(x_1x_3^2 - x_3x_4x_5 + x_2x_5^2)$$

defines a linear free divisor $\mathsf{Y} \subset \mathbb{C}^5$. A straightforward calculation using Saito's criterion (Theorem 2.3.3) shows that the linear vector fields

$$\begin{aligned}
Z_1 &= x_3\partial_{x_2} + x_5\partial_{x_4} \\
Z_2 &= 4x_1\partial_{x_1} - 6x_2\partial_{x_2} - x_3\partial_{x_3} - x_4\partial_{x_4} + 4x_5\partial_{x_5} \\
Z_3 &= x_5\partial_{x_1} + x_3\partial_{x_4} \\
Z_4 &= -6x_1\partial_{x_1} + 4x_2\partial_{x_2} + 4x_3\partial_{x_3} - x_4\partial_{x_4} - x_5\partial_{x_5}
\end{aligned}$$

together with the Euler vector field

$$E = \sum_{i=1}^{5} x_i\partial_{x_i}$$

form a global basis for $\mathcal{T}_{\mathbb{C}^5}(-\log \mathsf{Y})$. One readily computes that the only nontrivial Lie brackets between the vector fields are

$$[Z_2, Z_1] = 5Z_1$$

and

$$[Z_4, Z_3] = 5Z_3.$$

As a result, the bivector field $Z_1 \wedge Z_2 + Z_3 \wedge Z_4$ is a quadratic Poisson structure on $\mathbb{C}^5$. The elementary brackets have the form

$$\begin{aligned}
\{x_1, x_2\} &= 4x_2x_5 - 4x_1x_3 & \{x_3, x_5\} &= 0 \\
\{x_1, x_3\} &= 4x_3x_5 & \{x_2, x_5\} &= 4x_3x_5 \\
\{x_1, x_4\} &= 6x_1x_3 - 4x_1x_5 - x_4x_5 & \{x_2, x_4\} &= 6x_2x_5 - 4x_2x_3 - x_3x_4 \\
\{x_1, x_5\} &= -x_5^2 & \{x_2, x_3\} &= -x_3^2 \\
\{x_4, x_5\} &= 4x_5^2 - x_3x_5 & \{x_3, x_4\} &= x_3x_5 - 4x_3^2.
\end{aligned}$$

Let us denote by σ the corresponding Poisson structure on $\mathbb{P}^4$. Then σ is generically symplectic and degenerates along the hypersurface $\mathsf{D} = \mathsf{Dgn}_{2n-2}(\sigma) = \mathbb{P}(\mathsf{Y}) \subset \mathbb{P}^4$. This hypersurface is the union $\mathsf{D} = \mathsf{H}_1 \cup \mathsf{H}_2 \cup \mathsf{W}$ of the two hyperplanes $\mathsf{H}_1, \mathsf{H}_2$ defined by $x_3 = 0$ and $x_5 = 0$, and the singular cubic W defined by $x_1x_3^2 - x_3x_4x_5 + x_2x_5^2 = 0$. The set-theoretic singular locus of W is the intersection of H_1 and H_2

The hyperplanes H_1 and H_2 are strong Poisson subspaces. Moreover $\mathsf{W} \cap \mathsf{H}_1$ is also a strong Poisson subspace, and is the union of the plane defined by the ideal (x_3, x_2) and the triple plane defined by (x_3, x_5^3). A similar description holds for $\mathsf{W} \cap \mathsf{H}_2$.

The Poisson vector fields for this Poisson structure are readily computed: since D is a strong Poisson subspace, every Poisson vector field must be tangent to D and must therefore be a linear combination of Z_1, Z_2, Z_3 and Z_4. Using the bracket relations, one easily verifies that the only linear combinations that annihilate σ are the linear combinations Z_1 and Z_3. Hence

$$\mathsf{H}^1(\sigma) = \mathfrak{aut}(\sigma) = \mathbb{C}\langle Z_1, Z_3 \rangle$$

is an abelian Lie algebra of dimension two. This Lie algebra integrates to an action of $\mathbb{C}^2$ on $\mathbb{P}^4$ by translations, and so the closure of the orbit through a point $p \in \mathbb{P}^4 \setminus \mathsf{D}$ is a two-plane $\mathbb{P}^2_p \subset \mathbb{P}^4$. From the form of the Poisson structure, we see that this two-plane is coisotropic. In particular, its intersection with the symplectic manifold $\mathbb{P}^4 \setminus \mathsf{D}$ is Lagrangian.

If we let $\mathsf{G} = \mathsf{Gr}(3, 5)$ be the Grassmannian parameterizing projective two-planes in $\mathbb{P}^4$, there is a two-dimensional subvariety $\mathsf{V} \subset \mathsf{G}$ consisting of two-planes to which Z_1 and Z_2 are tangent. We therefore have a rational map

$$\Phi : \mathbb{P}^4 \dashrightarrow \mathsf{V}$$

which sends a point $p \in \mathbb{P}^4 \setminus \mathsf{D}$ to the closure of its $\mathbb{C}^2$-orbit, giving a Lagrangian fibration. $\square$

7.6 Feigin–Odesskii elliptic Poisson structures

In this section, we apply our methods to Poisson structures on projective space associated with elliptic normal curves. Let X be an elliptic curve, and let $\mathcal{E}$ be a stable vector bundle over X of rank r and degree d, with $\gcd(r,d)=1$. Feigin and Odesskii [56, 57] have explained that the projective space $\mathbb{P}^{d-1}=\mathbb{P}(\mathsf{Ext}^1_{\mathsf{X}}(\mathcal{E},\mathcal{O}_{\mathsf{X}}))$ inherits a natural Poisson structure; see also [118] for a generalization of their construction.

For the particular case when $\mathcal{E}=\mathcal{L}$ is a line bundle of degree d, the projective space is

$$\mathbb{P}^{d-1}=\mathbb{P}(\mathsf{H}^1(\mathsf{X},\mathcal{L}^\vee))=\mathbb{P}(\mathsf{H}^0(\mathsf{X},\mathcal{L})^\vee)$$

by Serre duality. Hence X is embedded in $\mathbb{P}^{d-1}$ as an elliptic normal curve of degree d by the complete linear system associated to $\mathcal{L}$.

The vector space $\mathsf{H}^1(\mathsf{X},\mathcal{L}^\vee)$ inherits a homogeneous Poisson structure [19, 56]. This homogeneous Poisson structure has a unique homogeneous symplectic leaf of dimension $2k+2$, given by the cone over the k-secant[1] variety $\mathsf{Sec}_k(\mathsf{X})$ [56]—that is, the union of all the k-planes in $\mathbb{P}^{d-1}$ which pass through $k+1$ points on X (counted with multiplicity). It follows that the induced Poisson structure σ on $\mathbb{P}^{d-1}$ has rank $2k$ on $\mathsf{Sec}_k(\mathsf{X})\setminus\mathsf{Sec}_{k-1}(\mathsf{X})$, and so

$$\mathsf{Dgn}_{2k}(\sigma)\cap\mathsf{Sec}_{k+1}(\mathsf{X})=\mathsf{Sec}_k(\mathsf{X})$$

for all $k<(d-1)/2$. When $d=2n+1$ is odd, we have the equality

$$\mathsf{Sec}_k(\mathsf{X})=\mathsf{Dgn}_{2k}(\sigma)_{red}$$

of the underyling reduced schemes, because σ is symplectic on $\mathbb{P}^{2n}\setminus\mathsf{Sec}_{n-1}(\mathsf{X})$. When d is even, though, $\mathsf{Dgn}_{2k}(\sigma)$ may have additional components. For example, when $d=4$, the Poisson structure on $\mathbb{P}^3$ vanishes on the elliptic curve $\mathsf{X}\subset\mathbb{P}^3$ together with four isolated points [117]. We believe that $\mathsf{Sec}_k(\mathsf{X})$ is always a component of $\mathsf{Dgn}_{2k}(\sigma)$ and that the latter scheme is reduced; see Conjecture 7.6.8.

The secant varieties have been well-studied:

Theorem 7.6.1 ([65, §8]). *The secant variety $\mathsf{Sec}_k(\mathsf{X})$ enjoys the following properties:*

1. $\mathsf{Sec}_k(\mathsf{X})$ *has dimension*
$$\dim\mathsf{Sec}_k(\mathsf{X})=2k+1$$
and degree
$$\deg\mathsf{Sec}_k(\mathsf{X})=\binom{d-k-2}{k}+\binom{d-k-1}{k+1}.$$

2. $\mathsf{Sec}_k(\mathsf{X})$ *is normal and arithmetically Gorenstein. Its dualizing sheaf is trivial.*

3. $\mathsf{Sec}_k(\mathsf{X})\setminus\mathsf{Sec}_{k-1}(\mathsf{X})$ *is smooth.*

[1] We warn the reader that our index k differs from other sources.

Remark 7.6.2. Since $\mathsf{Sec}_k(\mathsf{X})$ is contained in $\mathsf{Dgn}_{2k}(\sigma)$, we have that

$$\dim \mathsf{Dgn}_{2k}(\sigma) \geq 2k+1$$

for $2k < d-1$ in accordance with Conjecture 6.1.1. Indeed, Bondal cited these examples as motivation for his conjecture. $\square$

With this information in hand, we can now make a conclusion about the singular scheme of the highest secant variety of an odd-degree curve:

Theorem 7.6.3. *Let $\mathsf{X} \subset \mathbb{P}^{2n}$ be an elliptic normal curve of degree $2n+1$, and let $\mathsf{Y} = \mathsf{Sec}_{n-1}(\mathsf{X})$, a hypersurface of degree $2n+1$. Then the singular subscheme Y_{sing} of Y is a $(2n-3)$-dimensional Gorenstein scheme of degree $2\binom{2n+2}{3}$ whose dualizing sheaf is $\mathcal{O}_{\mathsf{Y}_{sing}}(2n+1)$.*

Proof. By the discussion above, Y is the reduced scheme underlying the degeneracy locus $\mathsf{Dgn}_{2n-2}(\sigma)$, which is an anti-canonical divisor. Since the degree of $\omega^{-1}_{\mathbb{P}^{2n}}$ is $2n+1$, we must have $\mathsf{Y} = \mathsf{Dgn}_{2n-2}(\sigma)$ as schemes. The singular subscheme is supported on the subvariety $\mathsf{Sec}_{n-2}(\mathsf{X})$, which has codimension three in $\mathbb{P}^{2n}$. By Theorem 6.6.1, Y_{sing} is Gorenstein with dualizing sheaf given by the restriction of $\omega^{-1}_{\mathbb{P}^{2n}} \cong \mathcal{O}_{\mathbb{P}^n}(2n+1)$. Its fundamental class is given by $c_1c_2 - c_3$, where c_j is the j^{th} Chern class of $\mathbb{P}^{2n}$. But c_j has degree $\binom{2n+1}{j}$, and hence one obtains the formula for the degree of Y_{sing} by a straightforward computation. $\square$

Corollary 7.6.4. *In the situation of the previous theorem, Y_{sing} is not reduced. Its geometric multiplicity (that is, the length of its structure sheaf at the generic point) is equal to eight.*

Proof. The reduced scheme underlying Y_{sing} is $\mathsf{Sec}_{n-2}(\mathsf{X})$. Using the formula in Theorem 7.6.1, we find that the degree of Y_{sing} is eight times that of $\mathsf{Sec}_{n-2}(\mathsf{X})$, and the result follows. $\square$

We now return to the case when d is arbitrary and study the modular residues:

Proposition 7.6.5. *The modular residue*

$$\mathrm{Res}^k_{mod}(\sigma) \in \mathsf{H}^0\Big(\mathsf{Dgn}_{2k}(\sigma)\,,\mathscr{X}^{2k+1}_{\mathsf{Dgn}_{2k}(\sigma)}\Big)$$

induces a non-vanishing multiderivation on $\mathsf{Sec}_k(\mathsf{X}) \subset \mathsf{Dgn}_{2k}(\sigma)$, in accordance with the fact that the dualizing sheaf of $\mathsf{Sec}_k(\mathsf{X})$ is trivial.

Proof. Because $\omega_{\mathbb{P}^{d-1}} \cong \mathcal{O}_{\mathbb{P}^{d-1}}(-d)$, the tautological bundle is a Poisson module. With respect to this Poisson module structure, the residues of $\mathcal{O}_{\mathbb{P}^{d-1}}(-1)$ are non-zero multiples of the residues of $\omega_{\mathbb{P}^{d-1}}$. We may therefore work with $\mathcal{O}_{\mathbb{P}^{d-1}}(-1)$ instead of $\omega_{\mathbb{P}^{d-1}}$.

Let $\mathsf{V} = \mathsf{H}^1(\mathsf{X}, \mathcal{L}^\vee)$ so that $\mathbb{P}^{d-1} = \mathbb{P}(\mathsf{V})$. Let V' be the total space of the tautological bundle $\mathcal{O}_{\mathbb{P}^{d-1}}(-1)$ and let $\pi : \mathsf{V}' \to \mathbb{P}^{d-1}$ be the projection. Being the total space of an invertible Poisson module, V' is a Poisson variety. For a subscheme $\mathsf{Y} \subset \mathbb{P}^{d-1}$, the preimage $\pi^{-1}(\mathsf{Y})$ in V' is a Poisson subscheme if and only if Y is a Poisson subscheme of $\mathbb{P}^{d-1}$ and $\nabla|_{\mathsf{Y}}$ is a Poisson module. (This fact follows from the definition of the Poisson bracket on V' in Section 5.4.)

The key point is that the blowdown map $\mathsf{V}' \to \mathsf{V}$ is a Poisson morphism; see Remark 7.6.6. Let $\mathsf{Y} = \mathsf{Sec}_k(\mathsf{X}) \setminus \mathsf{Sec}_{k-1}(\mathsf{X})$ and $\mathsf{Y}' = \pi^{-1}(\mathsf{Y})$. Then the blowdown identifies $\mathsf{Y}' \setminus 0$ with the cone over Y in V, which is a $2k$-dimensional symplectic leaf of the Poisson structure. Hence $\mathsf{Y}' \setminus 0$ is a symplectic leaf in V'. In particular, it is a Poisson subvariety, and so $\mathsf{Y} \subset \mathbb{P}^{d-1}$ is a Poisson subscheme to which $\mathcal{O}_{\mathbb{P}^{d-1}}(-1)$ restricts as a Poisson module. It follows that the residue is tangent to Y. By Remark 6.5.1, the residue is actually non-vanishing on Y.

To conclude that the residue gives a non-vanishing multiderivation on all of $\mathsf{Sec}_k(\mathsf{X})$, we note that the sheaf $\mathscr{X}^{2k+1}_{\mathsf{Sec}_k(\mathsf{X})}$ is reflexive. We know from Theorem 7.6.1 that $\mathsf{Sec}_k(\mathsf{X})$ is normal and $\mathsf{Sec}_{k-1}(\mathsf{X}) = \mathsf{Sec}_k(\mathsf{X}) \setminus \mathsf{Y}$ has codimension two, so it follows that the residue on Y extends to a non-vanishing multiderivation on all of $\mathsf{Sec}_k(\mathsf{X})$. □

Remark 7.6.6. The fact that the blowdown is a Poisson morphism in this case is not obvious; we sketch the proof here. One can that the canonical lift of a Poisson structure on $\mathbb{P}(\mathsf{V})$ to a unimodular Poisson structure on V as in Corollary 7.2.7 is given by blowing down the Poisson structure on the total space of $\mathcal{O}_{\mathbb{P}^{d-1}}(-1)$ obtained from the canonical module as in the proof of the Proposition 7.6.5.

On the other hand, the Poisson structures of Feigin and Odesskii have been proven to be unimodular as a consequence of their invariance under the action of the Heisenberg group [115]. Therefore Bondal's theorem implies that they must coincide with the Poisson structure obtained from the blow-down. □

Remark 7.6.7. This proposition shows that in every dimension, there are examples of Poisson manifolds whose modular residues are all nonzero. We therefore argue that these residues provide a possible explanation for the dimensions appearing in Bondal's conjecture. Since these residues are non-vanishing on the secant varieties, it may be reasonable to expect the degeneracy loci of Poisson structures to be highly singular Calabi-Yau varieties. □

We close with a conjecture regarding the degeneracy loci of the Feigin-Odesskii Poisson structures:

Conjecture 7.6.8. *The degeneracy loci of the Feigin-Odesskii Poisson structures on $\mathbb{P}^{d-1}$ are reduced.*

Evidence. We focus on the case of Poisson structures on $\mathbb{P}(\mathsf{H}^1(\mathsf{X}, \mathcal{L}^\vee))$ for $\mathcal{L}$ a line bundle of degree d.

According to Proposition 7.6.5, the modular residue on every degeneracy locus $\mathsf{Dgn}_{2k}(\sigma)$ is non-trivial. This is a sort of genericity condition on the one-jet of σ^{k+1}, and hence it may be reasonable to expect that the zero scheme of σ^{k+1} is reduced. For $d = 2n+1$ an odd number, the degeneracy divisor $\mathsf{Dgn}_{2n-2}(\sigma)$ is always reduced, as we saw in Theorem 7.6.3. In general, though, the non-triviality of the residue is not sufficient to conclude that $\mathsf{Dgn}_{2k}(\sigma)$ is reduced.

We also have explicit knowledge in low dimensions: for $d = 3$, the zero locus is the smooth cubic curve $\mathsf{X} \subset \mathbb{P}^2$, which is the reduced anti-canonical divisor.

For $d = 4$, the formulae in [112, 115] show that the two-dimensional symplectic leaves form a pencil of quadrics in $\mathbb{P}^3$ intersecting in a smooth elliptic curve. The zero scheme of the Poisson structure is reduced and consists of the elliptic curve, together with four isolated points [117]. (We shall see this Poisson structure again in the next Chapter 8.

For $d = 5$, we checked using `Macaulay2` [68] and the formulae in [112, 115] that the degeneracy loci are reduced. □

Chapter 8

Poisson structures on $\mathbb{P}^3$ and their quantizations

In this chapter, we study the geometry of Poisson structures on $\mathbb{P}^3$ and their quantizations. We relate Cerveau and Lins-Neto's celebrated classification [34] of degree-two foliations on $\mathbb{P}^3$ and use their classification to give normal forms for unimodular quadratic Poisson structures on $\mathbb{C}^4$. These normal forms are generic in the sense that their $\mathsf{GL}(4,\mathbb{C})$-orbits form a dense, Zariski open set in the space of all unimodular quadratic Poisson structures on $\mathbb{C}^4$.

Remarkably, the quantizations of these Poisson structures can be described explicitly. We give presentations for the corresponding noncommutative algebras in terms of generators and relations. We believe that these algebras (or degenerations thereof) describe all of the deformations of the classical, commutative $\mathbb{P}^3$ as a noncommutative projective scheme in the sense of Artin and Zhang [5], provided that the deformation takes place over an irreducible base; see Conjecture 8.2.2.

Finally, we focus our attention on a particular family of Poisson structures arising from group actions associated with rational normal curves in projective space, generalizing one of the Cervea–Lins Neto families to arbitrary dimension. Using the universal deformation formula of Coll, Gerstenhaber and Giaquinto [39], we are able to describe many features of the quantizations in terms of equivariant geometry. In particular, we show that the Schwarzenberger bundles [126]—the classic examples of rank-n vector bundles on $\mathbb{P}^n$—deform to bimodules over the quantization, and that the corresponding quantum version of $\mathbb{P}^{12}$ contains a quantum version of the Mukai–Umemura threefold [111].

Throughout this chapter $\mathsf{V} \cong \mathbb{C}^4$ will be a four-dimensional complex vector space, and $\mathbb{P}^3 = \mathbb{P}(\mathsf{V})$ its projectivization.

8.1 Foliations and the Cerveau–Lins Neto classification

Let $\sigma \in \mathsf{H}^0\big(\mathbb{P}^3, \mathscr{X}^2_{\mathbb{P}^3}\big)$ be a Poisson structure on $\mathbb{P}^3$. Using the isomorphism

$$\mathscr{X}^2_{\mathbb{P}^3} \cong \Omega^1_{\mathbb{P}^3} \otimes \omega^{-1}_{\mathbb{P}^3} \cong \Omega^1_{\mathbb{P}^3}(4),$$

we may view σ a one-form $\alpha \in \mathsf{H}^0\big(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(4)\big)$ with values in the line bundle $\mathscr{O}_{\mathbb{P}^3}(4)$. The integrability condition for σ is equivalent to the requirement that $\alpha \wedge d\alpha = 0$, i.e., that the kernel of α defines an integrable distribution; this distribution exactly defines the foliation of $\mathbb{P}^3$ by the symplectic leaves of σ.

The set of integrable one-forms is an algebraic subvariety of $\mathsf{H}^0\big(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(4)\big)$. In [34], Cerveau and Lins Neto describe the irreducible components of this variety under the additional assumption that α vanishes on a curve, and give formulae for the one-forms in homogeneous coordinates. The result is the following

Theorem 8.1.1 ([34])**.** *The variety parametrizing integrable $\mathscr{O}_{\mathbb{P}^3}(4)$-valued one-forms on $\mathbb{P}^3$ whose singular locus is a curve has six irreducible components, called* $\mathsf{L}(1,1,1,1)$*,* $\mathsf{L}(1,1,2)$*,* $\mathsf{R}(2,2)$*,* $\mathsf{R}(1,3)$*,* $\mathsf{S}(2,3)$ *and* $\mathsf{E}(3)$*.*

The first four components are of normal crossings type in the sense of Definition 7.4.2 and were proven to be irreducible components by Calvo-Andrade [30], but $\mathsf{S}(2,3)$ and $\mathsf{E}(3)$ have rather different flavours: $\mathsf{S}(2,3)$ is obtained by pulling back a foliation of $\mathbb{P}^2$ along a linear projection, and $\mathsf{E}(3)$ is associated with the geometry of twisted cubic curves.

Loray, Touzet and Pereira [101] recently gave a similar description of the spaces of Poisson structures on Fano threefolds of Picard rank one. Among the other examples is a Poisson structure on the Mukai–Umemura threefold that we shall discuss briefly in Section 8.9 in this chapter. As a consequence, they verified that every Poisson structure on $\mathbb{P}^3$ is a degeneration of a Poisson structure vanishing on a curve and therefore arrived at the following

Theorem 8.1.2 ([101])**.** *The variety parametrizing Poisson structures on $\mathbb{P}^3$ has the same six irreducible components as the Cerveau–Lins Neto classification.*

Figure 8.1 illustrates the six different families of Poisson structures on $\mathbb{P}^3$ by displaying some symplectic leaves, together with some curves on which the Poisson structure vanishes.

If we work in homogeneous coordinates x_0, x_1, x_2, x_3 on $\mathbb{C}^4$, we may view the one form α on $\mathbb{P}^3$ as a one-form on $\mathbb{C}^4$ with homogeneous cubic coefficients that is integrable ($d\alpha\wedge\alpha = 0$) and horizontal ($\iota_E\alpha = 0$ for the Euler vector field E). Let $\omega = d\alpha$ be its derivative. Then the formula

$$\{f, g\} = \frac{df \wedge dg \wedge \omega}{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3}$$

defines a Poisson bracket on $\mathbb{C}^4$; up to rescaling by a constant, it is the canonical unimodular lift of σ specified by Theorem 7.2.8. In this way, we obtain the following

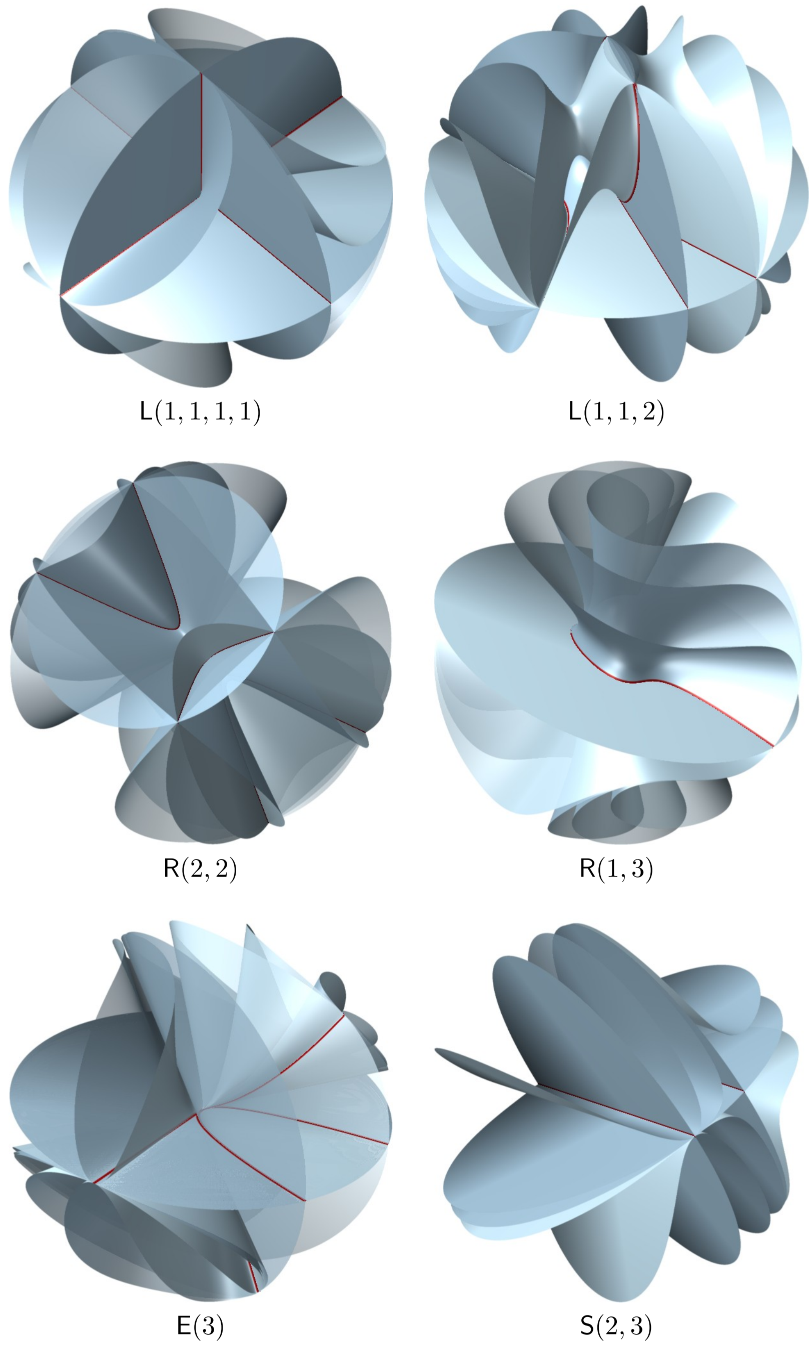


Figure 8.1: Real slices of Poisson structures on $\mathbb{P}^3$, one from each irreducible component in the classification. The blue surfaces represent symplectic leaves, while the red curves represent one-dimensional components of the zero locus.

Corollary 8.1.3. *The space of unimodular Poisson structures on $\mathbb{C}^4$ has the same six irreducible components as the Cerveau–Lins Neto classification.*

Notice that since $\mathsf{GL}(4,\mathbb{C})$ acts on $\mathbb{C}^4$ by linear automorphisms, it also acts on the space of unimodular quadratic Poisson structures. In this chapter, we give normal forms for the generic unimodular quadratic Poisson structures and their quantizations. By analyzing each of the six components separately in Section 8.3 through Section 8.8, we shall prove the

Theorem 8.1.4. *Every unimodular quadratic Poisson structure on $\mathbb{C}^4$ is a degeneration of one of the normal forms given in Table 8.1.*

By a degeneration, we mean that the Poisson structure lies in the closure of the $\mathsf{GL}(4,\mathbb{C})$-orbit of one of the normal forms in the table. It would be interesting to give a complete classification classify all of the degenerations:

Problem 8.1.5. *Complete the classification of unimodular quadratic Poisson structures on $\mathbb{C}^4$ by studying the degenerations of the normal form.*

We remark that while this classification deals with Poisson structures over $\mathbb{C}$, it may be used to to deduce the classification over $\mathbb{R}$. Indeed, the specification of a quadratic Poisson structure on $\mathbb{R}^4$ is the same as the specification of a quadratic Poisson structure on $\mathbb{C}^4$ together with an anti-holomorphic Poisson involution $\mathbb{C}^4 \to \mathbb{C}^4$. We therefore pose another

Problem 8.1.6. *Give a classification of unimodular quadratic Poisson structures on $\mathbb{R}^4$ using the normal forms in Table 8.1.*

See also the related computations in [88], where the non-unimodular case is studied. Just as the real Lie algebras $\mathfrak{sl}(3,\mathbb{R})$ and $\mathfrak{so}(3,\mathbb{R})$ are not isomorphic over $\mathbb{R}$ but nevertheless have isomorphic complexifications, we expect that some of the irreducible components in the classification, such as $\mathsf{R}(2,2)$, will split into multiple different components over the real numbers.

8.2 Quantization of quadratic Poisson structures

Recall that if A is a commutative algebra equipped with a Poisson bracket $\{\cdot,\cdot\}$, then a deformation quantization of the bracket is a deformation of the commutative product on A to a non-commutative associative "star product"

$$f \star_\hbar g = fg + \hbar\{f,g\} + O(\hbar^2),$$

depending on a deformation parameter $\hbar$. In this case $\{\cdot,\cdot\}$ is called the ***semi-classical limit*** of the noncommutative deformation.

Table 8.1: Normal forms for generic unimodular quadratic Poisson structures on $\mathbb{C}^4$

Type	Poisson brackets	Parameters
$\mathsf{L}(1,1,1,1)$	$\{x_i, x_{i+1}\} = (-1)^i(a_{i+3} - a_{i+2})x_i x_{i+1}$ $\{x_i, x_{i+2}\} = (-1)^i(a_{i+1} - a_{i+3})x_i x_{i+2}$ $i \in \mathbb{Z}/4\mathbb{Z}$	$a_0 + a_1 + a_2 + a_3 = 0$
$\mathsf{L}(1,1,2)$	$\{x_0, x_1\} = 0$ $\{x_0, x_2\} = c_0 x_0 x_2$ $\quad$ $\{x_1, x_2\} = -c_1 x_1 x_2$ $\{x_0, x_3\} = -c_0 x_0 x_3$ $\quad$ $\{x_1, x_3\} = c_1 x_1 x_3$ $\{x_2, x_3\} = (c_0 - c_1)\big(x_0^2 + \lambda x_0 x_1 + x_1^2 + x_2 x_3\big)$ $+2c_0 x_0^2 - 2c_1 x_1^2$	$c_0, c_1, \lambda \in \mathbb{C}$
$\mathsf{R}(2,2)$	$\{x_0, x_1\} = (a_3 - a_2)x_2 x_3$ $\quad$ $\{x_2, x_1\} = x_0 x_3$ $\{x_0, x_2\} = (a_1 - a_3)x_3 x_1$ $\quad$ $\{x_3, x_2\} = x_0 x_1$ $\{x_0, x_3\} = (a_2 - a_1)x_1 x_2$ $\quad$ $\{x_1, x_3\} = x_0 x_2$	$a_1 + a_2 + a_3 = 0$
$\mathsf{R}(1,3)$	$\{x_3, x_i\} = 0$ $\{x_{i+1}, x_i\} = \nu x_{i+2}^2 - \lambda x_{i+1} x_i + \sum_{j=0}^{2} b_{ij} x_j x_3$ $i \in \mathbb{Z}/3\mathbb{Z}$	$b_{ij} = b_{ji},\ i, j \in \mathbb{Z}/3\mathbb{Z}$ $\det(b_{ij}) = 1$ $\lambda \in \mathbb{C}$
$\mathsf{S}(2,3)$	$\{x_i, x_3\} = x_i^2 + x_i(b_i x_{i+1} + c_i x_{i-1}) + d_i x_{i+1} x_{i-1}$ $\{x_i, x_j\} = 0$ $i, j \in \mathbb{Z}/3\mathbb{Z}$	$b_{i-1} + c_{i+1} + 2 = 0$ $d_i \in \mathbb{C}$ $i \in \mathbb{Z}/3\mathbb{Z}$
$\mathsf{E}(3)$	$\{x_0, x_1\} = 5x_0^2$ $\quad$ $\{x_1, x_2\} = x_1^2 + 3x_0 x_2$ $\{x_0, x_2\} = 5x_0 x_1$ $\quad$ $\{x_1, x_3\} = x_1 x_2 + 7x_0 x_3$ $\{x_0, x_3\} = 5x_0 x_2$ $\quad$ $\{x_2, x_3\} = 7x_1 x_3 - 3x_2^2$	none

Suppose that $\mathsf{A} = \mathsf{Sym}^\bullet \mathsf{V}^*$ is the ring of algebraic functions on a vector space V. The Poisson bracket $\{\cdot,\cdot\}$ is compatible with the grading on A exactly when it is homogeneous quadratic; in this case we want to deform A as a graded algebra. According to the philosophy of noncommutative projective geometry, we should interpret the result as a homogeneous coordinate ring for a quantum version of the projective space $\mathbb{P}(\mathsf{V})$. We refer the reader to the works [4, 5, 19, 18, 62, 133] for an introduction to this viewpoint. In particular, [4, 19] give the classification of quantum deformations of the projective plane $\mathbb{P}^2$.

Instead of deforming the product on $\mathsf{Sym}^\bullet \mathsf{V}^*$ directly, we could view $\mathsf{Sym}^\bullet \mathsf{V}^*$ as a quotient of the tensor algebra $\bigotimes^\bullet \mathsf{V}^*$ by the ideal generated by the quadratic relations $\Lambda^2 \mathsf{V}^* \subset \mathsf{V}^* \otimes \mathsf{V}^*$, and deform the relations to a new subspace $\mathsf{I}_2^\hbar \subset \mathsf{V}^* \otimes \mathsf{V}^*$. In so doing, we must be careful to ensure that the resulting graded algebra $\mathsf{A}^\hbar$ is isomorphic as a vector space to $\mathsf{Sym}^\bullet \mathsf{V}^*$. In other words, we want the composition

$$\mathsf{Sym}^\bullet \mathsf{V}^* \to \bigotimes{}^\bullet \mathsf{V}^* \to \mathsf{A}^\hbar$$

to be an isomorphism. Then, we can transport the product on $\mathsf{A}^\hbar$ to a star product on $\mathsf{Sym}^\bullet \mathsf{V}^*$ in a canonical way, and speak of its semi-classical limit.

Recall that the Hilbert series of a graded algebra $\mathsf{A} = \bigoplus_{k=0}^\infty \mathsf{A}_k$ is

$$H_{\mathsf{A}}(t) = \sum_{k=0}^\infty \dim_{\mathbb{C}}(\mathsf{A}_k) t^k.$$

For the polynomial ring in $n+1$ variables we have $H(t) = \frac{1}{(1-t)^{n+1}}$. Thus, for a quadratic Poisson structure, we wish to deform the quadratic relations without changing the Hilbert series of the resulting ring. In this case, we certainly require that $\dim_{\mathbb{C}} \mathsf{I}_2^\hbar = \dim_{\mathbb{C}} \Lambda^2 \mathsf{V}^* = \binom{\dim \mathsf{V}}{2}$, but this restriction is not sufficient since there may still be too many relations in higher degree. However, the ***Koszul deformation principle*** indicates that, at least for formal deformations, we only need to worry about the dimensions of the space of relations of degree ≤ 3 [47].

This situation may be formalized as follows. For a vector space V, consider the space $\mathsf{PS}(\mathsf{V}) \subset \mathsf{Sym}^2 \mathsf{V}^* \otimes \Lambda^2 \mathsf{V}$ of quadratic Poisson structures. This space is a cone defined by a collection of homogeneous quadratic polynomials (the coefficients of the Schouten bracket).

For $k \in \mathbb{N}$, let $\mathsf{G}_k = \mathsf{Gr}\big(d_k, (\mathsf{V}^*)^{\otimes k}\big)$ be the Grassmannian parametrizing d_k-planes in $(\mathsf{V}^*)^{\otimes k}$, where

$$d_k = (\dim \mathsf{V}^*)^k - \dim \mathsf{Sym}^k \mathsf{V}^*$$

is the dimension of the space of degree-k relations for the symmetric algebra. Consider the subvariety $\mathsf{QA}(\mathsf{V}) \subset \mathsf{G}_2 \times \mathsf{G}_3$ consisting of pairs $(\mathsf{I}_2, \mathsf{I}_3)$ such that

$$\mathsf{I}_3 \supset \mathsf{I}_2 \otimes \mathsf{V}^* + \mathsf{V}^* \otimes \mathsf{I}_2$$

Thus, we require that all of the relations defined by I_2 lie in I_3; this prevents I_2 from defining an ideal that is too big to produce the correct Hilbert series. Clearly, the symmetric algebra defines a special point $c \in \mathsf{QA}(\mathsf{V})$. A (formal) neighbourhood of c in $\mathsf{QA}(\mathsf{V})$ therefore parametrizes the deformations of the polynomial ring $\mathsf{Sym}^\bullet \mathsf{V}^*$ to a noncommutative graded algebra with the same Hilbert series. Bondal [19] notes that the tangent cone of $\mathsf{QA}(\mathsf{V})$ at c is naturally identified with $\mathsf{PS}(\mathsf{V})$, but the connection is much deeper. In fact, Kontsevich proved the following result as a consequence of his formality theorem:

Theorem 8.2.1 ([90])**.** *Let* V *be a* $\mathbb{C}$*-vector space. Then there is a canonical* $\mathsf{GL}(\mathsf{V})$*-equivariant isomorphism between the formal neighbourhoods of* $0 \in \mathsf{PS}(\mathsf{V})$ *and* $c \in \mathsf{QA}(\mathsf{V})$*.*

Kontsevich conjectures that this formal isomorphism converges, at least in a small neighbourhood of 0. However, even if it does converge, it is given by a formal power series that appears to be very difficult to compute. So, it is not clear how to obtain an explicit description of the quantizations using this approach.

In this chapter, we attempt to describe the quantizations directly in the case when $\mathsf{V} = \mathbb{C}^4$, corresponding to quantum deformations of $\mathbb{P}^3$. Instead of using Kontsevich's formula, we use the intuition gained from a good understanding of the Poisson geometry. As a result, we arrive at the list of algebras shown in Table 8.2, giving honest (not formal) deformation quantizations for the generic unimodular quadratic Poisson structures on $\mathbb{C}^4$.

In light of Theorem 8.2.1 and Theorem 8.1.4, we make the following

Conjecture 8.2.2. *Let* $\mathbb{P}^3_{nc}$ *be a noncommutative projective scheme in the sense of [5] that arises as flat deformation [102] of the classical, commutative* $\mathbb{P}^3$ *over an irreducible base. Then* $\mathbb{P}^3_{nc}$ *is isomorphic to* $\mathsf{Proj}(\mathsf{A})$*, where* A *is a quadratic algebra that is a degeneration of one of the algebras in Table 8.2.*

Many of the algebras presented here have already appeared in the literature. Others may be less familiar, although their ring-theoretic properties do not appear to be particularly unusual. We pay special attention to the "exceptional" Poisson structure $\mathsf{E}(3)$, and use a formula of Coll, Gerstenhaber and Giaquinto to study the quantization in terms of equivariant geometry. As a result, we are able to construct quantum analogues of the famous Schwarzenberger bundles [126], which played an important historical role as early examples of indecomposable rank-n vector bundles on $\mathbb{P}^n$.

Up to now, we have ignored one key point: to make the correspondence between Poisson structures on $\mathbb{P}^n$ and quadratic Poisson structures on $\mathbb{C}^{n+1}$ bijective, we need to impose the additional constraint that the quadratic Poisson structure be unimodular: the canonical module $\omega_{\mathbb{C}^{n+1}}$ must be trivial as a Poisson module. Dolgushev [44] showed that under Kontsevich's quantization, unimodular Poisson structures correspond precisely to Calabi–Yau algebras in the sense of Ginzburg [60]. Since, according to Theorem 7.2.4, the difference between the non-unimodular lifts and the unique unimodular ones is controlled by a Poisson

Table 8.2: Quantum deformations of $\mathbb{P}^3$

Type	Quantization	Parameters
$\mathsf{L}(1,1,1,1)$	Skew polynomial ring $x_i x_j = p_{ij} x_j x_i$	$p_{ij} = p_{ji}^{-1} \in \mathbb{C}^*$
$\mathsf{L}(1,1,2)$	$x_1x_0 = x_0x_1$ $x_2x_0 = p_0^{-1}\, x_0x_2$ $x_3x_0 = p_0\, x_0x_3$ $x_2x_1 = p_1\, x_1x_2$ $x_3x_1 = p_1^{-1}\, x_1x_3$ $x_3x_2 = p_0^{-1}p_1\, x_2x_3 + (p_1 - p_0)(x_0^2 + \lambda x_0x_1 + x_1^2)$ $+(1-p_0^2)x_0^2 + (p_1^2 - 1)x_1^2$	$p_0, p_1 \in \mathbb{C}^*, \lambda \in \mathbb{C}$
$\mathsf{R}(2,2)$	Four-dimensional Sklyanin algebra [131]	
$\mathsf{R}(1,3)$	Central extension of a three-dimensional Sklyanin algebra [95]	
$\mathsf{S}(2,3)$	Ore extension of $\mathbb{C}[x_0,x_1,x_2]$ by a derivation	
$\mathsf{E}(3)$	$[x_0,x_1] = 5x_0^2$ $[x_1,x_2] = 3x_0x_2 + x_1^2 - \frac{3}{2}x_0x_1$ $[x_0,x_2] = 5x_0x_1 - \frac{45}{2}x_0^2$ $[x_1,x_3] = 7x_0x_3 + x_1x_2 - 3x_0x_2 - \frac{5}{2}x_1^2 + 5x_0x_1$ $[x_0,x_3] = 5x_0x_2 - \frac{45}{2}x_0x_1 + \frac{195}{2}x_0^2$ $[x_2,x_3] = 7x_1x_3 - \frac{77}{2}x_0x_3 - 3x_2^2 + \frac{21}{2}x_1x_2 - \frac{77}{2}x_0x_2$	none

vector field (an infinitesimal graded automorphism), we expect that for a given deformation of the homogeneous coordinate ring of $\mathbb{P}^n$, there should be a Calabi–Yau algebra that presents the same quantum projective scheme in the sense of [5]. Moreover, we expect this Calabi–Yau algebra to differ from the given one by a graded twist, as studied in [122].

8.3 The $\mathsf{L}(1,1,1,1)$ component

Poisson geometry

Choose four planes $\mathsf{D}_1,\ldots,\mathsf{D}_4 \subset \mathbb{P}^3$ in general position. Then the union $\mathsf{D} = \mathsf{D}_1 + \cdots + \mathsf{D}_4$ is normal crossings and is therefore a free divisor. The space $\mathfrak{h} = \mathsf{H}^0\big(\mathbb{P}^3, \mathscr{X}^1_{\mathbb{P}^3}(-\log \mathsf{D})\big)$ of vector fields tangent to D is a three-dimensional abelian Lie algebra; it is a Cartan subalgebra of the Lie algebra $\mathfrak{g} = \mathsf{H}^0\big(\mathbb{P}^3, \mathscr{X}^1_{\mathbb{P}^3}\big) \cong \mathfrak{sl}(4,\mathbb{C})$ of vector fields on $\mathbb{P}^3$.

Notice that $\mathscr{X}^1_{\mathbb{P}^3}(-\log \mathsf{D})$ is identified with the trivial bundle $\mathfrak{h} \otimes \mathcal{O}_{\mathbb{P}^3}$, and hence the natural map $\mathsf{\Lambda}^2\mathfrak{h} \to \mathsf{H}^0\big(\mathbb{P}^3, \mathscr{X}^2_{\mathbb{P}^3}(-\log \mathsf{D})\big)$ is an isomorphism in this case. Since $\mathfrak{h}$ is abelian, we see that every bivector field tangent to D is Poisson.

Let $\mathsf{Y} \subset \mathsf{Gr}(3,\mathfrak{g})$ be the closure of the space of Cartan subalgebras in $\mathfrak{g}$. Since all Cartan subalgebras differ by conjugation by elements of $\mathsf{SL}(4,\mathbb{C})$, the space Y is irreducible.

There is a tautological rank-three vector bundle $\mathcal{E} \to \mathsf{Y}$ whose fibre at $\mathfrak{h} \in \mathsf{Y}$ is $\Lambda^2\mathfrak{h}$. Let $\mathsf{X} = \mathbb{P}(\mathcal{E})$ be its projectivization, and $\mathcal{L} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$ the tautological line bundle. Then the construction above defines a $\mathbb{C}^*$-equivariant map $\phi : \mathsf{Tot}(\mathcal{L}) \to \mathsf{PS}(\mathbb{P}^3)$. Since X is projective and irreducible, the image of ϕ is a closed, irreducible conical subvariety in the vector space $\mathsf{H}^0(\mathbb{P}^3, \mathscr{X}^2_{\mathbb{P}^3})$.

Definition 8.3.1. The image of ϕ is the the component $\mathsf{L}(1,1,1,1)$ in the space of Poisson structures on $\mathbb{P}^3$.

The infinitesimal symmetries of the generic Poisson structures in this component are readily computed:

Lemma 8.3.2. *Suppose that $r \in \Lambda^2\mathfrak{h}$ is generic, in the sense that none of the roots of $\mathfrak{g}$ lie in the kernel of $r^\sharp : \mathfrak{h}^* \to \mathfrak{h}$. Then the first Poisson cohomology group is given*

$$\mathfrak{aut}(\sigma) = \mathsf{H}^1(\sigma) = \mathfrak{h}.$$

Proof. Since $\mathfrak{h}$ is three-dimensional, we may write $r = e_1 \wedge e_2$ for some $e_1, e_2 \in \mathfrak{h}$. Let us decompose

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\beta \in \mathsf{J}} \mathfrak{g}_\beta$$

into its root spaces. Since $[\sigma, \mathfrak{h}] = 0$, it remains to show that if $\xi = \sum_{\beta\in\mathsf{J}} \xi_\beta$ is a sum of elements with nonzero roots and $[\sigma, \xi] = 0$ then $\xi = 0$. But we have

$$\begin{aligned}
\mathscr{L}_\xi(e_1 \wedge e_2) &= [\xi, e_1] \wedge e_2 + e_1 \wedge [\xi, e_2] \\
&= \sum_{\beta\in\mathsf{J}} \xi_\beta \wedge (\beta(e_2)e_1 - \beta(e_1)e_2) \\
&= \sum_{\beta\in\mathsf{J}} r^\sharp(\beta) \wedge \xi_\beta
\end{aligned}$$

where $r^\sharp : \mathfrak{h}^* \to \mathfrak{h}$ is the contraction with r. If this sum vanishes, then by linear independence, so must every summand. But $r^\sharp(\beta) \neq 0$ for any β by assumption and hence we must have $\xi_\beta = 0$ for all β, as required. $\square$

Corollary 8.3.3. *The map ϕ is generically one-to-one.*

We obtain the normal form for these Poisson structures as follows: choose homogeneous coordinates $x_0, \ldots, x_3$ so that D_i is the vanishing set of x_i. Then the one-form $\alpha \in \mathsf{H}^0(\mathbb{P}^3, \Omega^1_{\mathbb{P}^3}(4))$ may be written in homogeneous coordinates as

$$\alpha = x_0x_1x_2x_3 \sum_{i=0}^{3} a_i \frac{dx_i}{x_i},$$

where $a_0, \ldots, a_3 \in \mathbb{C}$ are constants with $\sum_{i=0}^{3} a_i = 0$. A straightforward computation of the Poisson brackets in these coordinates yields the normal form in Table 8.1.

We claim that generically, the only isomorphisms between pairs of these normal forms come from permutations of the coordinates. Indeed, provided that the Poisson structure is generic in the sense of the lemma, any automorphism must preserve the Cartan subalgebra $\mathfrak{h}$. Since all of the elements of $\mathfrak{h}$ are infinitesimal symmetries, we may identify $\mathsf{Aut}(\sigma)$ with the Weyl group, which exactly acts by permuting the coordinates.

Since the space of parameters in this normal form is three-dimensional, we conclude that the dimension of the moduli space in the neighbourhood of a generic normal form is three, and we therefore have a canonical identification of the tangent space $\mathsf{H}^2\left(\mathbb{P}^3, \sigma\right)$ to the moduli space with $\mathsf{\Lambda}^2\mathfrak{h}$.

Computing the dimension of the third cohomology group using the fact that the Euler characteristic is -4, we arrive at the following

Proposition 8.3.4. *If $\sigma \in \mathsf{L}(1,1,1,1)$ is generic in the sense of Lemma 8.3.2, then*

$$\begin{aligned}
\mathsf{H}^0\left(\mathbb{P}^3, \sigma\right) &= \mathbb{C} \\
\mathsf{H}^1\left(\mathbb{P}^3, \sigma\right) &= \mathfrak{h} \cong \mathbb{C}^3 \\
\mathsf{H}^2\left(\mathbb{P}^3, \sigma\right) &= \mathsf{\Lambda}^2\mathfrak{h} \cong \mathbb{C}^3
\end{aligned}$$

and $\mathsf{H}^3\left(\mathbb{P}^3, \sigma\right)$ is 5-dimensional.

Quantization

From the form of the Poisson brackets, it is clear that the quantizations should be skew polynomial rings—that is, algebras with four degree-one generators x_0, x_1, x_2, x_3 and quadratic relations $x_i x_j = p_{ij} x_j x_i$, where $p_{ij} = p_{ji}^{-1}$. If we take $p_{i,i+1} = e^{(-1)^i (a_{i+3} - a_{i+2})\hbar}$, and $p_{i,i+2} = e^{(-1)^i (a_{i+1} - a_{i+3})\hbar}$ for $i \in \mathbb{Z}/4\mathbb{Z}$, then the semiclassical limit $\hbar \to 0$ recovers the normal form for the Poisson bracket. This product is essentially the Moyal–Vey quantization [108, 140].

8.4 The $\mathsf{L}(1,1,2)$ component

Poisson geometry

Choose homogeneous linear polynomial $f_0, f_1 \in \mathsf{V}^*$ and a homogeneous quadratic form $g \in \mathsf{Sym}^2\mathsf{V}^*$. Choose $a_0, a_1, b \in \mathbb{C}$ and define a one-form on V by

$$\alpha = f_0 f_1 g \left(a_0 \frac{df_0}{f_0} + a_1 \frac{df_1}{f_1} + b \frac{dg}{g} \right)$$

Then α is integrable. Moreover, $\iota_E \alpha = 0$ if and only if $a_0 + a_1 + 2b = 0$. Allowing the polynomials f_0, f_1 and g as well as the coefficients a_1, a_2 and b to vary, we obtain a subvariety of $\mathsf{H}^0(\mathbb{P}^3, \mathscr{X}^2_{\mathbb{P}^3})$ and its closure is the irreducible component $\mathsf{L}(1,1,2)$ in the space of Poisson structures.

Let σ be the Poisson structure on $\mathbb{P}^3$ associated with the above data. Then the planes $\mathsf{D}_0, \mathsf{D}_1 \subset \mathbb{P}^3$ defined by f_0 and f_1, and the quadric surface $\mathsf{Q} \subset \mathbb{P}^3$ defined by g are Poisson subspaces.

If f_0, f_1 and g are generic, then the union $\mathsf{D} = \mathsf{D}_0 + \mathsf{D}_1 + \mathsf{Q}$ will be a normal crossings divisor whose singular locus is the union of the line $\mathsf{D}_0 \cap \mathsf{D}_1$ and the two plane conics $\mathsf{D}_0 \cap \mathsf{Q}$ and $\mathsf{D}_1 \cap \mathsf{Q}$. If the coefficents a_0, a_1 and b are all nonzero, then these three rational curves are precisely the irreducible components of the zero locus of σ, and they all intersect at the two points $\{p, q\} = \mathsf{D}_0 \cap \mathsf{D}_1 \cap \mathsf{Q}$.

Let us now describe the normal form for such a Poisson structure. Requiring that Q be smooth is equivalent to requiring that g be non-degenerate. Generically, g^{-1} will restrict to a nondegenerate form on the span of f_1 and f_2. By adjusting the coefficients a_0, a_1 and b, we may rescale f_1 and f_2 so that the inner products are $g^{-1}(f_1, f_1) = g^{-1}(f_2, f_2) = 1$ and $g^{-1}(f_1, f_2) = \lambda \in \mathbb{C}$. We may then find coordinates $x_0, \ldots, x_3$ on V so that $f_1 = x_0$, $f_2 = x_1$ and

$$g = x_0^2 + \tfrac{\lambda}{2} x_0 x_1 + x_1^2 + x_2 x_3.$$

To compute the bracket from the one-form, we need to pick a constant volume form $\omega = C dx_0 \wedge \cdots \wedge dx_3 \in \det \mathsf{V}^*$. Setting $c_0 = C^{-1}(a_0 - b)$, $c_1 = C^{-1}(a_1 - b)$ and computing the Poisson brackets using this normal form leads directly to the formulae in Table 8.1.

Quantization

Consider the algebra $\mathsf{A} = \mathbb{C}\langle x_0, x_1, x_2, x_3 \rangle / \mathsf{I}$, where I is the homogeneous ideal generated by the quadratic relations

$$\begin{aligned}
x_1 x_0 &= x_0 x_1 \\
x_2 x_0 &= p_0^{-1}\, x_0 x_2 \\
x_3 x_0 &= p_0\, x_0 x_3 \\
x_2 x_1 &= p_1\, x_1 x_2 \\
x_3 x_1 &= p_1^{-1}\, x_1 x_3 \\
x_3 x_2 &= p_0^{-1} p_1\, x_2 x_3 + F
\end{aligned}$$

for $p_0, p_1 \in \mathbb{C}^*$ and $\lambda \in \mathbb{C}$. Here

$$F = (p_1 - p_0)(x_0^2 + \lambda x_0 x_1 + x_1^2) + (1 - p_0^2) x_0^2 + (p_1^2 - 1) x_1^2.$$

This algebra was found in conversations with Ingalls and Van den Bergh; the author thanks them for their interest and for explaining the proof of the proposition below.

The relations for A were obtained by a somewhat *ad hoc* procedure: since, for the most part, the Poisson brackets have a similar form to the brackets for $\mathsf{L}(1,1,1,1)$ case, we may guess that most of the relations should look like relations in a skew polynomial algebra. The only nontrivial relation to be added is the one corresponding $\{x_2, x_3\}$, which was obtained by examining the explicit form of the Poisson bracket. As a result, there may be a more natural set of relations that gives an equivalent quantum $\mathbb{P}^3$; for example, we have not determined if the algebra presented here is Calabi–Yau. Nevertheless, we may verify that it has the correct Hilbert series:

Proposition 8.4.1. *The monomials $x_0^i x_1^j x_2^k x_3^l$ with $i,j,k,l \geq 0$ form a basis for A as a $\mathbb{C}$-vector space. In particular, the Hilbert series of A is $H_{\mathsf{A}}(t) = \frac{1}{(1-t)^4}$.*

Proof. The proof is a straightforward appeal to Bergman's diamond lemma [12]. The given monomials are reduced words with respect to the lexicographical ordering, and the relations are written so that the leading term appears on the left hand side. There are no inclusion ambiguities, and the overlap amibiguities come from the expressions $x_2x_1x_0$, $x_3x_1x_0$ $x_3x_2x_0$ and $x_3x_2x_1$. We note that $x_0F = Fx_0$ in A since x_0 and x_1 commute. Moreover, x_0F is a linear combination of reduced words.

It is now straightforward to verify that these ambiguities resolve. For example, we can reduce $x_3x_2x_0$ in two different ways. The first reduction is

$$\begin{aligned}
(x_3x_2)x_0 &= (p_0^{-1}p_1\, x_2x_3 + F)x_0 \\
&= p_1x_2x_0x_3 + x_0F \\
&= p_0^{-1}p_1x_0x_2x_3 + x_0F,
\end{aligned}$$

while the second is

$$\begin{aligned}
x_3(x_2x_0) &= p_0^{-1}x_3x_0x_2 \\
&= x_0x_3x_2 \\
&= x_0(p_0^{-1}p_1x_2x_3 + F) \\
&= p_0^{-1}p_1x_0x_2x_3 + x_0F,
\end{aligned}$$

which agrees with the first. Similar calculations show that the other overlap ambiguities resolve. □

Notice that when $p_0 = p_1 = 1$, we recover the relations for the (commutative) polynomial ring. We therefore conclude that the family of algebras A gives a flat deformation of $\mathbb{C}[x_0, x_1, x_2, x_3]$ over the parameter space $(p_0, p_1) \in (\mathbb{C}^*)^2$. If we set $p_0 = e^{\hbar c_0}$ and

$p_1 = e^{\hbar c_1}$ then the semiclassical limit $\hbar \to 0$ recovers the normal form of the Poisson structure described in the previous section. We have therefore obtained the desired deformation quantization.

8.5 The $\mathsf{R}(2,2)$ component

Poisson geometry

Choose a pair of homogeneous quadratic forms $g_1, g_2 \in \mathsf{Sym}^2\mathsf{V}^*$ and define a one-form on V by

$$\alpha = g_1\, dg_2 - g_2\, dg_1.$$

Then α is integrable and $\iota_E\alpha = 0$. Clearly α depends only on the element $g_1 \wedge g_2 \in \Lambda^2\mathsf{Sym}^2\mathsf{V}^*$. Converting the two-form $d\alpha$ to a bivector field via an element of $\det \mathsf{V}$ we obtain a unimodular quadratic Poisson structure.

Let $\mathsf{G} = \mathsf{Gr}\big(2, \mathsf{Sym}^2\mathsf{V}^*\big)$ be the Grassmannian of two-planes in $\mathsf{Sym}^2\mathsf{V}^*$. Over G there is a tautological line bundle $\mathcal{L}$ whose fibre at a plane $\mathsf{W} \subset \mathsf{Sym}^2\mathsf{V}^*$ is the line $\Lambda^2\mathsf{W} \otimes \det \mathsf{V}$. The construction above therefore gives rise to a $\mathbb{C}^*$-equivariant map $\phi : \mathsf{Tot}(\mathcal{L}) \to \mathsf{PS}\big(\mathbb{P}^3\big)$. Since G is projective, its image is closed.

Definition 8.5.1. The image of ϕ is the irreducible component $\mathsf{R}(2,2)$ of $\mathsf{PS}\big(\mathbb{P}^3\big)$.

If $\sigma \in \mathsf{R}(2,2)$ is given by a pair of quadratic forms g_1, g_2, then the elements of the pencil spanned by g_1 and g_2 are quadric surfaces in $\mathbb{P}^3$ that give the closures of the symplectic leaves. If the pencil is generic, all but four of these surfaces will be smooth, and the base locus will be an elliptic normal quartic curve $\mathsf{X} \subset \mathbb{P}^3$. Furthermore, each of the four singular members $\mathsf{Q}_0, \ldots, \mathsf{Q}_3$ will have a single isolated singularity. Let $p_i \in \mathsf{Q}_i$ be the singular point. Then the zero locus of σ is given by

$$\mathsf{Dgn}_0(\sigma) = \mathsf{X} \cup \{p_0, p_1, p_2, p_3\}$$

Polishchuk [117] showed that every Poisson structure vanishing on an elliptic normal curve in $\mathbb{P}^3$ is necessarily of this form.

Let us assume that the quadratic form g_1 describes one of the singular members of the pencil, and that g_2 describes a smooth one. Thus $\operatorname{rank}(g)_1 = 3$ and a well-known result in linear algebra says that, provided that the pair is suitably generic, we may choose coordinates on V such that

$$\begin{aligned} g_1 &= \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \\ g_2 &= x_0^2 + x_1^2 + x_2^2 + x_3^2. \end{aligned}$$

The Poisson structure is determined by a tensor of the form $(g_1 \wedge g_2) \otimes \mu \in \Lambda^2 \mathsf{Sym}^2 \mathsf{V}^* \otimes \det \mathsf{V}$. By absorbing an overall constant into the definitions of λ_1, λ_2 and λ_3, we may assume that $\mu = \partial_{x_0} \wedge \partial_{x_1} \wedge \partial_{x_2} \wedge \partial_{x_3}$. A change of variables $x_i \mapsto t_i x_i$ for $1 \leq i \leq 3$, where $t_i = \sqrt{\frac{\lambda_1 \lambda_2 \lambda_3}{\lambda_i}}$ replaces this tensor with

$$g_1' \wedge g_2' \otimes \mu$$

where

$$\begin{aligned} g_1' &= x_1^2 + x_2^2 + x_3^2 \\ g_2' &= x_0^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 \end{aligned}$$

Now we note that we can add an arbitrary multiple of g_1' to g_2' without changing the Poisson tensor. In this way, we may assume without loss of generality that $a_1 + a_2 + a_3 = 0$. We may now compute the Poisson bracket by the formula

$$\{f_1, f_2\} = \frac{df_1 \wedge df_2 \wedge dg_1' \wedge dg_2'}{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3}$$

giving the normal form in Table 8.1. The resulting Poisson algebra is the famous Skylanin bracket [131]. Its Poisson cohomology has recently been computed by Pelap [116].

Quantization

The quantizations of the generic Poisson structures in this component are the Skyanin algebras [131], introduced by Sklyanin in the same paper as the Poisson brackets. The relations are given in Table 8.2. The algebas have been well studied: for example, they are known to be regular and Calabi–Yau [17, 132, 137] and various modules have been constructed that correspond to the projective geometry of the elliptic curve [96, 134].

8.6 The $\mathsf{R}(1,3)$ component

Poisson geometry

Given a linear form $f \in \mathsf{V}^*$ and a cubic form $g \in \mathsf{Sym}^3 \mathsf{V}^*$, we obtain an integrable horizontal one-form

$$\alpha = 3g df - f dg$$

on V. Let $\mathsf{G} = \mathsf{Gr}(2, \mathsf{Sym}^3 \mathsf{V}^*)$ be the Grassmannian of three-planes in G. Thus G parametrizes pencils of cubic surfaces in $\mathbb{P}(\mathsf{V})$. Let $\mathsf{X} \subset \mathsf{G} \times \mathbb{P}(\mathsf{V}^*)$ be the closed subvariety consisting of pairs (W, H) of a cubic pencil and a plane in $\mathbb{P}^3$ such that $3\mathsf{H} \in \mathsf{W}$. There is a tautological line bundle $\mathcal{L}$ over X whose fibre at (W, H) is $\mathsf{W}/3\mathsf{H} \otimes \mathsf{ann}(\mathsf{H}) \otimes \det \mathsf{V}$, where $\mathsf{ann}(\mathsf{H}) \subset \mathsf{V}^*$

is the space of linear functional vanishing on H. Viewing g as an element of W/H^3 and f as an element of $\mathsf{ann}(\mathsf{H})$, the formula above gives rise to a canonical $\mathbb{C}^*$-equivariant map $\phi : \mathsf{Tot}(\mathcal{L}) \to \mathsf{PS}(\mathbb{P}^3)$. Since X is projective, the image of this map is closed.

Definition 8.6.1. The image of ϕ is the component $\mathsf{R}(1,3)$ of $\mathsf{PS}(\mathbb{P}^3)$.

Let $\mathsf{X} \subset \mathbb{P}^3$ be the surface defined by g and $\mathsf{H} \subset \mathbb{P}^3$ the plane defined by f. Then the elements of the pencil $|\mathsf{X}+3\mathsf{H}|$ are Poisson subspaces. If g is generic, so that X and $\mathsf{X}\cap\mathsf{H}$ are smooth, then X and H intersect in an elliptic curve $\mathsf{Y} \subset \mathsf{H}$ of degree three. Polishchuk [117] showed that every Poisson structure vanishing on Y is of this form.

We obtain the normal form for this family as follows: we may choose homogeneous coordinates x_0, x_1, x_2 for the plane H so that the restriction of g to this subspace is given by the Hesse form

$$\tfrac{\nu}{3}(x_0^3 + x_1^3 + x_2^3) - \lambda x_0 x_1 x_2$$

for some $\nu, \lambda \in \mathbb{C}$. Extend these coordinates to coordinates on V by adding the linear form $f = x_3$ that vanishes on H. Then

$$g = \tfrac{\nu}{3}(x_0^3 + x_1^3 + x_2^3) - \lambda x_0 x_1 x_2 + Q(x_0, x_1, x_2)x_3 + L(x_0, x_1, x_2)x_3^2 + Cx_3^3$$

for L and Q homogeneous of degrees 1 and 2, respectively, and $C \in \mathbb{C}$. Provided that Q and L are suitably generic, a coordinate change of the form $x_i \mapsto x_i + t_i x_3$ for $0 \le i \le 2$ allows us to assume that $L = 0$, i.e., that g has the simpler form

$$g = \tfrac{\nu}{3}(x_0^3 + x_1^3 + x_2^3) - \lambda x_0 x_1 x_2 + Q'(x_0, x_1, x_2)x_3 + Cx_3^3.$$

(A similar argument was used in [95] to find normal forms in the quantum case.) Since Cx_3^3 is an equation for $3\mathsf{H}$, it may be subtracted from g without changing the Poisson structure. Hence, we may assume that

$$g = \tfrac{1}{3}(x_0^3 + x_1^3 + x_2^3) - \lambda x_0 x_1 x_2 + Q(x_0, x_1, x_2)x_3$$

without loss of generality. Let us write

$$Q'(x_0, x_1, x_2) = \tfrac{1}{2} \sum_{i,j=0}^{2} b_{ij} x_i x_j$$

for $b_{ij} \in \mathbb{C}$ with $b_{ij} = b_{ji}$. Once again, the Poisson structure is determined by the additional data of an element of $\det \mathsf{V}$, but the only effect of changing this element is to rescale the bracket, and we may absorb this effect into a rescaling of g. Thus the normal form in

Table 8.1 can be obtained by the formula

$$\{x_i, x_j\} = \frac{dx_i \wedge dx_j \wedge dg \wedge dx_3}{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3}$$

with g as above. Now we note that a coordinate change of the form $x_3 \mapsto tx_3$ for some $t \in \mathbb{C}^*$ allows us to set the determinant of the matrix (b_{ij}) equal to some fixed constant, which we can take to be one. This restriction gives the parametrization in the table.

Quantization

Since x_3 is a Casimir element for the Poisson bracket on $\mathbb{C}[x_0, x_1, x_2, x_3]$ (i.e., $\{x_3, \cdot\} = 0$), the bracket is a central extension of a Poisson bracket on $\mathbb{C}[x_0, x_1, x_2]$. Correspondingly, the quantization should be a regular algebra that is an extension of a quantum $\mathbb{P}^2$ by a central element of degree one. Such algebras were classified in [95, Theorem 3.2.6], where one sees that the relations in the algebra have are similar in form to the Poisson brackets.

8.7 The $\mathsf{S}(2,3)$ component

Poisson geometry

The twisted one-forms on $\mathbb{P}^3$ that define the Poisson structures of type $\mathsf{S}(2,3)$ are obtained by pulling back a section of $\Omega^1_{\mathbb{P}^2}(4)$ along a linear projection $\mathbb{P}^3 \to \mathbb{P}^2$. Notice that since $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, we have $\Omega^1_{\mathbb{P}^2}(4) \cong \mathscr{X}^1_{\mathbb{P}^2}(1)$ and hence the foliation on $\mathbb{P}^2$ may equivalently be thought of as the action of an $\mathcal{O}_{\mathbb{P}^2}(1)$-valued vector field $Z \in \mathsf{H}^0\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\big)$.

Let $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. Then Z defines a map $\mathcal{O}_{\mathbb{P}^2} \to \mathscr{X}^1_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(1)$, which we view as a nilpotent operator

$$Z : \mathcal{E} \to \mathscr{X}^1_{\mathbb{P}^2} \otimes \mathcal{E},$$

making the pair $(\mathcal{E}, Z)$ a co-Higgs bundle. We refer the reader to the thesis of Rayan [121] for a detailed discussion of co-Higgs bundles on $\mathbb{P}^2$. We interpret the Higgs field as defining a Poisson module structure on $\mathcal{E}$ for the zero Poisson structure on $\mathbb{P}^2$. As a result, we obtain a Poisson structure on the projective bundle $\mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ that projects to the zero Poisson structure on $\mathbb{P}^2$. Blowing down the section $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1)) \subset \mathbb{P}(\mathcal{E})$ we obtain a Poisson structure $\mathbb{P}^3$, and every Poisson structure of type $\mathsf{S}(2,3)$ arises in this manner.

If Z is chosen generically, its orbits in $\mathbb{P}^2$ will be non-algebraic (see Example 2.9.1). Furthermore, since $c_2(\mathcal{T}_{\mathbb{P}^2}(1)) = 7$, a generic section Z will vanish at exactly seven points. A computation in sheaf cohomology shows that these seven points determine Z up to rescaling [32].

It follows that in the generic situation, the Poisson structure on $\mathbb{P}(\mathcal{E})$ vanishes on exactly seven fibres. Since the Higgs field has an upper-triangular form with respect to the splitting

$\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, the only other component of the zero locus is the section $\mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1))$ that we blow down. As a result, the Poisson structure on $\mathbb{P}^3$ has non-algebraic symplectic leaves and its zero locus is the union of 7 lines that all intersect at a single point $p \in \mathbb{P}^3$.

This construction has the following interpretation in homogeneous coordinates: let x_0, x_1, x_2 be homogeneous coordinates on $\mathbb{C}^3$. Then Z corresponds to a divergence-free vector field X on $\mathbb{C}^3$ whose coefficients are quadratic polynomials. Now extend the coordinates to $\mathbb{C}^4$ by adding the variable x_3. Then corresponding unimodular Poisson structure on $\mathbb{C}^4$ has the form $X \wedge \partial_{x_3}$. In particular, $\{x_i, x_j\} = 0$ for $0 \leq i, j \leq 2$.

To obtain the normal form for the Poisson bracket, it is enough to find a normal form for X on $\mathbb{C}^3$. Let us write

$$X = q_0\partial_0 + q_1\partial_1 + q_2\partial_2$$

for homogeneous quadratic polynomials q_0, q_1, q_2. By applying a linear automorphism, we may assume that Z vanishes at the points $[1,0,0], [0,1,0], [0,0,1] \in \mathbb{P}^2$. Correspondingly, the coefficients of $E \wedge X$ must vanish on the lines through $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Here, as above, E is the Euler vector field. Using these constraints, a straightforward calculation shows that q_i must have the form

$$q_i = a_i x_i^2 + x_i(b_i x_{i+1} + c_i x_{i-1}) + d_i x_i x_{i-1}$$

where the indices are taken modulo three. Generically, we have $a_i \neq 0$ for all i. Applying a coordinate transformation of the form $x_i \mapsto t_i x_i$ with $t_i \in \mathbb{C}^*$ and we may assume that $a_i = 1$ for all i. Then the requirement that X be divergence-free imposes the equations

$$b_{i-1} + c_{i+1} + 2 = 0$$

in the table. The remaining Poisson brackets are given by $\{x_i, x_3\} = Z(x_i)$.

Quantization

The quantization of these Poisson structures is easily obtained. Recall that if A is a $\mathbb{C}$-algebra and $Y : \mathsf{A} \to \mathsf{A}$ is a derivation, then the ***Ore extension*** $\mathsf{A}[t; Y]$ is the algebra whose underlying vector space is the polynomial ring $\mathsf{A}[t]$, with the relation

$$tr - rt = Y(r)$$

for $r \in \mathsf{A}$. For the present situation, the ring A is the polynomial ring $\mathbb{C}[x_0, x_1, x_2]$, and the derivation is the quadratic vector field X. The algebra is given by the Ore extension $\mathbb{C}[x_0, x_1, x_2][x_3; X]$, which gives a deformation quantization of the Poisson structure $X \wedge \partial_{x_3}$. The semiclassical limit is obtained by rescaling X to zero.

8.8 The $\mathsf{E}(3)$ component

Poisson geometry

The final component in the classification is the one that Cerveau and Lins–Neto call the exceptional component $\mathsf{E}(3)$. It is part of a larger class of Poisson structures on projective space that we will discuss in the next section, so we shall delay a detailed discussion of its geometry. For now, we simply explain the normal form for the Poisson bracket, which is easily extracted from the original paper [34]. Let $x_0, \ldots, x_3$ be linear coordinates on $\mathbb{C}^4$ and define the vector fields

$$X = x_1\partial_{x_1} + 2x_2\partial_{x_2} + 3x_3\partial_3$$
$$Y = 4x_0\partial_{x_1} + 4x_1\partial_{x_2} + 4x_2\partial_{x_3}.$$

A straightforward computation using the identity $[Y, X] = Y$ shows that the quadratic bivector field

$$\sigma = Y \wedge (X - \tfrac{5}{4}E)$$

is a unimodular Poisson structure and gives the brackets in Table 8.1. Every Poisson structure in the $\mathsf{E}(3)$ component is a degeneration of this one.

Quantization

We leave a discussion of the quantization to Section 8.10, where it will be obtained as part of a larger class. It will follow from the general discussion that the $\mathsf{E}(3)$ algebra with the relations given in Table 8.2 has the same Hilbert series as the polynomial ring.

8.9 Poisson structures and rational normal curves

In this section, we discuss a generalization of the $\mathsf{E}(3)$ Poisson structure to projective spaces of higher dimension, in the spirit of [31].

Consider the group $\mathsf{G} \subset \mathsf{Aut}(\mathbb{P}^1)$ consisting of projective transformations that preserve the point $\infty \in \mathbb{P}^1$. Then $\mathsf{G} \cong \mathbb{C}^* \ltimes \mathbb{C}$ is the group of affine transformations of the complex line $\mathbb{C} \subset \mathbb{P}^1$, which is a two-dimensional non-abelian Lie group. Its Lie algebra $\mathfrak{g} \subset \mathsf{H}^0(\mathbb{P}^1, \mathscr{X}^1_{\mathbb{P}^1})$ has a basis $U, V \in \mathfrak{g}$ with bracket relation $[V, U] = V$. Here V is the generator of translations—a section of $\mathsf{H}^0(\mathbb{P}^1, \mathscr{X}^1_{\mathbb{P}^1}(-2\cdot\infty))$.

Throughout this section, we shall deal with a complex manifold X equipped with an infinitesimal action $\rho : \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathscr{X}^1_{\mathsf{X}}$ and refer to X as a ***$\mathfrak{g}$-space***. We note that since $\Lambda^3\mathfrak{g} = 0$, the bivector field

$$\sigma = \rho(V \wedge U) \in \mathsf{H}^0(\mathsf{X}, \mathscr{X}^2_{\mathsf{X}})$$

will always be a Poisson structure.

The main example we have in mind is as follows: we view $\mathbb{P}^n$ as the n^{th} symmetric power of the projective line $\mathbb{P}^1$. Thus, a point in $\mathbb{P}^n$ is a degree-n divisor $\mathsf{D} \subset \mathbb{P}^1$. The action of G on $\mathbb{P}^1$ induces an action $\mathsf{G} \times \mathbb{P}^n \to \mathbb{P}^n$, and a corresponding infinitesimal action $\rho : \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathscr{X}^1_{\mathbb{P}^n}$. For $n \geq 2$, this action has orbits of dimension 0, 1 and 2. Correspondingly the bivector field $\sigma = \rho(V \wedge U) \in \mathsf{H}^0(\mathbb{P}^n, \mathscr{X}^2_{\mathbb{P}^n})$ is non-zero and defines a Poisson structure on $\mathbb{P}^n$ whose rank is generically equal to two. The case $n = 3$ recovers the $\mathsf{E}(3)$ Poisson structure in the Cerveau–Lins Neto classification.

For general X, the anchor map of the Poisson structure factors as

$$\Omega^1_{\mathsf{X}} \to \mathfrak{g} \otimes \mathcal{O}_{\mathsf{X}} \to \mathscr{X}^1_{\mathsf{X}}$$

and both maps are morphisms of Lie algebroids on X. We immediately have the following observation:

Proposition 8.9.1. *Let X be a $\mathfrak{g}$-space. Then every $\mathfrak{g}$-invariant subspace is a Poisson subspace. Moreover, the rank of σ at a point $x \in \mathsf{X}$ is equal to two if and only if the orbit of $\mathfrak{g}$ through x is two-dimensional. Otherwise, the rank is zero.*

Furthermore, any $\mathfrak{g}$-equivariant sheaf on X (i.e., a module over the Lie algebroid $\mathfrak{g} \otimes \mathcal{O}_{\mathsf{X}}$) is a Poisson module. In particular, all of the natural bundles, such as $\mathcal{T}_{\mathsf{X}}$, are Poisson modules.

For the case when $\mathsf{X} = \mathbb{P}^n$ is a symmetric power of the projective line, we have a number of privileged Poisson subspaces: if $0 \leq j, k \leq n$ and $jk \leq n$, then there is a natural map

$$\begin{array}{cccc} \mu_{j,k} & : & \mathbb{P}^j & \to & \mathbb{P}^n \\ & & \mathsf{D} & \mapsto & k\mathsf{D} + (n - jk)\infty \end{array}$$

These maps are clearly G-equivariant and therefore the image $\Delta_{j,k} = \mu_{j,k}(\mathbb{P}^j) \subset \mathbb{P}^n$ is a Poisson subspace. If $k = 0$ and $j > 0$, then the image is the single point $n \cdot \infty \in \mathbb{P}^n$, but if $k > 0$ this map is a k-Veronese embedding of $\mathbb{P}^j$ into a linear subspace of $\mathbb{P}^n$. For example, the subspace $\Delta_{j,1}$ is a j-plane $\mathbb{P}^j \subset \mathbb{P}^n$, and the curve $\Delta_{1,j}$ is a rational normal curve inside this j-plane. One readily verifies that the zero locus of σ is the union $\Delta_{1,1} \cup \cdots \cup \Delta_{1,n}$ of these rational curves. The situation for $n = 3$ is illustrated in Figure 8.2.

The description of the two-dimensional symplectic leaves is somewhat more involved. The case $n = 3$ is described as follows: consider the rational map

$$C : (\mathbb{P}^1)^3 \dashrightarrow \mathbb{P}^1$$

defined by the formula

$$C(x, y, z) = \frac{x - z}{y - z},$$

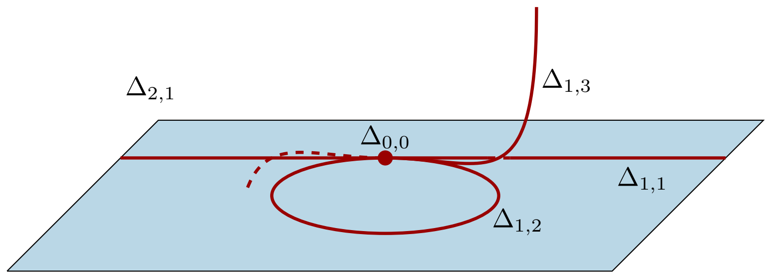


Figure 8.2: Poisson subspaces for the Poisson structure $\mathsf{E}(3)$ on $\mathbb{P}^3$. The plane $\Delta_{2,1} = \{p+q+\infty \mid p,q \in \mathbb{P}^1\}$ is a Poisson subspace that is generically symplectic. This plane contains the conic $\Delta_{1,2} = \{2\cdot p + \infty \mid p \in \mathbb{P}^1\}$ and the line $\Delta_{1,1} = \{p + 2\cdot\infty \mid p \in \mathbb{P}^1\}$, which meet at the single point $\Delta_{0,0} = \{n\cdot\infty\}$. The twisted cubic curve $\Delta_{1,3} = \{3\cdot p \mid p \in \mathbb{P}^1\}$ meets these two curves at the same point where it has a high-order tangency, and the union of the three curves is the zero locus of the Poisson structure.

which takes the cross-ratio of the four points (x, y, z, ∞). The level sets of this function are preserved by the G-action, but C is not invariant under the action of the symmetric group on $(\mathbb{P}^1)^3$ and therefore does not descend to a rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$. To remedy this problem, we use the j-invariant

$$j : \mathbb{P}^1 \to \mathbb{P}^1$$

defined by

$$j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2},$$

which has the property that $j \circ C$ actually *is* invariant under symmetrization. Hence, it descends to a rational map $\mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ of degree six—that is, a pencil of sextic surfaces in $\mathbb{P}^3$. The irreducible components of the surfaces in this pencil give the closures of the two-dimensional symplectic leaves of σ.

In higher dimension, there are several other special subspaces. For example, if $\mathsf{D} \subset \mathbb{P}^1$ is a divisor of degree twelve whose points form the vertices of a regular icosahedron, then the closure of the $\mathsf{Aut}(\mathbb{P}^1)$-orbit of D is a smooth, $\mathfrak{g}$-invariant Fano threefold $\mathsf{X} \subset \mathbb{P}^{12}$ called the Mukai–Umemura threefold [111]. It therefore inherits a Poisson structure, as observed in [101].

We may also construct some natural Poisson modules on $\mathbb{P}^n$ as follows: consider the symmetrization map

$$\begin{array}{rcccc} \pi & : & \mathbb{P}^{n-1} \times \mathbb{P}^1 & \to & \mathbb{P}^n \\ & & (\mathsf{D}, p) & \mapsto & \mathsf{D} + p. \end{array}$$

This map is clearly equivariant with respect to the G-actions on these varieties. It therefore defines an $n:1$ branched covering of $\mathbb{P}^n$ that is a Poisson map. Since, by definition the action of G stabilizes $\infty \in \mathbb{P}^1$, the subspace $\mathsf{Y} = \mathbb{P}^{n-1} \times \{\infty\} \subset \mathbb{P}^{n-1} \times \mathbb{P}^1$ is G-invariant, and

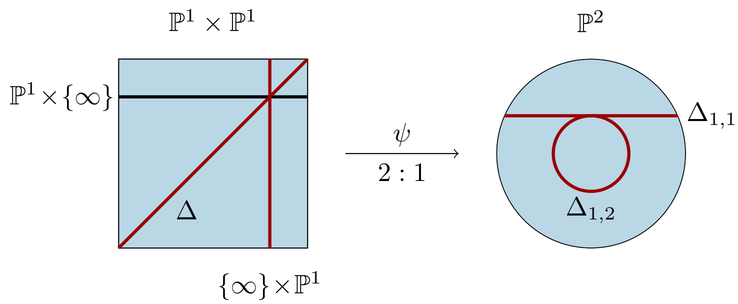


Figure 8.3: The divisor $\mathbb{P}^1 \times \{\infty\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a Poisson subspace and hence $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,1)$ is a Poisson module. Pushing the tensor powers of this module down to $\mathbb{P}^2$, we obtain Poisson module structures on the Schwarzenberger bundles.

hence the line bundle $\mathcal{L} = \mathcal{O}_{\mathbb{P}^{n-1}\times\mathbb{P}^1}(\mathsf{Y}) \cong \mathcal{O}_{\mathbb{P}^{n-1}\times\mathbb{P}^1}(0,1)$ is G-equivariant. The pushforward $\pi_*(\mathcal{L}^k) = \mathcal{E}_n^k$ is therefore a rank-n G-equivariant vector bundle on $\mathbb{P}^n$, giving rank-n Poisson module. These bundles are the Schwarzenberger bundles defined in [126]. The situation for $n = 2$ is illustrated in Figure 8.3.

Finally, we note that the infinitesimal symmetries of the Poisson structure are easily determined:

Proposition 8.9.2. *Every Poisson vector field for the Poisson structure σ on $\mathbb{P}^n$ is a scalar multiple of the generator $\rho(V)$ of translations:*

$$\mathsf{H}^1(\mathbb{P}^n, \sigma) = \mathbb{C}\{\rho(V)\} \subset \mathsf{H}^0\big(\mathbb{P}^n, \mathscr{X}^1_{\mathbb{P}^n}\big).$$

Moreover, $\rho(U)$ is a Liouville vector field, i.e.

$$\mathscr{L}_{\rho(U)}\sigma = -\sigma.$$

Proof. By the discussion in Section 4.3, any Poisson vector field must preserve the irreducible components of the zero locus of σ. In particular, such a vector field must preserve the rational normal curve $\Delta_{1,n} \subset \mathbb{P}^n$. But it is known (see [59, p. 154]) that the only vector fields on $\mathbb{P}^n$ that preserve the rational normal curve are those that arise from the action of $\mathsf{Aut}\big(\mathbb{P}^1\big)$, i.e., the vector fields that lie in the image of the Lie algebra homomorphism $\mathsf{H}^0\big(\mathbb{P}^1, \mathscr{X}^1_{\mathbb{P}^1}\big) \to \mathsf{H}^0\big(\mathbb{P}^n, \mathscr{X}^1_{\mathbb{P}^n}\big)$. Now the elements $U, V \in \mathfrak{g}$ may be extended to a basis for $\mathsf{H}^0\big(\mathbb{P}^1, \mathscr{X}^1_{\mathbb{P}^1}\big)$ by adding a generator W with $[W,U] = W$ and $[V,W] = 2U$. It remains to determine which linear combinations of U, V, W bracket trivially with $V \wedge U$. A straightforward calculation using the bracket relations shows that the only such linear combinations are multiples of V alone and also shows that $\rho(U)$ is Liouville. $\square$

8.10 The Coll–Gerstenhaber–Giaquinto formula

Let A be a commutative $\mathbb{C}$-algebra on which the Lie algebra $\mathfrak{g} = \mathbb{C}\{U, V\}$ from the previous section acts by derivations. We define an associative product

$$\mathsf{A}[[\hbar]] \otimes_{\mathbb{C}} \mathsf{A}[[\hbar]] \to \mathsf{A}[[\hbar]]$$

by the formula

$$f \star g = \sum_{k=0}^{\infty} \hbar^k V^k(f) \cdot \binom{U}{k}(g) = fg + \hbar V(f)U(g) + O(\hbar^2) \tag{8.1}$$

where

$$\binom{U}{k} = \frac{1}{k!} U(U-1)\cdots(U-k+1).$$

This product was introducted by Coll, Gerstenhaber and Giaquinto in [39], and we refer to it as the CGG formula. Its semi-classical limit is the Poisson structure $\{f, g\} = V(f)U(g) - U(f)V(g)$ defined by the bivector $V \wedge U$. Notice that if B is another algebra on which $\mathfrak{g}$ acts and $\phi : \mathsf{A} \to \mathsf{B}$ is a $\mathfrak{g}$-equivariant ring homomorphism, then the induced map $\widetilde{\phi}$: $\mathsf{A}[[\hbar]] \to \mathsf{B}[[\hbar]]$ is also a ring homomorphism with respect to the star products. Similarly, if M is an A-module on which $\mathfrak{g}$ acts by derivations, then $\mathsf{M}[[\hbar]]$ inherits the structure of an $\mathsf{A}[[\hbar]]$-bimodule via the formulae

$$f \star m = \sum_{k=0}^{\infty} \hbar^k (f) \cdot \binom{U}{k}(m)$$

and

$$m \star f = \sum_{k=0}^{\infty} \hbar^k (m) \cdot \binom{U}{k}(f)$$

for $f \in \mathsf{A}[[\hbar]]$ and $m \in \mathsf{M}[[\hbar]]$, and $\mathfrak{g}$-equivariant module maps $\mathsf{M} \to \mathsf{N}$ define bimodule homomorphisms $\mathsf{M}[[\hbar]] \to \mathsf{N}[[\hbar]]$.

As it stands, the power series is merely formal, while we seek actual, convergent deformations. However, there is a useful criterion that can be used to guarantee convergence. Recall that a derivation of $Z : \mathsf{A} \to \mathsf{A}$ is ***locally nilpotent*** if for every $a \in \mathsf{A}$ there exists a $k \in \mathbb{N}$ such that $Z^k(a) = 0$. The following observation is immediate from the CGG formula (8.1):

Lemma 8.10.1. *If the derivation $V \in \mathfrak{g}$ acts locally nilpotent on A, then the CGG formula for $f \star g$ truncates to a polynomial in $\hbar$ for any $f, g \in \mathsf{A}$, i.e., it defines a map*

$$\mathsf{A} \otimes_{\mathbb{C}} \mathsf{A} \to \mathsf{A}[\hbar].$$

Evaluation at a particular value $\hbar \in \mathbb{C}$ then gives an associative product

$$\star_\hbar : \mathsf{A} \otimes_{\mathbb{C}} \mathsf{A} \to \mathsf{A}.$$

We denote the corresponding deformed ring by $\mathsf{A}^\hbar$. If, in addition, M is an A-module on which $\mathfrak{g}$ acts by derivations and $V \in \mathfrak{g}$ acts locally nilpotently on M, then M becomes an $\mathsf{A}^\hbar$-bimodule $\mathsf{M}^\hbar$.

We are interested in the geometric situation in which $\mathsf{A} = \bigoplus_{k=0}^\infty \mathsf{H}^0(\mathsf{X}, \mathcal{L}^{\otimes k})$ is the homogeneous coordinate ring of a projective variety X with respect to a very ample, $\mathfrak{g}$-equivariant line bundle $\mathcal{L}$. Correspondingly, A is a finitely generated graded ring and each grading component $\mathsf{A}_k = \mathsf{H}^0(\mathsf{X}, \mathcal{L}^{\otimes k})$ is a finite-dimensional $\mathbb{C}$-vector space—i.e., A is ***finitely graded***. Notice that the grading components are also preserved by the action of $\mathfrak{g}$. We therefore have a further decomposition

$$\mathsf{A}_k = \bigoplus_{\lambda \in \mathbb{C}} \mathsf{A}_k^\lambda$$

into generalized eigenspaces for the action of $U \in \mathfrak{g}$. The commutation relation $[V, U] = V$ guarantees that V sends A_k^λ to $\mathsf{A}_k^{\lambda-1}$, and hence V acts nilpotently on A_k, so that it acts locally nilpotently on all of A.

Moreover, we obtain an $\mathbb{R}$-filtration $\mathsf{F}^\bullet \mathsf{A}_k$ defined by

$$\mathsf{F}^s \mathsf{A}_k = \bigoplus_{\mathrm{Re}\lambda \le s} \mathsf{A}_k^\lambda$$

with $\mathsf{F}^s\mathsf{A}_k \subset \mathsf{F}^t\mathsf{A}_k$ for $s \le t$, and this filtration is preserved by the action of $\mathfrak{g}$. It follows from the CGG formula that if $f \in F^s\mathsf{A}$ and $g \in F^t\mathsf{A}$ then

$$f \star_\hbar g = fg \mod \mathsf{F}^{s+t-1}\mathsf{A}.$$

Thus $\mathsf{A}^\hbar$ is an $\mathbb{R}$-filtered ring whose associated graded ring is isomorphic to A itself. In particular, if A is generated by $\bigoplus_{k=0}^n \mathsf{A}_k$, then the deformed ring $\mathsf{A}^\hbar$ is also generated by $\bigoplus_{k=0}^n \mathsf{A}_k$, and similar statements hold for other ring-theoretic properties; for example if A is a domain, then so is $\mathsf{A}^\hbar$.

By applying these observations to the case when $\mathsf{A} = \mathbb{C}[x_0, x_1, x_2, x_3]$ is the homogeneous coordinate ring of $\mathbb{P}^3$ with the action of $\mathfrak{g}$ described in the previous section, we obtain the quadratic algebra in Table 8.2, giving a quantization of the $\mathsf{E}(3)$ Poisson structure. The specific relations shown are obtained by setting $\hbar = 1$. In fact, corresponding to the fact that the Poisson structure can be rescaled using the flow of the Liouville vector field U, any value of $\hbar \neq 0$ will produce an isomorphic algebra:

Proposition 8.10.2. *Let A be a finitely graded algebra on which $\mathfrak{g}$ acts by homogeneous* *derivations. Then the $\mathbb{C}$-linear map $U : \mathsf{A} \to \mathsf{A}$ exponentiates to an algebra isomorphism*

$$e^{tU} : \mathsf{A}^{\hbar} \to \mathsf{A}^{e^{-t}\hbar}$$

for all $t \in \mathbb{C}$, so that the quantizations for different values of $\hbar \neq 0$ are all isomorphic.

Proof. We have the identity

$$e^{tU}(Wf) = (e^{t\mathrm{ad}U}W)(e^{tU}f)$$

for all $W \in \mathfrak{g}$ and $f \in \mathsf{A}$. Since $[U,V] = -V$, we have $e^{t\mathrm{ad}U}V = e^{-t}V$, giving the commutation rules

$$e^{tU}V^k = e^{-kt}V^k e^{tU}$$

and

$$e^{tU}\binom{U}{k} = \binom{U}{k}e^{tU}.$$

The result now follows from a direct calculation using the CGG formula (8.1). □

The nice thing about the universality of the CGG formula is that it allows us to promote all of the equivariant geometry to quantum geometry. For example, the formula for the product immediately gives the following

Proposition 8.10.3. *Let X be a projective variety on which $\mathfrak{g}$ acts, and let $\mathcal{L}$ be a very ample line bundle that is $\mathfrak{g}$-equivariant. Let $\mathsf{A} = \bigoplus_{k=0}^{\infty} \mathsf{H}^0\big(\mathsf{X}, \mathcal{L}^{\otimes k}\big)$ be the corresponding homogeneous coordinate ring. If $\mathsf{Y} \subset \mathsf{X}$ is a $\mathfrak{g}$-invariant subspace, then the homogeneous ideal $\mathsf{I} = \bigoplus_{k=0}^{\infty} \mathsf{H}^0\big(\mathsf{X}, \mathcal{I}_{\mathsf{Y}}\mathcal{L}^{\otimes k}\big) \subset \mathsf{A}$ is a two-sided ideal for the deformed product $\star_{\hbar}$. Similarly, if $\mathcal{E}$ is a $\mathfrak{g}$-equivariant sheaf then corresponding graded A-module $\mathsf{M}(\mathcal{E}) = \bigoplus_{k=0}^{\infty} \mathsf{H}^0\big(\mathsf{X}, \mathcal{E} \otimes \mathcal{L}^{\otimes k}\big)$ is a bimodule for the product $\star_{\hbar}$.*

In particular, for the case $\mathsf{X} = \mathbb{P}^n$ described in the previous section, we see that all of the embeddings $\mu_{j,k} : \mathbb{P}^j \to \mathbb{P}^k$ define two-sided ideals in $\mathsf{A}^{\hbar}$. In particular, we have the full flag

$$\Delta_{0,1} \subset \Delta_{1,1} \subset \cdots \subset \Delta_{n-1,1} \subset \mathbb{P}^n$$

of linear $\mathfrak{g}$-invariant subspaces. If we choose homogeneous coordinates such that $\Delta_{j,1}$ is the zero locus of $(x_n, \ldots, x_{j+1})$, we obtain a regular sequence $(x_n, x_{n-1}, \ldots, x_0)$ of elements in $\mathsf{A}^{\hbar}$ whose factor ring is the ground field $\mathbb{C}$. One can also verify that the points on the curve $\Delta_{1,1} \cup \cdots \cup \Delta_{n,1}$, which is the zero locus of the Poisson structure, give rise to point modules in the sense of [4].

As discused in the previous section, there are various other interesting features of the $\mathfrak{g}$-equivariant geometry. For example, the Mukai–Umemura threefold $\mathsf{X} \subset \mathbb{P}^{12}$ is $\mathfrak{g}$-invariant and hence we obtain a quantization of its homogeneous coordinate ring, which is a bimodule over the quantized homogeneous coordinate ring of $\mathbb{P}^{12}$. Similarly, since the Schwarzenberger bundles are $\mathfrak{g}$-equivariant the corresponding graded modules $\mathsf{M}(\mathcal{E}_n^k) = \bigoplus_{j\in\mathbb{Z}}^{\infty} \mathsf{H}^0\big(\mathbb{P}^n, \mathcal{E}_n^k(j)\big)$ quantize to bimodules. It would be interesting to study these modules in more detail—for example, to write down explicit presentations.

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