Resurgence in Geometry and Physics

Brent Pym

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Lecture 3

Abstract

We introduce the Borel transform as an operation that takes a germ of a function on a Riemann surface, and produces a one-form on a vector space. Its inverse is the Laplace transform. We then introduce the classes of Gevrey series and Gevrey function, and use the Borel and Laplace transforms to prove Watson's theorem on the existence and uniqueness of sums for Gevrey series.

Contents

At the end of the last lecture, we explained that every formal power series can be realized as the asymptotic expansion of a function defined in an arbitrary sectorial neighbourhood of a point. But because there are many functions whose asymptotic expansions are zero, this procedure does not give a unique sum for the asymptotic series. In this lecture, we will eventually single out a "nice" class of series—the series of Gevrey class—for which better uniqueness results can be obtained. But in order to motivate the introduction of this class of series, we will first introduce another key set of players in our story: the Borel and Laplace transform.

1 Integral transforms

An integral transform is a procedure that relates functions or tensors defined on different spaces X and T. In our case, X and T will both be Riemann surfaces, and we will construct a correspondence

$$\begin{tikzcd} \text{holomorphic functions on } \mathsf{X} \arrow[r, bend left] & \text{holomorphic one-forms on } \mathsf{T} \arrow[l, bend left] \end{tikzcd}$$

that will be used to approach the problem of resumming divergent series. In fact, it will be important for us later in the course that T has an additional structure: it is a one-dimensional complex vector space.

The constructions as presented here rely on the choice of a coordinate x on X centred at a point $p \in \mathsf{X}$. Meanwhile, the space T, being a one-dimensional vector space, is equipped with a global linear coordinate t. It seems to be conceptually helpful to think of T as the tangent space to X at p, and the coordinate t as the natural coordinate $t = dx|_p \in \mathsf{T}^*$ induced by x. We then imagine that the correspondence allows us to study functions on X by "zooming in" on X near p so drastically that X is replaced by its tangent space.

With T interpreted in this way, the constructions will be invariant under changes of coordinate in which x is rescaled by a constant. But arbitrary changes of coordinate will result in actually different integral transforms, so this geometric interpretation should not be taken too seriously. We remark, however, that if X is simply connected and non-compact, there is a special class of coordinates on X: the uniformizing ones. These are the coordinates that identify (X, p) with either $(\mathbb{C}, 0)$ or $(\Delta, 0)$, where $\Delta \subset \mathbb{C}$ is a disk centred at zero. Since the ratio of any two uniformizing coordinates is a constant, we can make our constructions canonical by requiring x to be uniformizing. This solution has the obvious drawback that uniformizations of a given abstract Riemann surface are rather difficult to find.

We therefore assume throughout these notes that the coordinates x and t have been fixed, and use the interpretation of T as the tangent space only as an intuitive guide.

1.1 The Borel transform

We begin by describing the procedure that takes a function on X to a one-form on T, called the ***Borel transform***. Like any integral transform, it is defined by integrating f against a "kernel" that depends on both x and t. In this case, the kernel is the two-form

$$\frac{e^{t/x}}{x^2} dx \wedge dt = -d(e^{t/x}) \wedge dt$$

defined on the product $\mathsf{X} \times \mathsf{T}$ using the coordinate functions x and t.

This form evidently has an essential singularity where $x = 0$, i.e. along the locus $\{p\} \times \mathsf{T}$. But if we integrate it along a path in X that avoids the singularity, we clearly obtain a holomorphic one-form defined on all of T.

More precisely, suppose that $f \in \mathcal{O}_{\mathsf{X},p}$ is the germ of a holomorphic function defined in a neighbourhood U of p. By shrinking U if necessary, we may assume without loss of generality that U is simply connected, so that $\mathsf{U} \setminus \{p\}$ is isomorphic to a punctured disk. Let γ be a loop in $\mathsf{U} \setminus \{p\}$ that wraps once around p in the positively oriented sense, i.e. a generator for the first homology of $\mathsf{U} \setminus \{p\}$, as shown in Figure 1. Then the ***Borel transform of f with respect to the coordinate*** x is the one-form defined by the formula

$$\mathscr{B}(f) = \frac{1}{2\pi i} \left(\int_\gamma f e^{t/x} \frac{dx}{x^2} \right) \cdot dt \in \Omega^1(\mathsf{T})$$

By Cauchy's theorem, the result is independent of the chosen loop γ. It is therefore apparent that the Borel transform only depends on the behaviour of f in an arbitrarily small neighbourhood of p, i.e. on the germ $f \in \mathcal{O}_{\mathsf{X},p}$.

Exercise 1. Show that the Borel transform is unchanged if the coordinates x and t undergo a simultaneous rescaling by a constant λ, as would happen if T were the tangent space of X at p and t were the linear coordinate $dx|_p \in \mathsf{T}^*$. What happens if we apply a more complicated change of coordinates? □

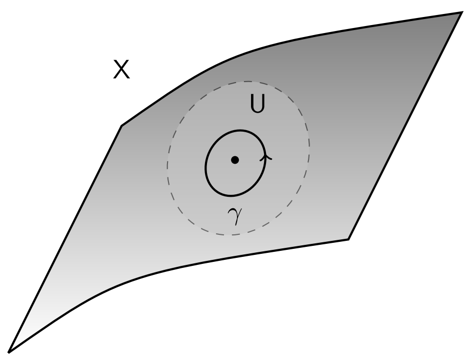

Figure 1: Integration contour for the Borel transform

The Borel transform may be rewritten in various different ways. For example, integration by parts using the identity

$$f \cdot \frac{e^{t/x} dx}{x^2} = -f \cdot d(e^{t/x}) = -d(f e^{t/x}) + e^{t/x} df$$

gives the formula

$$\mathscr{B}(f) = \frac{1}{2\pi i} \left(\int_\gamma e^{t/x} df \right) \cdot dt,$$

which shows that $\mathscr{B}(f)$ is unchanged if we add a constant to f. Meanwhile, the residue theorem gives

$$\mathscr{B}(f) = \left(\mathrm{Res}_{x=0} \frac{f e^{t/x} dx}{x^2}\right) \cdot dt.$$

which facilitates the computation of the transform.

Proposition 1. *Let $f \in \mathcal{O}_{\mathsf{X},p}$ be the germ of a holomorphic function at $p \in \mathsf{X}$. Then the following statements hold:*

1. *The Borel transform of f is a holomorphic one-form defined on all of T, with at most exponential growth at infinity. In other words,*

   $$\mathscr{B}(f) = g\, dt$$

   where g is an entire function such that

   $$|g| \leq Ce^{M|t|}$$

   for some constants $C, M > 0$.

2. *If f has the Taylor expansion*

   $$f = f(p) + \sum_{k=0}^{\infty} a_k x^{k+1}$$

   then

   $$\mathscr{B}(f) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} dt$$

3. *The Borel transform defines an injective $\mathbb{C}$-linear map*

   $$\mathscr{B} : \mathfrak{m}_{\mathsf{X},p} \to \Omega^1(\mathsf{T})$$

   where $\mathfrak{m}_{\mathsf{X},p} \subset \mathcal{O}_{\mathsf{X},p}$ denotes the ideal of functions that vanish at p.

4. *The following identity holds for all $f \in \mathcal{O}_{\mathsf{X},p}$:*

   $$\mathscr{B}(x^2 \partial_x f) = t\mathscr{B}(f)$$

Sketch of proof. The first statement follows easily from the fact that the integrand $fe^{t/x}x^{-2}dx$ depends analytically on t, and evidently has a uniform exponential bound on any circle of fixed radius $|x| = \epsilon > 0$.

For the second statement, we observe that by linearity, it is enough to show that $\mathscr{B}(x^{k+1}) = \frac{t^k}{k!}dt$. For this, we use the residue formula and calculate

$$\begin{aligned}
\mathscr{B}(x^{k+1}) &= \left(\mathrm{Res}_{x=0}\, x^{k+1}e^{t/x}\frac{dx}{x^2}\right)\cdot dt \\
&= \left(\mathrm{Res}_{x=0}\left(1+\frac{t}{x}+\frac{1}{2}\frac{t^2}{x^2}+\frac{1}{3!}\frac{t^3}{x^3}+\cdots\right)x^{k-1}dx\right)\cdot dt \\
&= \left(\mathrm{Res}_{x=0}\left(x^{k-1}+tx^{k-2}+\cdots+\frac{t^k}{k!}x^{-1}+\cdots\right)dx\right)\cdot dt \\
&= \frac{t^k dt}{k!}
\end{aligned}$$

as required.

The third statement is an immediate consequence of the second: clearly the only information that is lost about the Taylor series of f is the constant term $f(p)$, and this ambiguity can be removed by requiring f to vanish at p. Since the germ of an analytic function is determined by its Taylor series, the statement about injectivity follows.

The fourth statement is obtained easily using integration by parts. □

1.2 The Laplace transform

The Borel transform has given us an injective map

$$\mathscr{B} : \mathfrak{m}_{\mathsf{X},p} \to \Omega^1(\mathsf{T})$$

which takes a germ of a function that vanishes at p and produces an entire one-form on T with exponential growth at infinity. Let us denote the set of such one-forms by $\Omega^1_{\exp}(\mathsf{T})$.

We claim that, in fact, we have an isomorphism

$$\mathscr{B} : \mathfrak{m}_{\mathsf{X},p} \xrightarrow{\sim} \Omega^1_{\exp}(\mathsf{T}).$$

To see that this is true, we will explicitly construct the inverse map, which is given by the ***Laplace transform***

$$\mathscr{L} : \Omega^1_{\exp}(\mathsf{T}) \to \mathfrak{m}_{\mathsf{X},p},$$

The construction again uses our coordinates x and t: it is an integral transform whose kernel is given by the function

$$e^{-t/x}$$

defined on $\mathsf{X}\times\mathsf{T}$. But now some more care is required in defining the integration cycle; we will be integrating along an infinite contour in T, and we must ensure that the contour lies in a region where $e^{-t/x}$ decays.

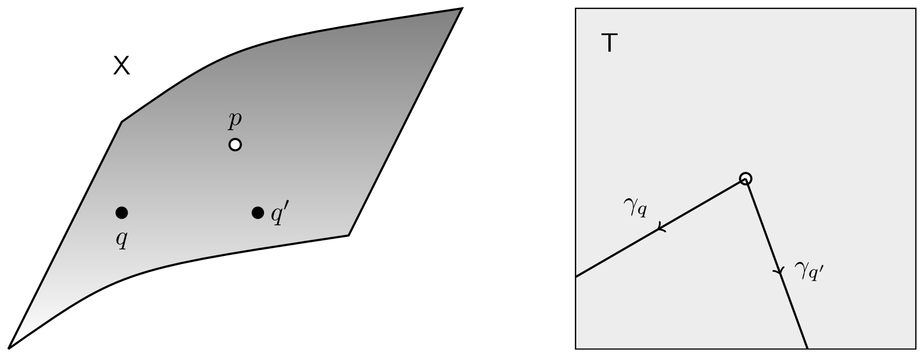

Figure 2: Integration contours for the Laplace transform

To this end, we observe that, given a point $q \in \mathsf{X}$, we can always find a ray γ_q from 0 to ∞ in T such that $t/x(q)$ takes on positive real values along γ_q. Here $x(q) \in \mathbb{C}$ is the fixed complex value assigned to q by the coordinate x. If we use our intuitive picture of T as the tangent space of X at p, then γ_q can be thought of as a tangent ray at p that points towards q as shown in Figure 2. In any case, with this choice of ray γ_q, the kernel function $e^{-t/x(q)}$ decays rapidly as $t \to \infty$ along γ_q.

Now suppose that $\omega = g\, dt \in \Omega^1_{\mathrm{exp}}(\mathsf{T})$ is a form on T that satisfies an exponential bound

$$|g| \leq Ce^{M|t|},$$

for some $C, M > 0$. Then the integral

$$\int_{\gamma_q} e^{-t/x(q)}\omega = \int_{\gamma_q} e^{-t/x(q)} g(t)\, dt$$

will be guaranteed to converge as long as $|x(q)| < M^{-1}$.

In this way, we obtain the ***Laplace transform*** $\mathscr{L}(\omega) \in \mathcal{O}_{\mathsf{X},p}$, defined by the formula

$$(\mathscr{L}(\omega))(q) = \int_{\gamma_q} e^{-t/x(q)}\omega.$$

This integral makes sense as long as q is sufficiently close to p. With a bit more work it is easy to see that the value of the integral depends holomorphically on q, and therefore defines a germ of a holomorphic function at p, as required.

Exercise 2. Show that the integral is independent of the ray γ_q, as long as this path goes to infinity in a direction such that $e^{-t/x(q)}$ decays. This can be done directly, by the usual trick of estimating the integral along an arc of large radius.

Or, for a more geometric approach, consider the manifold with boundary

$$\hat{\mathsf{T}} = \mathsf{T} \cup \mathsf{S}_\infty \mathsf{T},$$

obtained by forming the real-oriented blowup of the Riemann sphere $\mathsf{T} \cup \{\infty\}$ at infinity. Show that for fixed q, the one-form $e^{-t/x(q)}\omega$ vanishes on a sector

$\mathsf{A} \subset \mathsf{S}_\infty \mathsf{T} = \partial \hat{\mathsf{T}}$ in the boundary circle, and apply Stokes' theorem. Interpret the integration as a pairing on appropriate relative homology groups. □

Exercise 3. Use the path independence of the integral and "differentiation under the integral sign" to show that $\mathscr{L}(\omega)$ is holomorphic. □

Theorem 1. *Given a coordinate x on X centred at p, the corresponding Borel and Laplace transforms give mutually inverse isomorphisms*

$$\begin{tikzcd} \mathfrak{m}_{\mathsf{X},p} \arrow[r, bend left, "\mathscr{B}"] & \Omega^1_{\exp}(\mathsf{T}) \arrow[l, bend left, "\mathscr{L}"] \end{tikzcd}$$

between germs of holomorphic functions on X that vanish at p, and entire forms on T with at most exponential growth at infinity.

Proof. The proof is obtained by the usual trick of interchanging the order of integration. We leave it as an exercise to the reader. □

2 Formal transform and Gevrey functions

2.1 Formal Borel and Laplace transforms

In light of Proposition 1, which gave an explicit formula for the integral transform in terms of Taylor expansions, we may define the ***formal Borel and Laplace transforms*** $\widehat{\mathscr{B}}$, $\widehat{\mathscr{L}}$ by the formulae

$$\widehat{\mathscr{B}}\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right) = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!} dt$$

and

$$\widehat{\mathscr{L}}\left(\sum_{k=0}^{\infty} b_k t^k \, dt\right) = \sum_{k=0}^{\infty} k! b_k x^{k+1}.$$

using our chosen coordinates x and t. They give mutually inverse isomorphisms

$$\begin{tikzcd} \widehat{\mathfrak{m}}_{\mathsf{X},p} \arrow[r, bend left, "\widehat{\mathscr{B}}"] & \widehat{\Omega}^1(\mathsf{T}) \arrow[l, bend left, "\widehat{\mathscr{L}}"] \end{tikzcd}$$

where we denote by

$$\widehat{\mathfrak{m}}_{\mathsf{X},p} \cong x\mathbb{C}[[x]] \subset \mathbb{C}[[x]] = \widehat{\mathcal{O}}_{\mathsf{X},p}$$

the formal power series with zero constant coefficient, and

$$\widehat{\Omega}^1(\mathsf{T}) \cong \mathbb{C}[[t]]\, dt$$

the one-forms on T with formal power series coefficients.

2.2 Gevrey series

Notice that, because the Borel transform divides the kth coefficient by $k!$, it is perfectly possible for the Borel transform to have a nonzero radius of convergence, even if the original series diverges.

Definition 1. A formal power series has ***Gevrey class 1*** if its Borel transform has positive radius of convergence. We denote by $\widehat{\mathcal{G}}_{\mathsf{X},p} \subset \widehat{\mathcal{O}}_{\mathsf{X},p} = \mathbb{C}[[x]]$ the series of Gevrey class one.

Convergent power series have coefficients that grow at most exponentially, so the following lemma is immediate:

Lemma 1. *The formal power series*

$$\sum_{k=0}^{\infty} a_k x^{k+1}$$

has Gevrey class 1 if and only if there exists a constant $M > 0$ such that

$$|a_k| < M^k k!$$

for all $k \in \mathbb{Z}_{\geq 0}$.

As the name suggests, there are other Gevrey classes, which correspond to series with different bounds on their coefficients. We will not make much use of these other classes here, so we will typically drop the terminology "class 1" and simply say that f ***is a Gevrey series***. As the following exercises show, the Gevrey series define a natural subalgebra $\widehat{\mathcal{G}}_{\mathsf{X},p} \subset \widehat{\mathcal{O}}_{\mathsf{X},p}$ of the ring of formal functions at p, defined independently of any coordinate on X.

Exercise 4. Show that the product of two Gevrey series is Gevrey. □

Exercise 5. Suppose that $x = x(u)$ where u is another coordinate on X at p. Using Lemma 1 above, together with Faà di Bruno's formula for the composition of power series, show that $f(x) \in \mathbb{C}[[x]]$ is Gevrey if and only if $f(x(u)) \in \mathbb{C}[[u]]$ is Gevrey. □

The importance of Gevrey series is that they are the ones which we can hope to resum by the method of Borel summation, as follows. Given a Gevrey series $f \in \widehat{\mathcal{G}}_{\mathsf{X},p}$ we consider its formal Borel transform $\widehat{\mathscr{B}}(f) \in \widehat{\Omega}^1(\mathsf{T})$, which by definition converges to define a holomorphic form ω on T in a neighbourhood of zero. We then attempt to define a function of x by extending ω holomorphically to all of T, and taking its Laplace transform.

We will spend most of the rest of the course dealing with the fact that ω will not actually extend holomorphically to all of T. In good cases, it will have only a discrete collection of branch cuts and poles; these are the "resurgent" cases. But for the rest of this lecture, we will see that some interesting results can be obtained even when the form does not extend beyond its original disk of convergence.

2.3 Gevrey functions

Given that we have singled out the Gevrey series as particularly worthy of our attention, it is natural to try to realize these series as asymptotic expansions of some restricted class of functions.

Suppose that $f \in \widetilde{\mathcal{O}}(\mathsf{U})$ admits an asymptotic expansion

$$f \sim \sum_{k=0}^{\infty} a_k x^k$$

in a sectorial neighbourhood U of a point p. Recall from the previous lecture that the coefficients of the expansion in a coordinate x may be computed by a Taylor-type formula

$$a_k = \lim_{x \to 0} \frac{f^{(k)}(x)}{k!}$$

and the limit is uniform in any subsectorial neighbourhood $\mathsf{U}' \subset \mathsf{U}$ whose closure is contained in U. In fact, this statement can be upgraded so that the error term on U' is given by

$$\left| f - \sum_{k=0}^{N} a_k x^k \right| \leq \frac{1}{N!} \sup_{y \in \mathsf{U}', |y| \leq |x|} \left| f^{N+1}(y) \right| .$$

This estimate is proven in a fashion similar to the usual Taylor remainder theorem.

Definition 2. Let U be a sectorial neighbourhood of a point $p \in \mathsf{X}$. A function $f \in \mathcal{O}(\mathsf{U})$ is ***Gevrey*** if for every subsectorial neighbourhood $\mathsf{U}' \subset \mathsf{U}$ whose closure lies in U, there exists a constant $M > 0$ such that

$$\sup_{\mathsf{U}'} \left| \frac{f^{(k)}}{k!} \right| < M^k k!$$

for all $k \in \mathbb{Z}_{\geq 0}$.

This condition ensures that every Gevrey function admits an asymptotic expansion that is a Gevrey series. We therefore have a map

$$\mathcal{G}(\mathsf{U}) \to \widehat{\mathcal{G}}_{\mathsf{X},p}$$

defined by taking asymptotic expansions. Since we recall that asymptotic expansions are far from unique, we expect that the kernel of this map will be rather large. For example, we may expect it to contain any function that decays like $e^{-1/|x|^\beta}$ with $\beta > 0$, but the following lemma limits the possibilities:

Lemma 2. *The kernel of the expansion map*

$$\mathcal{G}(\mathsf{U}) \to \widehat{\mathcal{G}}_{\mathsf{X},p}$$

is the space of functions that decay exponentially in U*.*

Sketch of proof. See [1, p. 137] for details. Using the Gevrey condition, the fact that f is asymptotic to zero, and the Taylor remainder formula for the asymptotic expansion, we obtain in the coordinate x near p a bound of the form

$$|f| \leq C(M|x|)^n n!$$

on U'. With a bit of work starting from Stirling's approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

the equivalence between this bound and an exponential bound of the form

$$|f| \leq C' \exp(-M'/|x|)$$

can be established. □

Corollary 1. *The function e^{1/x^β} is not Gevrey for any $\beta < 1$.*

2.4 Resummation of Gevrey series

We may now combine our considerations to obtain the following key result on the resummation of Gevrey series, known as Watson's theorem:

Theorem 2 (Watson [2])**.** *Let $p \in \mathsf{X}$ be a point in a Riemann surface, and let A be a sector at p. Consider the map*

$$\mathcal{G}(\mathsf{A}) \to \widehat{\mathcal{G}}_{\mathsf{X},p},$$

which takes a germ of a Gevrey function defined in a sectorial neighbourhood of A, and produces its asymptotic expansion. Then

1. *If $|\mathsf{A}| > \pi$ this map is injective.*
2. *If $|\mathsf{A}| < \pi$ this map is surjective.*

Proof. (Taken from [1, p. 138–139].) For the first statement, we note that any function in the kernel of the expansion map must decay exponentially in A by Lemma 2. But since $|\mathsf{A}| > \pi$, the Fragmen–Lindelöf principle from Lecture 2 ensures that such a function is identically zero.

For the second statement, we apply Borel summation. Suppose that $f \in \widehat{\mathcal{G}}_{\mathsf{X},p}$ is a Gevrey series and choose a coordinate x centred at p. We denote by $\mathsf{U} \subset \mathsf{X}$ a small sectorial neighbourhood with opening A, and assume without loss of generality that U is contained in the domain of the coordinate x.

Subtracting a constant if necessary, we may assume that f has the form

$$f = \sum_{k=0}^{\infty} a_k x^{k+1}.$$

Let $\omega = \widehat{\mathscr{B}}(f)$ be the formal Borel transform of the series with respect to the coordinate x. Since f is Gevrey, ω defines a holomorphic one-form in some disk

$\mathsf{V} \subset \mathsf{T}$ centred at zero. In general, it may be impossible to extend this form beyond V, so we cannot apply the Laplace transform directly to ω. But for our purposes, it is sufficient to consider a sort of truncated Laplace transform, in which we integrate along a finite ray, rather than an infinite one. Indeed, consider the function

$$F(x) = \int_0^a e^{-t/x}\omega$$

where $a \in \mathsf{V}$ is chosen so that $\mathrm{Re}(a/x) > 0$ as $x \to 0$ in U; such an a can be chosen precisely because $|\mathsf{A}| < \pi$. The function F is evidently holomorphic in U; we claim that F is, in fact, asymptotic to the original series f.

To this end, we write $\omega = g(t)\, dt$ where

$$g = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!}.$$

Integrating by parts $N+1$ times gives the equation

$$F(x) = \sum_{k=0}^{N} a_k x^{k+1} - e^{-a/x} \sum_{k=0}^{N} a_k x^{k+1} + x^{N+1}\psi_N(x)$$

where ψ_N is a bounded function. Since the function $e^{-a/x}$ decays exponentially in U, this equation gives the desired bound

$$\left| F - \sum_{k=0}^{N-1} a_k x^{k+1} \right| < C|x|^{N+1}$$

uniformly on subsectors. With a bit more work one can check that the function is Gevrey. □

References

[1] B. Y. Sternin and V. E. Shatalov, *Borel-Laplace transform and asymptotic theory*, CRC Press, Boca Raton, FL, 1996. Introduction to resurgent analysis.

[2] G. N. Watson, *A Theory of Asymptotic Series*, Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences **211** (1912), no. 471-483, 279–313.