1. Quasimorphisms: definition and examples

Let $\Gamma$ be a group. A map $\phi : \Gamma \to \mathbb{R}$ is a quasimorphism if it satisfies
$$\sup_{g,h \in \Gamma} |\phi(g) + \phi(h) - \phi(gh)| < \infty,$$
in which case one defines the defect $D_\phi := \sup_{g,h \in \Gamma} |\phi(g) + \phi(h) - \phi(gh)|$.

A bounded perturbation of a quasimorphism of $\Gamma$ is again a quasimorphisms of $\Gamma$. In order to eliminate this “bounded noise”, we consider the following relation of asymptotic equivalence on the quasimorphisms of $\Gamma$:
$$\phi \sim \phi' \iff \sup_{g \in \Gamma} |\phi(g) - \phi'(g)| < \infty.$$  

We denote by $X(\Gamma)$ the (real) vector space of quasimorphisms modulo asymptotic equivalence.

It turns out (see the next Lemma) that each asymptotic equivalence class has a natural representative, namely the unique quasimorphism with the property that it is an actual morphism on every cyclic subgroup. Let us make a proper definition as follows. A quasicharacter is a quasimorphism $\phi : \Gamma \to \mathbb{R}$ satisfying $\phi(g^n) = n\phi(g)$ for all $g \in \Gamma$, $n \in \mathbb{Z}$. Note that a quasicharacter $\phi$ is constant on conjugacy classes:
$$|\phi(x^{-1}gx) - \phi(g)| \\leq \frac{1}{n}|\phi(x^{-1}g^nx) - \phi(g^n)| \leq \frac{2D_\phi}{n}$$
and we get $\phi(x^{-1}gx) = \phi(g)$ by letting $n \to \infty$.

**Lemma 1.1.** Every asymptotic equivalence class in $X(\Gamma)$ contains a unique quasicharacter.

**Proof.** We start by showing the uniqueness part. Let $\phi, \phi'$ be asymptotically equivalent quasicharacters. For each $g \in \Gamma$ we have $|\phi(g) - \phi'(g)| = \frac{1}{n}|\phi(g^n) - \phi'(g^n)|$. Since $|\phi(g^n) - \phi'(g^n)|$ is bounded as $n \to \infty$, it follows that $\phi(g) = \phi'(g)$.

Next, we show the existence part. Let $\phi$ be a quasimorphism and put
$$\overline{\phi}(g) = \lim_{n \to \infty} \frac{\phi(g^n)}{n}.$$  

First, we claim that $\overline{\phi}(g)$ is well-defined. Recall that, for a non-negative sequence $(a_n)$ satisfying $a_{m+n} \leq a_m + a_n$, the limit $\lim_{n \to \infty} a_n/n$ is well-defined. From
$$\phi(g^{m+n}) \leq \phi(g^m) + \phi(g^n) + D_\phi$$
we have that the sequence with general term $b_n = \phi(g^n) + D_\phi$ is subadditive. We modify $b_n$ by a linear term so as to guarantee non-negative values. An obvious induction yields
$$|\phi(g^n) - n\phi(g)| \leq (n-1)D_\phi$$
making $a_n = b_n - n(\phi(g) - D_\phi)$ non-negative and subadditive. Since $\lim_{n \to \infty} a_n/n$ is well-defined, it follows that $\overline{\phi}(g)$ is well-defined.
Second, we show that \( \overline{\phi} \) is a quasicharacter. First one gets a bound on the price paid for interchanging two group elements:

\[
|\phi(xghy) - \phi(xhgy)| \leq 6D_\phi
\]

It follows that

\[
|\phi(gh)^n - \phi(g^n h^n)| \leq 6(n - 1)D_\phi
\]

hence

\[
|\phi(g^n) + \phi(h^n) - \phi((gh)^n)| \leq D_\phi + |\phi(g^n h^n) - \phi((gh)^n)| \leq 6nD_\phi.
\]

Thus \( |\overline{\phi}(g) + \overline{\phi}(h) - \overline{\phi}(gh)| \leq 6D_\phi \). This proves that \( \overline{\phi} \) is a quasimorphism. We still have to show that \( \overline{\phi}(g^n) = k\overline{\phi}(g) \) for all integers \( k \). For \( k \geq 0 \), this is clear from the definition of \( \overline{\phi} \). Therefore, it suffices to check \( \overline{\phi}(g^{-1}) = -\overline{\phi}(g) \); this follows immediately from \( |\phi(g^n) + \phi(g^{-n})| \leq D_\phi + |\phi(1)| \).

Finally, \( \overline{\phi} \) is equivalent to \( \phi \): \( |\phi(g^n) - n\phi(g)| \leq (n - 1)D_\phi \) yields \( |\overline{\phi}(g) - \phi(g)| \leq D_\phi \). \( \square \)

Let \( \chi(\Gamma) \) be the (real) vector space of characters of \( \Gamma \), i.e., morphisms \( \Gamma \to \mathbb{R} \). Identifying \( X(\Gamma) \) with the space of quasicharacters, we have that \( \chi(\Gamma) \) is a subspace of \( X(\Gamma) \). We view the (real) vector space

\[
Q(\Gamma) := X(\Gamma)/\chi(\Gamma)
\]

as a measure for the “non-triviality” of quasimorphisms on \( \Gamma \).

In the following propositions, we compute \( Q(\Gamma) \) for some groups \( \Gamma \).

**Proposition 1.2.** If \( \Gamma \) is amenable then \( Q(\Gamma) = 0 \).

**Proof.** We show that every quasimorphism \( \phi : \Gamma \to \mathbb{R} \) is asymptotic to a morphism. Since the real-valued map on \( \Gamma \) given by \( x \mapsto \phi(gx) - \phi(x) \) is bounded, we may define \( \overline{\phi} : \Gamma \to \mathbb{R} \) as follows:

\[
\overline{\phi}(g) = \int_\Gamma (\phi(gx) - \phi(x))dx
\]

Then \( \overline{\phi} \) is a morphism. Indeed:

\[
\overline{\phi}(gh) = \int_\Gamma (\phi(ghx) - \phi(x))dx
\]

\[
= \int_\Gamma (\phi(ghx) - \phi(hx))dx + \int_\Gamma (\phi(hx) - \phi(x))dx = \overline{\phi}(g) + \overline{\phi}(h)
\]

Furthermore, we have

\[
|\overline{\phi}(g) - \phi(g)| = \left| \int_\Gamma (\phi(gx) - \phi(x))dx \right| \leq \int_\Gamma |\phi(gx) - \phi(x)|dx \leq D_\phi
\]

which shows that \( \phi \) and \( \overline{\phi} \) are asymptotically equivalent. \( \square \)

**Proposition 1.3.** If \( \Gamma \) is boundedly generated then \( Q(\Gamma) \) is finite dimensional.

**Proof.** Let \( S = \{ s_1, \ldots, s_n \} \subseteq \Gamma \) be a set that boundedly generates \( \Gamma \). We claim that the linear map from the space of quasicharacters \( X(\Gamma) \) to \( \mathbb{R}^n \), given by \( \phi \mapsto (\phi(s_1), \ldots, \phi(s_n)) \), is injective. It will follow that \( X(\Gamma) \), hence \( Q(\Gamma) \) as well, is finite dimensional.

Let \( \overline{\phi} \) be a quasicharacter with \( \overline{\phi}(s) = 0 \) for every \( s \in S \). Then \( \overline{\phi}(s^k) = 0 \) for every \( s \in S \) and every \( k \in \mathbb{Z} \). By the bounded generation hypothesis, there is a positive integer \( N \) such that each element \( g \in \Gamma \) can be written as \( g = s_{i_1}^{k_{i_1}} \cdots s_{i_N}^{k_{i_N}} \) for some \( s_{i_j} \in S \) and integers \( k_{i_j} \). Then

\[
|\overline{\phi}(g)| = |\overline{\phi}(g) - \phi(s_1^{k_{i_1}}) - \cdots - \phi(s_N^{k_{i_N}})| \leq (N - 1)D_\phi
\]

which shows that \( \overline{\phi} \) is bounded, hence \( \phi = 0 \). \( \square \)

**Proposition 1.4.** If \( \Gamma = \text{SL}_n(\mathbb{Z}) \), where \( n \geq 3 \), then \( Q(\Gamma) = 0 \).
Proof. We show that \(X(\Gamma) = 0\), i.e., that every quasimorphism on \(\Gamma\) vanishes identically. In general, a quasicharacter is bounded on commutators: from

\[
\phi(ghg^{-1}h^{-1}) = \phi(ghg^{-1}h^{-1}) - \phi(h) - \phi(h^{-1}) = \phi(ghg^{-1}h^{-1}) - \phi(gh^{-1}) - \phi(h^{-1})
\]

one sees that \(|\phi(ghg^{-1}h^{-1})| \leq D_0\) for any group elements \(g, h\). When \(n \geq 3\), every elementary matrix in \(\text{SL}_n(\mathbb{Z})\) is a commutator of elementary matrices, so every quasicharacter on \(\text{SL}_n(\mathbb{Z})\) is bounded on elementary matrices. As \(\text{SL}_n(\mathbb{Z})\) is boundedly generated by elementary matrices, it follows that every quasicharacter on \(\text{SL}_n(\mathbb{Z})\) is bounded, hence vanishing. \(\square\)

**Proposition 1.5** (Brooks). **If** \(\Gamma = F_2\) **then** \(Q(\Gamma)\) **is infinite dimensional.**

Many more results in this vein are currently known. Let us record here only one such result, due to Epstein and Fujiwara [1]: if \(\Gamma\) is a non-elementary hyperbolic group then \(Q(\Gamma)\) is infinite dimensional.

Proof. Let \(a, b\) be the generators of \(F_2\). In what follows, words are assumed to be reduced.

For a non-trivial word \(w\), let \(\#w(x)\) denote the number of appearances of \(w\) in \(x\). Define

\[
\phi_w(x) = \#w(x) - \#w^{-1}(x).
\]

Note that \(\phi_a\) is the morphism \(a \mapsto 1, b \mapsto 0\) and \(\phi_b\) is the morphism \(a \mapsto 0, b \mapsto 1\), and that they form a basis for \(\chi(\Gamma)\).

We claim that \(\phi_w\) is a quasimorphism. If there is no cancelation in the product \(xy\), then

\[
\#w(x) + \#w(y) \leq \#w(xy) \leq \#w(x) + \#w(y) + |w| - 1
\]

and consequently

\[
\#w^{-1}(x) + \#w^{-1}(y) \leq \#w^{-1}(xy) \leq \#w^{-1}(x) + \#w^{-1}(y) + |w| - 1
\]

which together yield

\[
|\phi_w(xy) - \phi_w(x) - \phi_w(y)| \leq |w| - 1.
\]

In general, there is a subword \(z\) that is canceled in the product \(xy\). Write \(x = x'z, y = z^{-1}y', \) and bound

\[
|\phi_w(xy) - \phi_w(x) - \phi_w(y)| = |\phi_w(x'y') - \phi_w(x'z) - \phi_w(z^{-1}y')|
\]

by

\[
|\phi_w(x'y') - \phi_w(x') - \phi_w(y')| + |\phi_w(x'z) - \phi_w(z) - \phi_w(x'z)| + |\phi_w(y') - \phi_w(z^{-1}) - \phi_w(z^{-1})|.
\]

By the first part, the above expression is bounded by \(3(|w| - 1)\). We conclude that \(\phi_w\) is, indeed, a quasimorphism.

Next we claim that \(\{\phi_{a^n b^m}\}_{n \geq 1}\) are linearly independent in \(Q(\Gamma)\). Otherwise, we would have

\[
\sup_{g \in \Gamma} |\phi_{a^{n+1}b^{m+1}} + c_n \phi_{a^n b^m} + \cdots + c_1 \phi_a + c_0 \phi_b| < \infty
\]

for some \(c_n, \ldots, c_1, c_0 \in \mathbb{R}\). Evaluating on \(a^k\) and \(b^k\), where \(k \geq 1\), gives \(c_0 = 0\) and \(c_1 = 0\). Evaluating on \((ab)^k\), where \(k \geq 1\), gives \(c_1 = 0\). And so on. Finally, evaluating on \((a^{n+1}b^{m+1})^k\), where \(k \geq 1\), gives a contradiction. \(\square\)

See [2] for a detailed investigation of quasimorphisms on free groups, particularly the construction of the Faiziev quasicharacters which are more natural than the Brooks quasimorphisms we considered.
2. BOUNDED COHOMOLOGY FOR GROUPS

Let \( V \) be a Banach space. The group cohomology \( H^* (\Gamma, V) \) arises from the complex
\[
V = C^0 (\Gamma, V) \rightarrow C^1 (\Gamma, V) \rightarrow C^2 (\Gamma, V) \rightarrow \ldots
\]
where \( C^n (\Gamma, V) = \{ \phi : \Gamma^n \rightarrow V \} \) and the differential \( d \) is given by the following formula:
\[
d\phi (g_1, \ldots, g_{n+1}) = \phi (g_2, \ldots, g_{n+1})
+ \sum_{i=1}^n (-1)^i \phi (g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \phi (g_1, \ldots, g_n)
\]
To get the bounded cohomology \( H^*_b (\Gamma, V) \), we consider the subcomplex
\[
V = C^0_b (\Gamma, V) \rightarrow C^1_b (\Gamma, V) \rightarrow C^2_b (\Gamma, V) \rightarrow \ldots
\]
where \( C^n_b (\Gamma, V) = \{ \phi : \Gamma^n \rightarrow V | \phi \text{ is bounded} \} \) and the differential \( d \) is the same. There is a natural comparison map \( H^*_b (\Gamma, V) \rightarrow H^* (\Gamma, V) \).

As \( H^*_b (\Gamma, V) \) consists of the bounded maps \( \phi : \Gamma \rightarrow V \) which satisfy \( \phi (gh) = \phi (g) + \phi (h) \), we have \( H^*_b (\Gamma, V) = 0 \). We focus on the second bounded cohomology group \( H^*_b (\Gamma, V) \).

**Proposition 2.1.** There is an isomorphism \( Q (\Gamma) \simeq \ker (H^2_b (\Gamma, \mathbb{R}) \rightarrow H^2 (\Gamma, \mathbb{R})) \).

**Proof.** Start with the well-defined map \( X (\Gamma) \rightarrow H^2_b (\Gamma, \mathbb{R}) \) given by \( \phi \mapsto [d\phi] \); here \( \phi \) actually stands for an asymptotic equivalence class. If \( [d\phi] = 0 \) then \( d\phi = d\beta \) for some bounded \( \beta : \Gamma \rightarrow \mathbb{R} \).

Hence \( d\phi = 0 \), so \( \phi \in \chi (\Gamma) \). Conversely, \( \chi (\Gamma) \) is in the kernel.

We get an injective map \( X (\Gamma)/\chi (\Gamma) \rightarrow \ker (H^2_b (\Gamma, \mathbb{R}) \rightarrow H^2 (\Gamma, \mathbb{R})) \) given by \( \phi + \chi \Gamma \mapsto [d\phi] \). For surjectivity, an element in \( \ker (H^2_b (\Gamma, \mathbb{R}) \rightarrow H^2 (\Gamma, \mathbb{R})) \) is of the form \( [d\phi] \) where \( \phi \) is defined up to \( \chi (\Gamma) \) and up to perturbations by bounded maps \( \beta : \Gamma \rightarrow \mathbb{R} \).

In the case of \( F_2 \), the vanishing of \( H^2 (F_2, \mathbb{R}) \) implies that \( H^*_b (F_2, \mathbb{R}) \) is described entirely by the space \( Q (F_2) \) of non-trivial quasimorphisms. In particular:

**Corollary 2.2.** \( H^*_b (F_2, \mathbb{R}) \) is infinite dimensional.

3. HOCHSCHILD COHOMOLOGY FOR ALGEBRAS

Let \( A \) be a complex algebra and \( V \) an \( A - A \) bimodule. Let \( L^n (A, V) \) denote the \( n \)-linear maps from \( A^n \) to \( V \). The Hochschild cohomology \( H^* (A, V) \) is given by the complex
\[
V = L^0 (A, V) \rightarrow L^1 (A, V) \rightarrow L^2 (A, V) \rightarrow \ldots
\]
with differential
\[
d\phi (a_1, \ldots, a_{n+1}) = a_1 \phi (a_2, \ldots, a_{n+1})
+ \sum_{i=1}^n (-1)^i \phi (a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \phi (a_1, \ldots, a_n) a_{n+1}.
\]
For instance, \( H^1 (A, V) \) consists of derivations (linear maps \( \phi : A \rightarrow V \) satisfying \( \phi (ab) = a \phi (b) + \phi (a) b \) for all \( a, b \in A \) modulo inner derivations (linear maps \( \phi : A \rightarrow V \) of the form \( \phi (a) = av - va \) for some \( v \in V \)).

For topological algebras, Hochschild cohomology is taken with respect to a subcomplex of \( L^n (A, V) \) obtained by imposing suitable continuity conditions. Let \( A \) be a Banach algebra and \( V \) a Banach \( A - A \) bimodule, meaning that \( V \) is a Banach space and \( A \) acts by bounded operators on both sides. One may take \( V = A \) or some Banach super-algebra, e.g. \( A = C_r^* \Gamma \) or \( L^1 \Gamma \) and \( V = B (l^2 \Gamma) \). Let \( B^n (A, V) \) denote the bounded \( n \)-linear maps from \( A^n \) to \( V \). The (continuous) Hochschild cohomology \( H^* (A, V) \) is given by the complex
\[
V = B^0 (A, V) \rightarrow B^1 (A, V) \rightarrow B^2 (A, V) \rightarrow \ldots
\]
with differential as above.
In the next theorem, it is natural to consider complex-valued bounded group cohomology. The results we had for the real-valued situation remain unchanged. Recall that the group algebra $\ell^1\Gamma$ carries a trace given by the following formula:

$$\text{tr}(\sum a_ng) = a_1$$

**Theorem 3.1** (Johnson). $H_1^b(\Gamma, \mathbb{C})$ embeds in $H^1(\ell^1\Gamma, \ell^1\Gamma)$.

**Proof.** We construct chain maps $M : C^b_0(\Gamma, \mathbb{C}) \to B^*(\ell^1\Gamma, \ell^1\Gamma)$ and $m : B^*(\ell^1\Gamma, \ell^1\Gamma) \to C^b_0(\Gamma, \mathbb{C})$ with $mM = \text{id}$. It will follow that $H_1^b(\Gamma, \mathbb{C})$ embeds in $H^1(\ell^1\Gamma, \ell^1\Gamma)$.

The map $M : C^b_0(\Gamma, \mathbb{C}) \to B^*(\ell^1\Gamma, \ell^1\Gamma)$ is the multiplication operator $\phi \mapsto M\phi$, where $M\phi$ is determined from

$$(M\phi)(g_1, \ldots, g_n) = \phi(g_1, \ldots, g_n)g_1 \ldots g_n$$

Then $\|M\phi\| = \|\phi\|_\infty$, so $M$ itself is linear and continuous of norm 1 if we equip $C^b_0(\Gamma, \mathbb{C})$ with the sup-norm and $B^*(\ell^1\Gamma, \ell^1\Gamma)$ with the operator norm.

The map $m : B^*(\ell^1\Gamma, \ell^1\Gamma) \to C^b_0(\Gamma, \mathbb{C})$ is defined as follows:

$$(m\Phi)(g_1, \ldots, g_n) = \text{tr}(\Phi(g_1, \ldots, g_n)(g_1 \ldots g_n)^{-1})$$

Then $|(m\Phi)(g_1, \ldots, g_n)| \leq \|\Phi(g_1, \ldots, g_n)\| \leq \|\Phi\|$, so $\|m\Phi\|_\infty \leq \|\Phi\|$. Thus $m$ is also linear and continuous.

One checks that $M$ and $m$ are chain maps, i.e., $dM = Md$ and $dm = md$. Furthermore:

$$mM\phi(g_1, \ldots, g_n) = \text{tr}(M\phi(g_1, \ldots, g_n)(g_1 \ldots g_n)^{-1}) = \text{tr}(\phi(g_1, \ldots, g_n)1) = \phi(g_1, \ldots, g_n)$$

This means that $mM = \text{id}$ on $C^b_0(\Gamma, \mathbb{C})$, as desired. \qed

**Corollary 3.2.** $H^2(\ell^1F_2, \ell^1F_2)$ is infinite dimensional.

As hinted by the generic vanishing of $H^1_b(\Gamma, \mathbb{C})$, it turns out that $H^1(\ell^1\Gamma, \ell^1\Gamma) = 0$ for every group $\Gamma$ (due to Johnson and Ringrose). For group von Neumann algebras, there is a similar vanishing result: $H^1(L^1\Gamma, L^1\Gamma) = 0$, a particular case of a theorem due to Kadison and Sakai. The computation of $H^2(LF_2, LF_2)$ is still an open problem, though a vanishing result is expected.

We end with a problem raised in [5]:

[Problem 8.3.4] How is the bounded cohomology of $\Gamma$ related to the Hochschild cohomology of $C^*_r\Gamma$, or to the Hochschild cohomology of $L\Gamma$?

**References**