ISBELL DUALITY FOR MODULES

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Abstract. The purpose of this paper is to extend the results of [Barr et al. (2008), Section 8] from the case of abelian groups (**Z**-modules) to that of modules over a large class of not necessarily commutative rings.

1. Introduction

The theme of the paper [Barr et al. (2008)] was the duality that may arise when you have, in Isbell's words, "One object living in two categories". For example, the Lefschetz duality between vector spaces and linearly compact vector spaces over the same field arises from the fact that the ground field lives in both categories.

One of the principal examples in that paper was the duality between a certain subcategory of the category of abelian groups and a certain subcategory of the category of topological abelian groups. This "Isbell duality" was described in each direction as the group of homomorphisms (respectively continuous homomorphisms) into the group **Z** of integers, see Section 8 of that paper.

The purpose of this paper is to extend most of those results from the category of abelian groups to categories of R-modules, for suitable rings R. It is more-or-less clear that the same arguments will work for any (commutative) integral domain. To generalize beyond that, we need some replacement for the field Q of quotients. The two properties of Q that were crucial were that it be R-injective (left R-injective, as it will turn out) and that Q/R be torsion (as a right R-module).

If two categories are dual, then the dual of a monomorphism is an epimorphism. This suggests, if the duality is mediated by a object in both categories, that the dualizing object be injective. Thus it was at least slightly surprising to discover, in [Barr et al. (2008), Section 8], that there is a duality between a full subcategory of topological abelian groups and discrete abelian groups that can be described—in each direction—as homomorphisms into the group **Z** of integers. It was crucial, in that case, that the field **Q** of quotients was **Z**-injective, but the dual was still taken in **Z**.

The first candidate for **Q** is the (classical) ring of quotients, gotten by inverting all the non zero-divisors. When R is commutative, this ring always exists but is not, in general, R-injective. When R is non-commutative, not even the existence is guaranteed. Even if you impose the *right Ore condition* (see 2.1 below), you get a right ring of quotients, but there is no reason for it to be left injective.

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Consider the case that R is a ring without zero-divisors. The left Ore condition implies that the ring of left quotients is a division ring and is therefore left and right self-injective. It is also right R-flat from which we can show that it is left R-injective as well, see Theorem 2.5. On the other hand, we also require that Q/R be a right torsion module and that requires (actually is equivalent to) the right Ore condition. Thus we must impose both Ore conditions in this case.

If there are zero-divisors, a better idea is to use *complete ring of right quotients*, see [Lambek (1986), 4.3]. This is much more likely to be R-injective, at least on the right (so left injectivity, which is the condition we actually need, is still a separate hypothesis). The torsion condition has to be weakened, but we will see that it is satisfied. We should point out that the complete ring of left quotients is likely to be left injective, but not likely to satisfy the weak torsion condition. You win some and you lose some.

Definitions, notation, and preliminary remarks. Here are some definitions and results from [Barr et al. (2008), Section 3] that we will need. If Z is an object of a complete category $\mathcal{A}$, an object A of $\mathcal{A}$ will be called Z-**cogenerated** if A has a regular monomorphism into a power of Z. It will be called **Z-sober** if there is an equalizer diagram $A \longrightarrow Z^X \rightrightarrows Z^Y$ for sets X and Y and **canonically** Z**-sober** if

$$A \longrightarrow Z^{\mathrm{Hom}(A,Z)} \rightrightarrows Z^{\mathrm{Hom}(Z^{\mathrm{Hom}(A,Z)},Z)}$$

is an equalizer. The diagram is the canonical one from the contravariant adjunction

$$\mathcal{A} \underset{Z^-}{\overset{\mathrm{Hom}(-,Z)}{\rightleftarrows}} \mathcal{S}et$$

It is known that A is canonically Z-sober if and only if there is an equalizer $A \longrightarrow Z^{\mathrm{Hom}(A,Z)} \rightrightarrows Z^Y$ in which the first map is the canonical one [Barr et al. (2008), Proposition 3.4].

We will be dealing below with modules and topological modules. If C is a topological module, we will denote by $\|C\|$ the underlying discrete module. If A is any object of $\mathcal{C}$ or $\mathcal{D}$ we will denote by $|A|$ the underlying set of A.

2. Rings, modules, and complete ring of quotients

All rings will be unital. A right ideal I of the ring R is called **dense** if for all elements $a, b \in R$ with $a \neq 0$, there is a $c \in R$ such that $ac \neq 0$ and $bc \in I$ ([Lambek (1986), p. 96]).

2.1. Definition. *A ring R satisfies the* **right Ore condition** *if for all $r \in R$ and all non zero-divisors $n \in R$, there are elements $s, m \in R$ with m a non zero-divisor such that $rm = ns$. Category theorists call this a* calculus of right quotients. *There is obviously a left Ore condition that we leave to the reader to formulate.*

For any subset $I \subseteq R$, define the **left annihilator of** I as

$$I^{\mathfrak{l}} = \{r \in R \mid rI = 0\}$$

When $I = \{s\}$ is a singleton, we will write $s^{\mathfrak{l}}$ instead of $\{s\}^{\mathfrak{l}}$. It is not hard to show that when R is commutative, then I is dense if and only if $I^{\mathfrak{l}} = 0$, but the noncommutative situation is more complicated. By taking $b = 1$ in the definition, we do see that when I is dense, then $I^{\mathfrak{l}} = 0$ even in the non-commutative case.

If R is a ring, the **complete ring of right quotients** of R consists of equivalence classes of partial R-linear maps $R \dashrightarrow R$ whose domains are dense right ideals. The equivalence relation is that $p = q$ if they agree on their common domain. This works because the intersection of two dense right ideals is dense. Addition is pointwise. As for multiplication, the class of dense right ideals has the property that if q is such a partial function and I a dense right ideal, then $q^{-1}(I)$ is also a dense right ideal. This allows one to define multiplication as the usual composition of partial functions. We denote the resultant ring by Q. We embed R as the subring of left multiplications by its elements. Details are found in [Lambek (1986), Section 4.3].

Weak torsion and weak torsion free. Let D be a right R-module. Then we say that an element $a \in D$ is a **weak torsion** element if there is a dense right ideal $I \subseteq R$ with $aI = 0$. We say that D is a weak torsion module if every element is weak torsion and that D is **weak torsion free** if the only weak torsion element is 0.

2.2. Proposition. *Let D be a right R-module. Then*

1. *the weak torsion elements of D form a submodule;*
2. *a submodule or quotient module of a weak torsion module is weak torsion;*
3. *a submodule of weak torsion free module is weak torsion free;*
4. *if D is weak torsion and D' is weak torsion free, then* $\mathrm{Hom}_{\mathcal{D}}(D, D') = 0$*;*
5. *if* $0 \longrightarrow D' \xrightarrow{f} D \xrightarrow{g} D''$ *is exact with D' and D'' weak torsion free, so is D.*

Proof.

1. If $aI = 0$ and $bJ = 0$, then $(a+b)(I \cap J) = 0$ and the meet of two dense right ideals is dense, [Lambek (1986), Lemma 4.3.3]. Let a be weak torsion and I be a dense right ideal with $aI = 0$. For any $r \in R$, it follows from [Lambek (1986), Lemma 4.3.2], by taking q to be left multiplication by r, which is a total function, that $J = \{s \in R \mid rs \in I\}$ is also dense and then $arJ \subseteq aI = 0$ and we see that ar is also weak torsion.
2. Immediate.
3. Immediate.

4. Let $f \in \mathrm{Hom}_{\mathcal{D}}(D, D')$. For $a \in D$, choose a dense right ideal I so that $aI = 0$. Then $0 = f(aI) = f(a)I$ and, since D' is weak torsion free, $f(a) = 0$.

5. Let $a \in D$ and suppose there is a dense right ideal I with $aI = 0$. Then $g(a)I = 0$ so that $g(a) = 0$ and $a = f(a')$ for some $a' \in D'$. But Then $f(a'I) = 0$ and f is injective, whence $a'I = 0$ and so $a = 0$. ∎

The classes of weak torsion and weak torsion free modules do not seem to form a torsion theory in the usual categorical sense (but it will be if, for example, R is Noetherian, see 3.18). The class of weak torsion modules is not closed under extension and the quotient mod the subobject of weak torsion elements is not necessarily weak torsion free.

2.3. THEOREM. *The quotient Q/R is weak torsion.*

PROOF. Let $q \in Q$ be represented by $\varphi : I \longrightarrow R$. We claim that $qI \subseteq R$. In fact, for $a \in I$, define $\mu_a : R \longrightarrow R$ to be left multiplication by a (which is right R-linear). According to [Lambek (1986), proof of Proposition 4.3.6], the product qa is represented by $\varphi\mu_a \in \mathrm{Hom}(\mu_a^{-1}(I), R)$. But $\mu_a^{-1}(I) = R$ since I is a right ideal so that left multiplication by a takes R into I. Thus $qa \in R$ and, since a was an arbitrary element of I, we see that $qI \subseteq R$. ∎

2.4. COROLLARY. *If $\{q_1, \ldots, q_k\}$ is a finite set of elements of Q, then there is a dense right ideal I with $q_l I \subseteq R$ for $l = 1, \ldots, k$.*

PROOF. This is immediate from the fact that a finite intersection of dense right ideals is dense ([Lambek (1986), Lemma 4.3.3]). ∎

INJECTIVES. We will require that the complete right ring of quotients Q of R be left R-injective. See Section 3.28 for some discussion of these hypotheses. In some cases, it will be known that Q is left self-injective and also right R-flat (for example, when R satisfies the left Ore condition and Q is also the classical ring of quotients). Thus the following result is interesting.

2.5. THEOREM. *Suppose Q is left self-injective and right R-flat. Then Q is left R-injective.*

PROOF. Let I be a left ideal of R and $\varphi : I \longrightarrow Q$ be left R-linear. Since Q is right R-flat, $Q \otimes_R I$ is a left ideal of $Q \otimes_R R \cong Q$ and $Q \otimes \varphi : Q \otimes_R I \longrightarrow Q \otimes_R Q$ is left Q-linear. Let $\mu : Q \otimes_R Q \longrightarrow Q$ be multiplication which is left (and right) Q-linear. Then there is a map $\psi : Q \longrightarrow Q$ such that $\psi|(Q \otimes I) = \mu(Q \otimes \varphi)$ and the composite $R \longrightarrow Q \otimes_R R \xrightarrow{\psi} Q$ extends φ. ∎

2.6. THEOREM. *Suppose that $R = \prod_{\alpha\in A} R_\alpha$ is a product of rings. If for each α, a left R_α-injective Q_α is given, then $Q = \prod_{\alpha\in A} Q_\alpha$ is R-injective.*

PROOF. It is clearly sufficient to show that each Q_α is R-injective. Let $e_\alpha \in R$ be the central idempotent that arises from the projection on Q_α. For any ideal $I \subseteq R$, $e_\alpha I$ is a left ideal of R_α contained in I. If $\varphi : I \longrightarrow Q_\alpha$ is given, then one easily sees that $\varphi = \varphi e_\alpha$ and therefore extends to R_α. But φ factors through R_α and hence also extends to R. ■

2.7. COROLLARY. *Suppose $R = \prod R_\alpha$ and the complete right ring of quotients of each R_α is left self-injective. Then the same is true of R.*

PROOF. This follows from the above combined with Utumi's theorem [Lambek (1986), Proposition 9].

3. The duality

When R is a ring, let $\mathcal{D}(R)$ and $\mathcal{C}(R)$ denote the categories, respectively of right R-modules and topological left R-modules. With the exception of Section 4, R will not vary and we will simply write $\mathcal{D}$ and $\mathcal{C}$, respectively.

It is clear that, using the right R-action of R on itself, there is a functor $\hom_{\mathcal{C}}(-, R) : \mathcal{C} \longrightarrow \mathcal{D}$. If D is an object of $\mathcal{D}$, we denote by $\hom_{\mathcal{D}}(D, R)$ the abelian group $\mathrm{Hom}_{\mathcal{D}}(D, R)$, topologized as a subobject of R^D with the left R-module structure coming from the left action of R on itself. We will show all the details we need about these functors, including the fact that they are adjoint on the right in Section 8 below.

For $C \in \mathcal{C}$, we will denote by C^*, the object $\hom(C, R) \in \mathcal{D}$ and, similarly for $D \in \mathcal{D}$, we denote by D^* the object $\hom_{\mathcal{D}}(D, R) \in \mathcal{C}$. This useful notation, used with care, will not cause trouble. An object $C \in \mathcal{C}$ is said to be **fixed** if the adjunction homomorphism $\eta C : C \longrightarrow C^{**}$ is an isomorphism and, similarly, an object $D \in \mathcal{D}$ is fixed if the canonical $\eta D : D \longrightarrow D^{**}$ is. It follows by standard arguments that the full subcategories of fixed objects, called $\mathrm{Fix}(\mathcal{C})$ and $\mathrm{Fix}(\mathcal{D})$ are dual. The main purpose of this paper is to identify these subcategories under certain assumptions on the ground ring. In the case of $\mathrm{Fix}(\mathcal{C})$ our assumptions are natural and mild. Having a good description of $\mathrm{Fix}(\mathcal{D})$ seems to require some special hypotheses. Examples indicate that these constraints are not necessary, but also that our results do not hold for all rings.

Note: In the rest of this paper, we will use the following notation. The free R-module on the basis X is denoted $X \cdot R$. We often use the letter "P" to denote a free module when we don't wish to emphasize the basis. They will always have the discrete topology, whether considered as an object of $\mathcal{C}$ or of $\mathcal{D}$. A power R^X of R could also be an object of $\mathcal{C}$ or of $\mathcal{D}$; context will make it clear. When it is considered an object of $\mathcal{C}$ it will always have the product topology arising from the discrete topology on R. Both $X \cdot R$ and R^X are two-sided modules, but will be considered as left modules when objects of $\mathcal{C}$ and as right modules when considered as objects of $\mathcal{D}$. It is quite evident that $(X \cdot R)^* \cong R^X$ whether $(X \cdot R)$ is considered as an object of $\mathcal{C}$ or $\mathcal{D}$. When X is finite, the $R^X \cong X \cdot R$ is free and discrete even in $\mathcal{C}$ since a finite product of discrete spaces is discrete.

One fact that will be used extensively follows.

3.1. PROPOSITION. *Suppose $D \in \mathcal{D}$ and X is a set such that there is a surjection $X \cdot R \twoheadrightarrow D$, Then the induced $D^* \hookrightarrow (X \cdot R)^* \cong R^X$ is a topological embedding.*

PROOF. Let $f : X \cdot R \longrightarrow D$ be the given map. Then the induced $f^* : D^* \longrightarrow (X \cdot R)^*$ is given by $f^*(\varphi) = \varphi f$. When composed with the isomorphism $g : (X \cdot R)^* \longrightarrow R^X$, we get that $gf^*(\varphi)(x) = \varphi f(x)$ for $x \in X$. The topology is determined by subbasic open neighbourhoods defined for each $d \in D$ by $\{\varphi \in D^* \mid \varphi(d) = 0\}$. Now suppose that $d = f(r_1x_1 + r_2x_2 + \cdots + r_kx_k)$ with $r_1, r_2, \ldots, r_k \in R$ and $x_1, x_2, \ldots, x_k \in X$. Then it is clear that

$$\{\varphi \in D^* \mid \varphi(d) = 0\} \supseteq (f^*)^{-1}\left(\bigcap_{i=1}^{k}\{\varphi \in D^* \mid \varphi(x_i) = 0\}\right)$$

which implies that the topology induced on D^* by R^X is finer than the one from R^D. The reverse inclusion is easier and we omit it. ■

Fixed modules in $\mathcal{C}$

ASSUMPTION. *Throughout this section we will suppose that Q is left R-injective.*

There is an obvious necessary condition that a topological module be fixed. If D is an right R-module, it has a presentation by free modules:

$$P_1 \longrightarrow P_0 \longrightarrow D \longrightarrow 0$$

Dualizing, we get an exact sequence

$$0 \longrightarrow D^* \longrightarrow P_0^* \longrightarrow P_1^*$$

which enables us to conclude:

3.2. PROPOSITION. *A necessary condition that a topological left R-module be fixed is that it be R-sober.* ■

In order to prove that this condition is also sufficient, we begin with:

3.3. THEOREM. *Free right R-modules are fixed as objects of $\mathcal{D}$.*

PROOF. Let $P = X \cdot R$ be free. Since $P^* = R^X$, we must show that $(R^X)^* \cong X \cdot R$. The kernel of a map $f : R^X \longrightarrow R$ has to be an open submodule since R is discrete. Every open submodule contains one of the form $R^{X - X_0}$ for some finite subset $X_0 \subseteq X$. This means that f factors as $R^X \longrightarrow R^{X_0} \xrightarrow{f_0} R$, which means that

$$\hom_{\mathcal{C}}(R^X, R) \cong \operatorname{colim} \hom_{\mathcal{C}}(R^{X_0}, R) \cong \operatorname{colim} X_0 \cdot R \cong X \cdot R$$ ■

3.4. PROPOSITION. *Suppose C is a topological left submodule of R^X. For any continuous left R-linear map $\varphi : C \longrightarrow R$, there is a finite subset $X_0 \subseteq X$ such that φ factors through the image of C in R^{X_0}.*

Proof. Since the kernel of φ is open and a basic neighbourhood of R^X has the form R^{X-X_0} as X_0 ranges over the finite subsets of X, the kernel of φ must contain a set of the form $C \cap R^{X-X_0}$ for some finite subset $X_0 \subseteq X$. The conclusion follows from this diagram in which $C_0 = C/(C \cap R^{X-X_0})$:

$$\begin{tikzcd}
C \arrow[r, hook] \arrow[d] \arrow[dd, bend right, "\varphi"'] & R^X \arrow[d] \\
C_0 \arrow[r, hook] \arrow[d, dashed, "\varphi_0"] & R^{X_0} \\
R &
\end{tikzcd}$$

■

3.5. Proposition. *Suppose C is a left submodule of a finitely generated free module P, both topologized discretely. Let $\varphi : C \longrightarrow R$ be a left R-linear map and $\psi : P \longrightarrow Q$ extend φ. Then there is a dense right ideal I such that ψI takes values in R.*

Proof. Let $a_1, a_2, \ldots, a_k$ enumerate the values of ψ on the generators of P, then from Corollary 2.4, it follows that there is a dense right ideal I for which all of $a_1 I, a_2 I, \ldots,$ $a_k I$ belong to R, whence ψI takes all its values in R. ■

3.6. Proposition. *Suppose P is free and C is a topological left submodule of P^*. For any continuous left R-linear map $\varphi : C \longrightarrow R$, there is a dense right ideal I such that every element of φI extends to an R-linear map $P^* \longrightarrow R$.*

Proof. We know from 3.4 that φ factors through the image C_0 of C in P_0^* for some finitely generated free submodule $P_0 \subseteq P$. We now apply the previous proposition to the inclusion $C_0 \longrightarrow P_0^*$, both topologized discretely. ■

3.7. Corollary. *If C is a topological left submodule of P^*, then the cokernel of $P \longrightarrow C^*$ is weak torsion.* ■

3.8. Corollary. *If $C \hookrightarrow C'$ is an inclusion of R-cogenerated topological left modules, then the cokernel of $C'^* \longrightarrow C^*$ is weak torsion.*

Proof. Embed $C' \hookrightarrow R^X$ for some X. The composite $C \longrightarrow C' \longrightarrow R^X$ is also an embedding. The snake lemma applied to

$$\begin{tikzcd}
0 \arrow[r] & (R^X)^* \arrow[r] \arrow[d] & C'^* \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (R^X)^* \arrow[r] & C^* \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

gives that $\operatorname{coker}(C'^* \longrightarrow C^*) \cong \operatorname{coker}(T' \longrightarrow T)$, which is weak torsion. ■

3.9. THEOREM. *Suppose the complete ring of right quotients of R is left R-injective. Then a topological left R-module is fixed if and only if it is R-sober.*

PROOF. We have one direction in 3.2. Suppose that $0 \longrightarrow C \longrightarrow P_0^* \longrightarrow P_1^*$ is exact. If we let $T = \operatorname{coker}(P_0 \longrightarrow C^*)$, then we have a sequence

$$P_1 \longrightarrow P_0 \longrightarrow C^* \longrightarrow T \longrightarrow 0$$

with $P_0 \longrightarrow C^* \longrightarrow T \longrightarrow 0$ exact. The subsequence $P_1 \longrightarrow P_0 \longrightarrow C^*$ is not exact, but the composite is 0. From Corollary 3.7, we know that T is weak torsion, and hence, by Proposition 2.2.4, has no non-zero right R-linear maps to R. Thus we have a sequence

$$0 \longrightarrow C^{**} \longrightarrow P_0^* \longrightarrow P_1^*$$

The initial subsequence is exact and the remaining composite is 0 so that C^{**} is a topological left submodule of the kernel of $P_0^* \longrightarrow P_1^*$, which is C, while C is canonically embedded in it. The composite $C \longrightarrow C^{**} \longrightarrow C$ is the identity since when followed by the inclusion into P_0^* it is the inclusion. When the composite of monics is an isomorphism, both factors are isomorphisms as well. ■

3.10. THEOREM. *For any object D of $\mathcal{D}$, D^* is fixed.*

PROOF. Let $P_1 \longrightarrow P_0 \longrightarrow D \longrightarrow 0$ be a free resolution of D. This gives an exact sequence $0 \longrightarrow D^* \longrightarrow P_0^* \longrightarrow P_1^*$ and the conclusion follows from 3.9. ■

3.11. THEOREM. *A module in $\mathcal{C}$ is fixed if and only if it is canonically R-sober.*

PROOF. Recall that when D is an object of $\mathcal{D}$, we denote by $|D|$ the underlying set of D. According to [Barr et al. (2008), Proposition 3.4] an R-cogenerated module $C \in \mathcal{C}$ is canonically R-sober if and only the cokernel C' of $C \longrightarrow R^{|C^*|}$ is R-cogenerated. And that happens if and only if $C' \longrightarrow C'^{**}$ is monic. The significance of the map $D \longrightarrow R^{|D^*|}$ is that it dualizes to $|C^*| \cdot R \longrightarrow C^*$ which is obviously surjective. Thus from $0 \longrightarrow C \longrightarrow R^{|C^*|} \longrightarrow C' \longrightarrow 0$, we get the exact sequence $0 \longrightarrow C'^* \longrightarrow |C^*| \cdot R \longrightarrow C^* \longrightarrow 0$ whose second dual is the second row of the commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r] \arrow[d] & R^{|C^*|} \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^{**} \arrow[r] & R^{|C^*|} \arrow[r] & C'^{**} &
\end{tikzcd}$$

The snake lemma implies that $\ker(C' \longrightarrow C'^{**}) \cong \operatorname{coker}(C \longrightarrow C^{**})$, from which the equivalence is immediate. ■

Fixed modules in $\mathcal{D}$

We begin the discussion of Fix($\mathcal{D}$) with

3.12. PROPOSITION. *For any object $C \in \mathcal{C}$, C^* is fixed in $\mathcal{D}$.*

PROOF. Let $P_1 \longrightarrow P_0 \longrightarrow C^* \longrightarrow 0$ be exact with P_1 and P_0 free. The dual sequence $0 \longrightarrow C^{**} \longrightarrow P_0^* \longrightarrow P_1^*$ is also exact. The second dual $P_1^{**} \longrightarrow P_0^{**} \longrightarrow C^{***}$ is not exact, but the composite is 0 and the cokernel of $P_0^{**} \longrightarrow C^{***}$ is a weak torsion module T. In the diagram

$$\begin{tikzcd}
P_1 \arrow[r] \arrow[d, "\eta P_1"'] & P_0 \arrow[r] \arrow[d, "\eta P_0"] & C^* \arrow[d, "\eta C^*"] & & \\
P_1^{**} \arrow[r] & P_0^{**} \arrow[r] & C^{***} \arrow[r] & T \arrow[r] & 0
\end{tikzcd}$$

ηP_1 and ηP_0 are isomorphisms. Since $P_0 \twoheadrightarrow C^*$ is surjective, one sees that $\mathrm{coker}(\eta C^*) = T$. But ηC^* is split by $(\eta C)^*$ and so $C^{***} \cong C^* \oplus T$. But T is torsion and C^{***} is R-cogenerated so that $T = 0$ and ηC^* is an isomorphism.

3.13. COROLLARY. *Any power of R is fixed in $\mathcal{D}$.*

PROOF. $R^X = (X \cdot R)^*$. ■

3.14. PROPOSITION. *For any R-cogenerated right R-module D, the cokernel of $D \longrightarrow D^{**}$ is weak torsion.*

PROOF. Let $P \twoheadrightarrow D \longrightarrow 0$ be exact with P free. Then $0 \longrightarrow D^* \longrightarrow P^*$ is exact, whence we have, from 3.7 that there is an exact sequence $P \longrightarrow D^{**} \longrightarrow T \longrightarrow 0$ with T a right weak torsion module. The map $P \longrightarrow D^{**}$ factors as $P \twoheadrightarrow D \rightarrowtail D^{**}$, so that $D^{**}/D \cong T$. ■

3.15. THEOREM. *Let D be an R-cogenerated right module and suppose that X is a set of generators for D^* such that the cokernel of the canonical embedding $D \longrightarrow R^X$ is weak torsion free. Then D is fixed.*

PROOF. The fact that X generates D^* implies that the map $X \cdot R \longrightarrow D^*$, induced by the embedding $D \hookrightarrow R^X$, is surjective and hence D^{**} is also canonically embedded in R^X. This gives a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d, "f"'] & R^X \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & D^{**} \arrow[r] & R^X \arrow[r] & D'' &
\end{tikzcd}$$

and the snake lemma implies that $\ker(g) \cong \mathrm{coker}(f)$. Since D' is weak torsion free, so is $\ker(g)$ and hence so is $\mathrm{coker}(f)$. But we have just seen that the latter is weak torsion and hence must be 0 and thus f is an isomorphism. ■

In the case of abelian groups the converse of this theorem is true. For an example that shows that the converse does not hold in general see 6.1 below. The point at which the problem arises will become clearer during from the study of the cases in which we do have the converse.

3.16. PROPOSITION. *Suppose that every dense right ideal of R contains a finite set J for which $J^{\mathfrak{l}} = 0$. Then for any $D \in \mathcal{D}$, $\|D^*\|^*/D^{**}$ is weak torsion free.*

PROOF. Suppose $\varphi : \|D^*\| \longrightarrow R$ is an element of $\|D^*\|^*$ and I is a dense right ideal such that $\varphi I \subseteq D^{**}$. This means that for all $r \in I$, $\varphi r \in D^{**}$ so that $\ker(\varphi r)$ is open in D^*. Choose $J = \{r_1, \cdots, r_k\} \subseteq I$ such that $J^{\mathfrak{l}} = I^{\mathfrak{l}}$ It is trivial to see that

$$\ker(\varphi) = \bigcap_{n=1}^{k} \ker(\varphi r_n)$$

which is thereby open in D^* and then $\varphi \in D^{**}$. ∎

3.17. THEOREM. *Suppose that D is a fixed right R-module and X is a set of generators for the abelian group $\mathrm{Hom}_{\mathcal{D}}(D, R)$. Then the cokernel of $D \longrightarrow R^X$ is weak torsion free if and only if the cokernel of $D = D^{**} \longrightarrow \|D^*\|^*$, induced by the identity $\|D^*\| \longrightarrow D^*$, is weak torsion free.*

PROOF. Since X generates $\mathrm{Hom}(D, R)$ the induced map $X \cdot R \longrightarrow \|D^*\|$ is surjective so we have an exact sequence $0 \longrightarrow C \longrightarrow X \cdot R \longrightarrow \|D^*\| \longrightarrow 0$ of discrete left modules in $\mathcal{C}$. Applying the duality functor gives us a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & D \arrow[r] \arrow[d, "f"] & R^X \arrow[r] \arrow[d] & D' \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \|D^*\|^* \arrow[r] & R^X \arrow[r] & C^* &
\end{tikzcd}$$

The snake lemma gives us that $\ker f = 0$ and that $\mathrm{coker} f \cong \ker g$. Since D is fixed, $D \cong D^{**}$. If D' is weak torsion free, this immediately implies that $\mathrm{coker} D \longrightarrow \|D^*\|^*$ is weak torsion free, while if the latter holds, then we have the exact sequence $0 \longrightarrow \ker g \longrightarrow D' \longrightarrow C^*$. Since $C^* \subseteq R^C$ is R-cogenerated, it is weak torsion free and then so is D', see Proposition 2.2.5.

3.18. COROLLARY. *Let D be a fixed right R-module and suppose that X is a set of generators for $\mathrm{Hom}(D, R)$. Suppose that every dense right ideal of R contains a finite set J such that $J^{\mathfrak{l}} = 0$. Then the cokernel of $D \longrightarrow R^X$ is weak torsion free.* ∎

3.19. Proposition. *Suppose* $D \overset{f}{\hookrightarrow} D' \overset{g}{\hookrightarrow} D^{**}$. *Then* f^* *and* g^* *are isomorphisms.*

Proof. By 3.14, D^{**}/D is weak torsion and it follows immediately that D'/D is weak torsion so that f^* is injective. Then from $D \xrightarrow{f} D' \xrightarrow{g} D^{**}$ and the fact that D^* is fixed, we get that the composite $D^{***} \xrightarrow{g^*} D'^* \xrightarrow{f^*} D^*$ is an isomorphism. It is standard now that when f^*g^* is an isomorphism and f^* is monic, then f^* and g^* are isomorphisms. ∎

3.20. weak closure. Let $D \subseteq D'$ be an inclusion in $\mathcal{D}$. Say that an element $b \in D'$ lies in the **weak closure** of D if there is a finite set $F = \{d_1, d_2, \ldots, d_k\} \subseteq D$ such that any $\varphi \in D'^*$ that vanishes on F also vanishes at b. One readily sees that the set of these elements is a right submodule of D' that contains D and we call it $\mathrm{wc}_{D'}(D)$.

Until further notice, we will assume that D is R-cogenerated and that we have chosen a set X of generators for $\mathrm{Hom}(D, R)$. It follows that $D \subseteq R^X$ and that $X \cdot R \twoheadrightarrow D^*$ is surjective. We denote $\mathrm{wc}_{R^X}(D)$ by $\mathrm{wc}(D)$ and the inclusion of $D \hookrightarrow \mathrm{wc}(D)$ by f.

3.21. Proposition. $f^* : \mathrm{wc}(D)^* \longrightarrow D^*$ *is an isomorphism.*

Proof. From $D \overset{f}{\hookrightarrow} \mathrm{wc}(D) \hookrightarrow R^X$, we get $(R^X)^* \longrightarrow \mathrm{wc}(D)^* \xrightarrow{f^*} D^*$. Since the composite is a surjection, so is f^*. Next we claim it is injective. If not, there is some $\varphi \in \mathrm{wc}(D)^*$ for which $f^*(\varphi) = \varphi f = 0$. But it is obvious that for any $b \in \mathrm{wc}(D)$, if $\varphi f = 0$, then $\varphi(b) = 0$. Hence f^* is bijective. A subbasic open neighbourhood of 0 in $\mathrm{wc}(D)^*$ is given by $U(b) = \{\varphi \in \mathrm{wc}(D)^* \mid \varphi(b) = 0\}$. If $F = \{d_1, d_2, \ldots, d_k\}$ is the set described in the definition, then $U(b) \supseteq U(d_1) \cap U(d_2) \cap \cdots \cap U(d_k)$. This shows that f^* is open. ∎

3.22. Corollary. $\mathrm{wc}(D) \subseteq D^{**}$.

Proof. The preceding shows that $D^{**} = \mathrm{wc}(D)^{**}$ and of course $\mathrm{wc}(D) \subseteq \mathrm{wc}(D)^{**}$. ∎

3.23. Theorem. *If D is R-cogenerated and X is a set of generators for* $\mathrm{Hom}(D, R)$, *then* $\mathrm{wc}_{R^X}(D) = D^{**}$.

Proof. It is sufficient to show that every element of D^{**} is in the weak closure of D. We know that the inclusion $f : D \longrightarrow D^{**}$ induces an isomorphism $D^{***} \longrightarrow D^*$. Since this map is open it must be that for any $b \in D^{**}$, the set $U(b)$, as defined in the proof of 3.21, is an open neighbourhood of 0 in D^*. This means that there is a finite set $F = \{d_1, d_2, \ldots, d_k\}$ such that $U(b) \supseteq U(d_1) \cap U(d_2) \cap \cdots \cap U(d_k)$. But this just means that any function that vanishes on F vanishes at b. Note that we use the surjectivity of $(R^X)^* \twoheadrightarrow D^{***}$ to be able to restrict to functions defined on D^{**}. ∎

3.24. Theorem. *Suppose that* $D' \in \mathrm{Fix}(\mathcal{D})$ *and* $D \subseteq D'$ *is a submodule such that* $D = \mathrm{wc}_{D'}(D)$. *Then* $D \in \mathrm{Fix}(\mathcal{D})$.

Proof. Let $\eta D : D \longrightarrow D^{**}$ be the adjunction morphism. Since D' is fixed, double dualization yields a map $f : D^{**} \longrightarrow D'$ such that $f \circ \eta D$ is the inclusion. Let $b \in D^{**}$. Then there is a finite subset $F \subseteq D$ such that any $\varphi : D^{**} \longrightarrow R$ that vanishes on F also vanishes at b. It follows that if $\psi : D' \longrightarrow R$ is a left R-linear map that vanishes on F, it also vanishes at $f(b)$. But this means that $f(b) \in D$. Thus D is a retract of D^{**} from which it follows that $D \in \mathrm{Fix}(\mathcal{D})$ and ηD is an isomorphism. ∎

Reflexive subcategories. We will show that $\mathrm{Fix}(\mathcal{C})$ is a reflective subcategory of $\mathcal{C}$ and $\mathrm{Fix}(\mathcal{D})$ is a reflective subcategory of $\mathcal{D}$.

3.25. Theorem. $\mathrm{Fix}(\mathcal{D})$ *is a reflective subcategory of* $\mathcal{D}$.

Proof. Suppose $f : D \longrightarrow D'$ is right R-linear and that $D' \in \mathrm{Fix}(\mathcal{D})$. Double dualization gives a commutative square

$$\begin{tikzcd} D \arrow[r, hook, "\eta D"] \arrow[d, "f"'] & D^{**} \arrow[d, "f^{**}"] \\ D' \arrow[r, hook, "\eta D'"'] & D'^{**} \end{tikzcd}$$

and $\eta D'$ is an isomorphism. This gives at least one lifting of f to a map $D^{**} \longrightarrow D'$. If there were two, the difference would induce a non-zero map $D^{**}/D \longrightarrow D'$. But this is impossible because D^{**}/D is weak torsion, while D' is R-cogenerated and thus torsion free. ∎

3.26. Theorem. $\mathrm{Fix}(\mathcal{C})$ *is a reflective subcategory of* $\mathcal{C}$.

Proof. It follows from 3.12 that for every $C \in \mathcal{C}$, the module C^* is fixed. Now if $f : C \longrightarrow C'$ is given and $C' \in \mathrm{Fix}(\mathcal{C})$, we get, as in the preceding proof, a map $(\eta C')^{-1} f^{**}$ whose composite with ηC is f. If there were another such map, the difference would induce a map on C^{**}/C. From the exactness of $C \longrightarrow C^{**} \longrightarrow C^{**}/C \longrightarrow 0$, we get that $0 \longrightarrow (C^{**}/C)^* \longrightarrow C^{***} \longrightarrow C^*$ is exact. But with C^* fixed, we see that $(C^{**}/C)^* = 0$ so that C^{**}/C has no non-zero maps to R and hence none to any R-cogenerated objects. ∎

From these two theorems, we conclude:

3.27. Corollary. *Both* $\mathrm{Fix}(\mathcal{C})$ *and* $\mathrm{Fix}(\mathcal{D})$ *are closed in* $\mathcal{C}$ *and* $\mathcal{D}$*, respectively, under limits.* ∎

3.28. The conditions. There are various special conditions we have used. One is that the complete ring of *right* quotients be *left* injective. The question of right injectivity has been studied and conditions are well known. Little is known about left injectivity. Obviously if R is commutative, the two conditions are the same. If a domain, not necessarily commutative, satisfies both Ore conditions then the right and left classical rings of quotients coincide and it forms a division ring. In that case the classical and complete rings of quotients coincide and this ring is also R-flat on both sides ($Q = \mathrm{colim}_{r\neq 0}\, Rr^{-1}$, a filtered colimit and similarly on the other side). Thus it is also left R-injective. In addition, every dense ideal obviously contains a non zero-divisor, whose annihilator is 0 and thus our characterizations of $\mathrm{Fix}(\mathcal{C})$ and $\mathrm{Fix}(\mathcal{D})$ both hold. With slight modification, this also holds for a finite product of domains that satisfy both Ore conditions.

Another condition is the one used in the proof of 3.18: that every dense right ideal I contains a finite subset J for which $J^{\mathfrak{l}} = 0$. This condition clearly holds in any domain and it also holds in any right Noetherian ring.

A trivial example is that of a ring which is left self-injective. In that case, the conclusion of Theorem 3.9 is valid, while that of Proposition 3.14 holds with $T = 0$. That shows that for any object $D \in \mathcal{D}$, the map $D \longrightarrow D^{**}$ is surjective while it is injective whenever D is R-cogenerated. Thus we conclude,

3.29. PROPOSITION. *If the ring R is left self-injective, then every R-cogenerated object of $\mathcal{D}$ is fixed.* ■

There is an interesting (but known) consequence to this. Let R be a left self-injective ring. Then we know that every right ideal is fixed. If I is a right ideal the inclusion $I \hookrightarrow R$ induces a surjection $R \twoheadrightarrow I^*$ and it follows that $\mathrm{wc}(I) = I$. Thus for any $b \in R - I$ and any finite set $F \subseteq I$ there must be a right R-linear map $R \longrightarrow R$ that vanishes on F and not at b. Any right R-linear map $R \longrightarrow R$ is left multiplication by an element of R so that this says that if $J = FR$, then an element $b \notin J$ is not annihilated by any element of the left annihilator $J^{\mathfrak{l}}$. In other words, $b \notin J^{\mathfrak{lr}}$, which is possible for all $b \notin I$ if and only if $J^{\mathfrak{lr}} = J$. Compare [Stenström (1975), XIV.2.2(ii)].

In 6.1 below, we will see that a fixed D might not have the property that the cokernel of $D \longrightarrow R^{D^*}$ is weak torsion free.

An interesting class of rings is that of strongly regular rings. For our purposes we use a definition different from, but equivalent to the usual one, which can be found in [Stenström (1975), Page 40]. We will say that a ring is **strongly regular** if it is von Neumann regular and every idempotent is central. If Q is the complete ring of right quotients of such a ring, then Q is left and right self-injective. All R-modules are flat, so that Q is also R-injective and Theorem 3.9 is satisfied. See [Stenström (1975), Proposition 5.2] for details. Unfortunately, such rings are unlikely to have finitely generated left annihilators.

Let R be the ring of upper triangular matrices over a field K. In this case Q is the full matrix ring ([Stenström (1975), Problem 4, p. 260]). It is a finite right (and left) localization and therefore flat on both sides (*op. cit.* p. 239). Then Q is R-injective and, obviously, Noetherian so that both Theorems 3.9 and 3.15 hold.

4. Morita equivalence

For a complete statement and proof of the Morita theorem, we refer to [Stenström (1975), IV.10]

In this section, we will be dealing with two rings R and S and will denote the categories of discrete right modules by $\mathcal{D}(R)$ and $\mathcal{D}(S)$ and the categories of topological left modules by $\mathcal{C}(R)$ and $\mathcal{C}(S)$. Also we will use the letters P and Q to denote modules; in particular Q is not a ring of quotients. We write ${}_RP_S$ to indicate that P is a left R-module and a right Q-module that satisfies the "associative" (really commutative) law $r(ps) = (rp)s$ for $r \in R$, $p \in P$, and $s \in S$. Analogously, we write ${}_SQ_R$.

A Morita equivalence between the rings R and S is mediated by a pair of modules, ${}_RP_S$ and ${}_SQ_R$ such that $P \otimes_S Q \cong R$ as a two-sided R-module and $Q \otimes_R P \cong S$ as a two sided S-module. Under these circumstance, P is finitely generated left R-projective and finitely generated right S-projective and, *mutatis mutandi*, the same is true for Q. In that case

$$Q \otimes_R - = \hom_R(P, -) : {}_R\mathrm{Mod} \longrightarrow {}_S\mathrm{Mod}$$

is an equivalence of categories whose inverse is given by

$$P \otimes_S - = \hom_S(Q, -)$$

and

$$- \otimes_R P = \hom_{R^{\mathrm{op}}}(Q, -) : \mathrm{Mod}_R \longrightarrow \mathrm{Mod}_S$$

is an equivalence of categories whose inverse is given by

$$- \otimes_S Q = \hom_{S^{\mathrm{op}}}(P, -)$$

4.1. Theorem. *The diagram*

$$\begin{tikzcd}
\mathcal{D}(R) \arrow[r, "- \otimes_R P"] \arrow[d, "\hom_{R^{\mathrm{op}}}(-,R)"'] & \mathcal{D}(S) \arrow[d, "\hom_{S^{\mathrm{op}}}(-,S)"] \\
\mathcal{C}(R) \arrow[r, "Q \otimes_R -"'] & \mathcal{C}(S)
\end{tikzcd}$$

commutes.

Proof. Both paths are contravariant and turn the colimits into limits. For the upper right path, this is evident. For the lower left, it becomes clear once we observe that $Q \otimes_R - \cong \hom_R(P, -)$. For every $D \in \mathcal{D}(R)$, we have that $D = \mathrm{colim}_{\mathrm{Hom}(R,D)} R$, so that it suffices to show that this diagram commutes at R itself. Going the upper left path, we get

$$\hom_{S^{\mathrm{op}}}(P \otimes_R R, S) \cong \hom_{S^{\mathrm{op}}}(P \otimes_R R, P \otimes_R Q) \cong \hom_{R^{\mathrm{op}}}(R, Q) \cong Q$$

the second isomorphism coming from the fact that $P \otimes_R - : \mathcal{D}(R) \longrightarrow \mathcal{D}(S)$ is an equivalence of categories. The lower left path gives

$$Q \otimes_R \hom_R(R, R) \cong Q \otimes_R R \cong Q$$

. ∎

Since when right adjoints commute, so do left adjoints, we conclude that:

4.2. THEOREM. *Suppose R and S are Morita equivalent. Then* $\mathrm{Fix}(\mathcal{C}(R))$ *and* $\mathrm{Fix}(\mathcal{C}(S))$ *are equivalent and so are* $\mathrm{Fix}(\mathcal{D}(R))$ *and* $\mathrm{Fix}(\mathcal{D}(S))$. ■

5. Complete subproducts

5.1. THEOREM. *Suppose the ring R is right Noetherian and has right global dimension at most 1. Then a topological right R-module is R-sober if and only if it is R-cogenerated and complete as a uniform space.*

PROOF. The necessity of the condition is obvious and requires no hypothesis on R. The given conditions imply that an R-cogenerated module C is topologically embedded in $R^{\mathrm{Hom}(C,R)}$ and that a submodule of a finitely generated free (or even projective) left R-module is finitely generated projective. Let X denote $\mathrm{Hom}(C, R)$. Since continuous homomorphisms of topological abelian groups are uniformly continuous for the canonical uniformities, we see that C is embedded as closed submodule of R^X. Sobriety will be demonstrated by showing that for any $w \in R^X - C$ there is a left R-linear map $\varphi : R^X \longrightarrow R$ that vanishes on C but for which $\varphi(w) \neq 0$. If not, then it is immediate that whenever n is finite two continuous maps $\sigma, \tau : R^X \longrightarrow R^n$ that are equal on C are also equal at w. Since C is closed there is a basic neighbourhood of w that does not meet C. This means that there is a finite subset $Y \subseteq X$ such that if $\pi_Y : R^X \longrightarrow R^Y$ is the canonical projection, then $w + \ker \pi_Y$ does not meet C. This implies that $\pi_Y(w) \notin C_0 = \pi_Y(C)$; for if $\pi_Y(w) = \pi_Y(c)$ for some $c \in C$, then $c - w \in \ker \pi_Y$ and then $c \in w + \ker \pi_Y$. Since C_0 is a left submodule of a finitely generated free module, it is finitely generated projective and hence there is a split monic $\alpha : C_0 \longrightarrow R^n$ for some integer n. Since C_0 is discrete, α is continuous and hence so is $\alpha \circ \pi_Y|C$. Since R^n is also discrete the splitting map β is also continuous. Since $\alpha \circ \pi_Y|C$ is a continuous map from C to a power of R, its components, say $\{\mu_z \mid z \in Z\}$ are elements of X. Let $M = \{\mu_z \mid z \in Z\} \subseteq X$ and $\pi_M : R^X \longrightarrow R^M$ be the corresponding projection. Then for $c \in C$, $\pi_M(c) = (\mu_z(c))_{z \in Z} = \alpha \circ \pi_Y|C(c) = \alpha\pi_Y(c)$ and then $\pi_Y(c) = \beta\pi_M(c)$. Since this holds for all $c \in C$ we conclude that $\pi_Y(w) = \beta\pi_M(w) \in C_0$, a contradiction. ■

6. Examples

6.1. EXAMPLE. *This example shows that the conclusion of Corollary 3.18 may fail if a dense ideal I does not contain a finite subset J with $J^{\mathfrak{l}} = 0$.*

Suppose that $\{R_\alpha \mid \alpha \in A\}$ is an infinite family of left self-injective rings, all with at least two elements. Let $R = \prod R_\alpha$. Then R is also self-injective, by Corollary 2.7. It follows from 3.29 that every R-cogenerated module, in particular, every ideal of R, is fixed. Each R_α is embedded in R as a two sided ideal we call I_α. The ideal $I = \sum I_\alpha$ is readily seen to be dense. But then R/I is annihilated by I and is therefore weak torsion, while I is fixed. It is also true that the singleton consisting of the inclusion map is a subset of $\operatorname{Hom}(I,R)$ for which $1 \cdot R \twoheadrightarrow I^*$. ■

6.2. EXAMPLE. *Theorem 5.1 fails for $R = \mathbf{Z}[x]$.*

Let M be the ideal of R generated by 2 and x. Then M is a submodule of R, necessarily closed and complete (as R is discrete). But we claim that every R-linear map from M to R has a unique extension to a map from R to R. Let $h : M \longrightarrow R$ be such a map. Let $h(2) = p(x)$ and $h(x) = q(x)$. Then $h(2x) = 2h(x) = 2q(x)$ and $h(x2) = xp(x)$ so $2q(x) = xp(x)$ which implies that all of the coefficients of $p(x)$ are even, so there exists a unique $t(x)$ with $2t(x) = p(x)$ and we can clearly extend h to a homomorphism from R to R which maps 1 to $t(x)$.

From this, it readily follows that any R-linear $h : M \longrightarrow R^X$ extends uniquely to a map $h^e : R \longrightarrow R^X$ and any R-linear $R^X \longrightarrow R^Y$ which is zero on $h(M)$ will be zero on $h^e(R)$ contradicting the sobriety of M. ■

6.3. EXAMPLE. *Theorem 5.1 may hold even though R is not a PID.*

Let K be any field and $R = A_1(K)$, the Weyl algebra in one variable defined as $K[x,y]/(xy - yx - 1)$. The reason it is described as the one variable algebra is that y is thought of as representing the differential operator d/dx. It is known that this ring is a right and left hereditary Noetherian domain, see [Rinehart (1962)]. The commutation relation implies that the Ore conditions are satisfied and so the ring of quotients is a division ring and therefore R-injective, see Theorem 2.5. ■

6.4. EXAMPLE. *Every ideal in a strongly regular ring R is fixed. But there are strongly regular rings R (in particular, Boolean rings) for which not every R-cogenerated module is fixed.*

PROOF. Recall that a ring R is strongly regular if it is von Neumann regular and every idempotent is central. We use Theorem 3.24 and show that for any ideal $I \subseteq R$, $\mathrm{wc}_R(I) = I$. Suppose $r \in \mathrm{wc}_R(I)$. Then there a finite set $B = \{d_1, \ldots, d_n\}$ of elements of I such that any R-linear map $f : R \longrightarrow R$ that vanishes on B vanishes at r. It is clear that an R-linear map vanishes at d if and only if it vanishes at the idempotent $e = dd'$. Thus we may assume that B is a finite set of idempotents $e_1, \ldots, e_n$. Let $e = e_1 \vee \cdots \vee e_n$ and $f : R \longrightarrow R$ be multiplication by $1 - e$. Then f vanishes on B, hence also at r. This means that $r(1-e) = 0$ which is possible only if $re = r$ which implies $r \in I$.

For an example that not every R-cogenerated module is fixed, we let S be the Boolean ring of all subsets of $\mathbf{N}$ and R be the subring of finite/cofinite subsets. Since R is a subring of S, we have that S is an R-module. Let $I \subseteq R$ be the ideal of all finite subsets of $\mathbf{N}$.

6.5. PROPOSITION. *Suppose D is an R-submodule of S that strictly contains R. Let $\varphi : D \longrightarrow R$ be R-linear. Then for any $d \in D$, $\varphi(d) = d\varphi(1)$. In particular $\varphi(1)$ must be a finite subset of $\mathbf{N}$.*

Proof. We begin with the observation that for any $k \in \mathbf{N}$, $\{k\}$ is an atom so that for any $s \in S$, either $\{k\}s = 0$ or $\{k\}s = \{k\}$. Suppose that $k \in \varphi(d)$. Then $\{k\} = \{k\}\varphi(d) = \varphi(\{k\}d)$ so that $k \in d$. But then $\{k\}d = \{k\}$ so that $\{k\} = \varphi(\{k\}d) = \varphi(\{k\}) = \varphi(\{k\}1) = \{k\}\varphi(1)$ and so $k \in \varphi(1)$ and thus $k \in d\varphi(1)$. Conversely, if $k \in d\varphi(1)$, then $\{k\} = \{k\}\varphi(1) = \varphi(\{k\}) = \varphi(\{k\}d) = \{k\}\varphi(d)$ and hence $k \in \varphi(d)$. Suppose that $\varphi(1)$ is cofinite. Since D is strictly larger than R, it contains an element d that is neither finite nor cofinite. But then $\varphi(d) = d\varphi(1) = d$ is infinite, but not cofinite, and hence not in R. ∎

6.6. Corollary. *$D^* = I$ with the discrete topology.*

Proof. We have just seen that $D^* = I$. For each $d \in D$, the evaluation at d must be continuous so that for each $d \in S$, $\{r \in I \mid rd = 0\}$ is open. But when $d = 1$, this is just the 0 element. ∎

Now let D lie strictly between R and S. We see that $D^* = S^* = I$ and hence D cannot be fixed. ∎

7. Actions

This section is not really about modules, but should be considered as an addition to [Barr et al. (2008)] since it gives more instances of the general duality theory of that paper.

Let E be a monoid. If E is considered as a category with one object, then for any category $\mathcal{A}$, the functor category $\mathcal{A}^E$ has for objects pairs (A, σ) with A and object of $\mathcal{A}$ and $\sigma : E \longrightarrow \mathrm{Hom}(A, A)$ is a monoid homomorphism to the endomorphism monoid of A. A morphism $f : (A, \sigma) \longrightarrow (A'\sigma')$ is a morphism $f : A \longrightarrow A'$ such that

$$\begin{tikzcd} E \arrow[r, "\sigma"] \arrow[d, "\sigma'"'] & \mathrm{Hom}(A, A) \arrow[d, "\mathrm{Hom}(A,f)"] \\ \mathrm{Hom}(A', A') \arrow[r, "\mathrm{Hom}(f,A')"'] & \mathrm{Hom}(A, A') \end{tikzcd}$$

commutes. Note that this is equivalent to the statement that for all $x \in E$, the diagram

$$\begin{tikzcd} A \arrow[r, "\sigma(x)"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A' \arrow[r, "\sigma'(x)"'] & A' \end{tikzcd}$$

commutes. In concrete cases, this can be interpreted as $f(xa) = xf(a)$ which means that A is a *left* E-action in the usual sense. It is easy to see that the category $\mathcal{A}^{E^{\mathrm{op}}}$ can similarly be interpreted as the category of *right* E-actions. Incidentally, had we chosen to write composition in the opposite order, the notions of left and right E-actions would be reversed.

Suppose $\mathcal{C}$ and $\mathcal{D}$ are dual categories. From $\mathcal{C}^{\mathrm{op}} \cong \mathcal{D}$ we easily see that $\mathcal{C}^{E^{\mathrm{op}}}$ is dual to $\mathcal{D}^E$.

The duality of the preceding sections is between categories $\mathcal{C}_0^{\mathrm{op}}$ of certain topological R^{op}-modules and $\mathcal{D}_0$ of certain discrete R-modules and is mediated in both directions by functors we may denote $\hom(-,R)$. In this section, we need to be a bit more careful, so we will denote them $\hom_R(-,R) : \mathcal{D}_0 \longrightarrow \mathcal{C}_0$ and $\hom_{R^{\mathrm{op}}}(-,R) : \mathcal{C}_0 \longrightarrow \mathcal{D}_0$. As indicated above, these functors extend to dualities between $\mathcal{C}_0^{E^{\mathrm{op}}}$ and $\mathcal{D}_0^E$. However, this is not in itself an Isbell duality in the sense of [Barr et al. (2008)] since the object R has no E-action and the homfunctors are not in the categories of E-actions. What we want to do is show that it is an Isbell duality, with the dual object being the ring R^E with left action given by $(xf)(y) = f(yx)$ and right action given by $(fx)(y) = f(xy)$ for $x, y \in E$ and $f \in R^E$. As an object of $\mathcal{C}^{E^{\mathrm{op}}}$, R^E is given the product topology from discrete R. Notice that these objects, being powers of R are actually in $\mathcal{C}_0$ and $\mathcal{D}_0$.

Another observation we need is that when $\mathcal{D}$ is the category of R-modules, then $\mathcal{D}^E$ is just the category of $R[E]$-modules, the left modules over the monoid algebra $R[E]$. Similarly $\mathcal{C}^{E^{\mathrm{op}}}$ is the category of right topological $R[E]$-modules (which is to say, $R^{\mathrm{op}}[E^{\mathrm{op}})]$-modules.

We turn first to $\mathcal{D}_0$. The inclusion of $R[E]$-modules into R-modules has a right adjoint that takes an R-module M to $\mathrm{Hom}_R(R[E], M)$ with left action induced by the right action of $R[E]$ on itself. This gives, for an $R[E]$-module M,

$$\hom_R(M,R) \cong \hom_{R[E]}(M, \mathrm{Hom}_R(R[E],R)) \cong \hom_{R[E]}(M, R^E)$$

In $\mathcal{C}$, we have to take the topology into account. It is well known that when E is discrete and X and Y are topological spaces, then a map $X \times E \longrightarrow Y$ is continuous if and only if its transpose $X \longrightarrow Y^E$ is continuous into the product topology. Applying that here we see that for an $R[E]^{\mathrm{op}}$-module M and an R-module N, the groups of continuous homomorphisms $\mathrm{Hom}_{R^{\mathrm{op}}}(M,N)$ and $\mathrm{Hom}_{R[E]^{\mathrm{op}}}(M,N^E)$ are isomorphic and so we have the same isomorphisms as in $\mathcal{D}$. Thus we conclude:

7.1. THEOREM. *Assume that R is a ring for which the duality of the first section is valid. Let $\mathcal{C}_0$ and $\mathcal{D}_0$ be the fixed categories. Then for any monoid E, the object R^E (with the product topology in $\mathcal{C}$) gives an Isbell duality between $\mathcal{C}_0^{E^{\mathrm{op}}}$ and $\mathcal{D}_0^E$.*

8. Appendix on adjunction between $\mathcal{C}$ and $\mathcal{D}$

We relegate to this appendix the proof that $\hom_{\mathcal{C}}(-,R)$ is adjoint on the right to $\hom_{\mathcal{D}}(-,R)$.

What we will show is that for any $C \in \mathcal{C}$ and $D \in \mathcal{D}$, both $\mathrm{Hom}_{\mathcal{C}}(C, \hom_{\mathcal{D}}(D,R))$ and $\hom_{\mathcal{D}}(D, \hom_{\mathcal{C}}(C))$ can be identified as the set of all functions $\varphi : \|C\| \times \|D\| \longrightarrow R$ that are additive in both variables, such that $\varphi(-,d)$ is continuous on C for all $d \in D$ and such that $\varphi(rc, ds) = r\varphi(c,d)s$ for all $r, s \in R$, $d \in D$ and $c \in C$.

Suppose $f : C \longrightarrow \hom_{\mathcal{D}}(D, R)$ is given. Clearly we get from this a function $\varphi : \|C\| \times \|D\| \longrightarrow R$ and it is obviously additive in both variables. Now f is assumed to be a morphism of left R-modules. This means that $f(rc) = rf(c)$ for every $r \in R$ and $c \in C$. The left action of R on $\hom_{\mathcal{D}}(D, R)$ comes from the left action of R on itself so that for any $d \in D$, $\varphi(rc, d) = f(rc)(d) = rf(c)(d) = r\varphi(c, d)$. Since f takes values in the group of right R-linear maps, we must also have, for every $s \in R$ that $f(c)(ds) = f(c, d)s$ so that $\varphi(c, ds) = \varphi(c, d)s$. We also have that f is continuous. The topology on $\hom_{\mathcal{D}}(D, R)$ embeds it as a subspace of $R^{|D|}$. For f to be continuous, it is necessary and sufficient that the composite $C \longrightarrow \hom_{\mathcal{D}}(D, R) \longrightarrow R^{|D|}$ be continuous. This precisely means that for all $d \in D$, the map $\varphi(d, -)$ be continuous on C, which means that its kernel is open.

Now let $g : D \longrightarrow \hom_{\mathcal{C}}(C, R)$ be given with induced map $\psi : \|C\| \times \|D\| \longrightarrow R$. The additivity in each variable as well as the preservation of right and left action from R are exactly the same as in the preceding paragraph. Finally, in order that g land in the set of continuous maps, it is necessary and sufficient that $g(d)$ be continuous for all $d \in D$.

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