Local minimum spanning tree optimization
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Motivation
Imagine a city manager trying to improve the global state of the transportation system of a city, from building new roads, to changing bus routes, or even adding new public transit. The city is currently laid out in a way that does not necessarily match the population and its organization and they might want to optimize its structure while affecting the least people. Given this problem, how local can the changes be so that they lead to the desired global optimum?

Although we do not directly address this exact optimization problem, our problem falls under the same framework, wondering how local optimizations can lead to the global optimum.

Optimizations
Our work focuses on studying properties of optimizations, as defined below.

Definition [Optimization]. Given a complete weighted graph \((K_n, w)\), an optimization \(X = (H_0, S_0, H_1, S_1, \ldots, H_m)\) is an alternating sequence where:

- \(H_i\) is a spanning subgraph of \(K_n\);
- \(S_i\) is a connected subset of \(H_i\);
- \(H_{i+1}\) is obtained from \(H_i\) by replacing its induced subgraph \(H_i[S_i]\) on \(S_i\) by its corresponding minimum spanning tree.

Since the weights of \(H_0, H_1, \ldots, H_m\) decreases, we are mainly interested in optimizations that eventually reach the minimum spanning tree. Write \(\mathcal{O}^+(H_0)\) for the set of such optimizations, where \(H_m\) is the minimum spanning tree of \(K_n\).

Given an optimization \(X = (H_0, S_0, \ldots, H_m)\), we want to define a measure on its efficiency. To do so, we define the cost of an optimization to be the maximal weight of the replaced subgraph when going from \(H_i\) to \(H_{i+1}\):

\[
\text{cost}(X) := \max_{0 \leq i < m} \left\{ w(H_i[S_i]) \right\}.
\]

In the case of the city manager problem, an optimization corresponds to a sequence of local changes implemented to improve the road layout, and the cost of an optimization corresponds to the largest size of any single change.

Example
Given our current graph \(H_i\), consider a connected subset \(S_i\); replace the subgraph on \(S_i\) by its minimum spanning tree; and obtain the new graph \(H_{i+1}\).

The cost of this transition from \(H_i\) to \(H_{i+1}\) is the sum of the weights of the yellow edges.

Results
Under mild assumptions on \((K_n, w)\) and \(H_0\), the minimal value of the cost to reach the minimum spanning tree starting from \(H_0\) nicely converges when \(n \to \infty\), as stated in the following theorem.

Theorem [Addario-Berry, Barrett, and C.]. Let \((K_n, w)\) be the complete graph with independent \(\text{Uniform}(0,1)\) edge weights and let \(H_0\) be a spanning subgraph of \(K_n\) chosen independently of \(w\). Then, as \(n \to \infty\),

\[
\min \left\{ \text{cost}(X) : X \in \mathcal{O}^+(H_0) \right\} \xrightarrow{P} 1.
\]

The fact that this minimum is larger than \(1 - o(1)\) is not difficult to check, simply by considering the largest edge in \(H_0\). The interesting part of this result is the upper bound, stating that there actually exists optimizations whose cost is \(1 + o(1)\). Below is a quick explanation of our main argument.

Method
The main method of our proof consists of finding an optimization with small cost able to go from the minimum spanning tree on a graph of size \(k\) to the minimum spanning tree on a graph of size \(k + 1\).

Assume that we managed to obtain \(H_i\) such that there exists \(|S| = k\) with \(H_i[S]\) being the minimum spanning tree on \(S\). Then, by choosing \(S_i = S \cup \{v\}\), where \(v\) is a neighbour of \(S\) in \(H_i\), it follows that \(H_{i+1}\) contains the minimum spanning tree on \(S \cup \{v\}\). Moreover, the weight of such transition is

\[
w(H_i[S]) \geq 1 + w(MST_k) \geq 1 + \zeta(3) \approx 2.211\ldots
\]

where the \(1\) comes from the maximal possible weight from \(S\) to \(v\) and \(\zeta(3)\) comes from the asymptotic weight of large minimum spanning trees. This method gives an upper bound of \(\approx 2.2\) for the minimal cost, which is not tight. However a similar but finer approach leads to the desired \(1 + o(1)\).