RELATIONAL ALGEBRAS

Michael Barr

Received October 20, 1969

Introduction:

All notation not otherwise specified in this paper is taken from the introduction to the [ZTB]. Let $\underline{T}$ denote the category of topological spaces and continuous mappings, $\underline{C}$ be the full subcategory whose objects are the compact hausdorff spaces and $\underline{S}$ the category of sets. The obvious functor U: $\underline{C} \to \underline{S}$ has a left adjoint F which is best described by saying that FX is the set of all ultrafilters on X with the hull-kernel topology. If $\mathbb{T} = (T, \eta, \mu)$ is the triple coming from the adjoint pair, the natural functor $\underline{C} \to \underline{S}^{\mathbb{T}}$ is an equivalence. This associates to a compact hausdorff space C the pair (UC,c) where c: TUC $\to$ UC is given by $c(\underline{u}) = \lim \underline{u}$ for $\underline{u}$ an ultrafilter on UC.

In an arbitrary topological space V there is still given a relation x: TUV $\to$ UV which associates to an ultrafilter $\underline{u}$ the set $x(\underline{u})$ of all its limits (possibly empty). Moreover it is well known that this convergence relation determines the topology uniquely and that continuity of mappings can also be described by it. It thus seemed plausible that by a suitable axiomatization of the notion of relational algebra of a triple (or of a theory) it could be shown that $\underline{T}$ is the category of relational algebras of the theory whose algebras are $\underline{C}$. That this is so is the main result of this paper.

Section 1 tabulates a few properties of the category $\underline{R}$ of sets and relations. Section 2 includes the definitions of relational pre-algebras and establishes their basic properties. In section 3 the main theorem 3.1 about $\underline{T}$ is proved and in section 4 a few additional examples are given.

Everything here is done for triples over the category $\underline{S}$. Presumably much of this could be done over other categories, at least those in which there is a good notion of what are relations. Manes conjectures (and is attempting to prove) that the analog of (3.1) is true for any varietal category and its compact algebra triple (see [Ma], § 5).

In carrying out this work I had several stimulating discussions with Basil Rattray who independently proved (3.1). I would also like to thank the National Research Council of Canada for its support.

1. Relations.

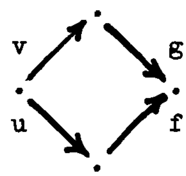
Before talking about relational algebras we will here tabulate some properties of the category $\underline{R}$ of sets and relations. If $r: X \twoheadrightarrow Y$ is a relation it has a standard factorization

$$X \xrightarrow{dr^{-1}} \Gamma_r \xrightarrow{cr} Y$$

where $\Gamma_r \subset X \times Y$ is the graph of r and the functions cr, dr are the restrictions to Γ_r of the coordinate projections. We write

$r^{-1} = dr.cr^{-1}$. We call r epi, mono, e.d. (everywhere defined) or a p.f. (partial function) according as cr is epi, cr is mono, dr is epi or dr is mono, respectively. To define the composite of two relations it is only necessary to define a composition of the form $f^{-1}g$ where f and g are functions, for then we define $r.s = cr.dr^{-1}.cs.cs^{-1}$. Consider a commutative diagram:

(1.1)



where u, v, f, g are all functions. In general we have only $uv^{-1} \subset f^{-1}g$ with equality if (1.1) is a weak pullback. In particular we define $f^{-1}g$ to be uv^{-1} when (1.1) is a pullback.

It is well known that the category $\underline{R}$ is a 2-category, the hom-sets being partially ordered by $r \subset s$ in $\underline{R}(X,Y)$ if $\Gamma_r \subset \Gamma_s$. The following proposition is also well known (see e.g. , [Mac]).

<u>Proposition 1.2</u>. For any relation $r: X \twoheadrightarrow Y$

i) r is epi iff $rr^{-1} \supset Y$

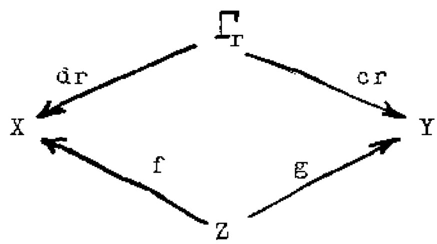
ii) r is mono iff $r^{-1}r \subset X$

iii) r is e.d. iff $r^{-1}r \supset X$

iv) r is a p.f. iff $rr^{-1} \subset Y$.

Of course combinations of these can be worked out as well. The most important characterizes r as a function iff $r^{-1}r \supset X$ and $rr^{-1} \subset Y$.

If T: $\underline{S} \to \underline{S}$ is a functor it induces a pseudofunctor, also denoted by T: $\underline{R} \to \underline{R}$ given by $T(cr.dr^{-1}) = Tcr.T(dr)^{-1}$. This means that for r·s composable relations, $T(r,s) \subset Tr.Ts$ and that $r \subset s \Rightarrow Tr \subset Ts$. Notice that $T(r^{-1}) = (Tr)^{-1}$ under this definition and will be written Tr^{-1}. If we have a diagram



then $gf^{-1} \subset r$ iff there is a mapping $Z \to \Gamma_r$ making both triangles commute and $gf^{-1} = r$ iff that mapping is epi. In that case, since endofunctors on $\underline{S}$ preserve epimorphisms, it follows that the induced map $TZ \to T\Gamma_r$ is also epi. We have defined Tr to have the graph Γ_{Tr} which is the image of $T\Gamma_r$ in $TX \times TY$. The result is that the induced map $TZ \to \Gamma_{Tr}$ is still epi and so $Tr = Tg.Tf^{-1}$ is independent of the factorization. Then to see that T is a pseudofunctor it is sufficient to show that $T(f^{-1}g) \subset Tf^{-1}.Tg$, for then we have, after factoring $dr^{-1}.cs = p.q^{-1}$,

$$T(r.s) = T(cr.dr^{-1}.cs.ds^{-1}) = T(cr.p.q^{-1}.ds^{-1})$$
$$= T(cr.p).T(q^{-1}.ds^{-1}) = Tcr.Tp.T((ds.q)^{-1})$$
$$= Tcr.Tp.(T(ds.q))^{-1} = Tcr.Tp.(Tds.Tq)^{-1}$$
$$= Tcr.Tp.Tq^{-1}.Tds^{-1} = Tcr.T(pq^{-1}).Tds^{-1}$$
$$= Tcr.T(dr^{-1}.cs).Tds^{-1} \subset Tcr.Tdr^{-1}.Tcs.Tds^{-1}$$
$$= Tr.Ts$$

Clearly equality holds if either r is a function or s is an inverse function. Now suppose that u and v are the pullback of f and g as in 1.1. After we apply T the diagram will still commute, but might fail to be a pullback. Thus, $T(f^{-1}g) = T(u.v^{-1}) = Tu.Tv^{-1} \subset Tf^{-1}.Tg$. If $\alpha: T \to T_1$ is a natural transformation of functors on S, it becomes pseudonatural on $\underline{R}$, which in this context means that $\alpha Y.Tr \subset T_1r.\alpha X$.

2. Relational Algebras

We recall some of the definitions from the introduction to [ZTB]. A triple $\mathbb{T} = (T,\eta,\mu)$ on $\underline{X}$ is an endofunctor T of $\underline{X}$ together with natural transformations $\eta: T \to \underline{X}$ (= identity functor) and $\mu: T^2 \to T$ satisfying $\mu.T\mu = \mu.\mu T$, $\mu.T\eta = \mu.\eta T = T$. The category $\underline{X}^{\mathbb{T}}$ of $\mathbb{T}$-algebras has as objects (X,x) where X is an object and $x: TX \to X$ a morphism of $\underline{X}$ such that

$$x.Tx = x.\mu X$$

and

$$x.\eta X = X .$$

By a <u>relational</u> $\mathbb{T}$- <u>prealgebra</u> we mean a pair (X,x) where $x\colon TX \to X$ is a relation. A mapping $f\colon (X,x) \to (Y,y)$ of relational prealgebras is a function $f\colon X \to Y$ such that $f.x \subset y.Tf$. The category of relational prealgebras and mappings of prealgebras is denoted by $\underline{S}^{P(\mathbb{T})}$. A relational pre-algebra (X,x) is called a relational algebra if

$$x.Tx \subset x.\mu X$$

and

$$x.\eta X \supset X .$$

This definition is a precise generalization of algebra because any inclusion between actual maps in $\underline{S}$ is an equality. We let $\underline{S}^{R(\mathbb{T})} \subset \underline{S}^{P(\mathbb{T})}$ denote the full subcategory consisting of relational algebras. That $\underline{S}^{\mathbb{T}} \subset \underline{S}^{R(\mathbb{T})} \subset \underline{S}^{P(\mathbb{T})}$ are full inclusions follows from the fact that an inclusion among functions is an equality. We will show that each is a coreflective inclusion (by which we mean it has a left adjoint). First we show that limits and colimits are easily computed in $\underline{S}^{P(\mathbb{T})}$.

<u>Proposition 2.1</u>. $\underline{S}^{P(\mathbb{T})}$ is complete and cocomplete.

<u>Proof</u>: Limits (= projective limits) are computed in $\underline{S}^{P(\mathbb{T})}$ exactly as in $\underline{S}^{\mathbb{T}}$. For example, if (X_i,x_i) is a family of relational

prealgebras their product in $\underline{S}^{P(\mathbb{T})}$ is $(X = \prod X_i, \prod x_i.a)$ where $a\colon T(\prod X_i) \to \prod TX_i$ is determined by $p_i.a = Tp_i$ and p_i is the ith coordinate projection. Equalizers are computed similarly. The same holds for relational algebras. As for colimits, first consider the same family (X_i, x_i). Let $b\colon \coprod TX_i \to T(\coprod X_i)$ be determined by $b.u_i = Tu_i$, where u_i denotes the ith coordinate injection. Then I claim that $(\coprod X_i, \coprod x_i.b^{-1})$ is the coproduct. First we must show the $u_i\colon X_i \to \coprod X_i$ are morphisms, i.e., that $u_i.x_i \subset \coprod x_i.b^{-1}.Tu_i$. Now $b.u_i = Tu_i$ (by 1.2.iii) $= b^{-1}.Tu_i$; then $u_i.x_i = \coprod x_i.u_i \subset \coprod x_i.b^{-1}.Tu_i$. (Note: we have been using u_i for different coproduct injections.) Now if $f_i\colon (X_i, x_i) \to (Y,y)$ is given for each i, then the f_i, extend to a unique $f\colon \coprod X_i \to Y$ such that $f.u_i = f_i$. Then $f.\coprod x_i.u_i = f.u_i.x_i = f_i.x_i \subset y.Tf_i =$ $= y.Tf.Tu_i = y.Tf.bu_i$ for all i and so by uniqueness of morphisms from a coproduct, $f.\coprod x_i \subset y.Tf.b$. Then $f.\coprod x_i.b^{-1} \subset y.Tf.b.b^{-1} \subset y.Tf$ (by 1.2.iv). Thus f is a morphism of relational prealgebras. Uniqueness is clear.

Now suppose $d^0, d^1\colon (X,x) \to (Y,y)$ are given. Let $d\colon Y \to Z$ be the coequalizer of the set morphisms d^0 and d^1 and $z = d.y.Td^{-1}\colon TZ \to Z$. Then $d.y \subset d.y.Td^{-1}.Td$ (by 1.2iii) $= z.Td$ and so d is a morphism. If $f\colon (Y,y) \to (W,w)$ coequalizes d^0 and d^1 then there is induced a unique $g\colon Z \to W$ with $g.d = f$. Moreover, $g.z = g.d.y.Td^{-1} = f.y.Td^{-1} \subset w.Tf.Td^{-1} = w.Tg.Td.Td^{-1} \subset w.Tg$ (by 1.2.iv).

Remark: Note that the limits and colimits preserve the underlying sets. This could have been predicted from the fact that the underlying set functor $\underline{S}^{P(\mathbb{T})} \to \underline{S}$ has both adjoints, namely $X \mapsto (X, \eta X^{-1})$ and $X \mapsto (X, X \times TX)$ being left and right adjoint, respectively. Each of these is a relational algebra so the same remarks apply to $\underline{S}^{R(\mathbb{T})} \to \underline{S}$.

Proposition 2.2. (X,x) is a relational algebra iff $x \supset \eta X^{-1}$ and $x.Tx.\mu X^{-1} \subset x$.

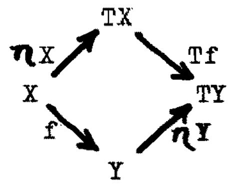
Proof: If (X,x) is a relational algebra then $X \subset x.\eta X$ so $\eta X^{-1} \subset x.\eta X.\eta X^{-1} \subset x$. Also $x.Tx.\mu X^{-1} \subset x.\eta X.\eta X^{-1} \subset x$. To see the converse we suppose that $x \supset \eta X^{-1}$; we have $x.\eta X \supset \eta X^{-1}.\eta X \supset X$. Similarly, if $x.Tx.\mu X^{-1} \subset x$, then $x.Tx \subset x.Tx.\mu X^{-1}.\mu X \subset x.\mu X$.

Proposition 2.3. The obvious functor $\underline{S}^{R(\mathbb{T})} \to \underline{S}^{P(\mathbb{T})}$ has a left adjoint.

Proof: Let $\eta = \eta X$ and $\mu = \mu X$. If (X,x) is a relational pre-algebra, we define an ordinal sequence of relations $x_n\colon TX \to X$ as follows. Let $x_o = x \cup \eta^{-1}$. Having defined x_m for all $m < n$, define $x_n = \bigcup_{m<n} x_m$ if m is a limit ordinal and $x_n = x_{n-1}.Tx_{n-1}.\mu^{-1}$ otherwise. Clearly $x_o \supset x$ and $x_o \supset \eta^{-1}$. Assuming $x_m \supset x_k$ for all $k < m < n$, we have $x_n \supset x_k$ for all $k < n$ if n is a limit ordinal and otherwise

$$x_n = x_{n-1}.Tx_{n-1}.\mu^{-1} \supset x_{n-1}.T\eta^{-1}.\mu^{-1} =$$
$$= x_{n-1}.(\mu.T\eta)^{-1} = x_{n-1}.$$ Now since $TX \times X$ is a set this

ascending chain of subsets represented by the graphs of these x_n must eventually terminate, at which point $x_n = x_{n-1} = \bar{x}$ and so by 2.2 $(Tx, \bar{x})$ is a relational algebra. If $f: (X,x) \to (Y,y)$ is a morphism of (X,x) into the relational algebra (Y,y), then $f.x \subset y.Tf$. Also considering the diagram



as a diagram like 1.1, we have $f.\eta^{-1} \subset \eta^{-1}.Tf \subset y.Tf$ so that $f.x_o \subset y.Tf$. Similarly if n is a limit ordinal and $f.x_m \subset y.Tf$ for all $m \subset n$ then certainly $f.x_n \subset y.Tf$. Finally, if $f.x_{n-1} \subset y.Tf$ then $f.x_n = f.x_{n-1}.Tx_{n-1}.\mu^{-1} \subset y.T(f.x_{n-1}).\mu^{-1}$ (since f is a function)

$\subset y.T(y.Tf).\eta^{-1} \subset y.Ty.T^2f.\mu^{-1} \subset y.Ty.\mu^{-1}.\mu.T^2f.\mu^{-1} \subset y.Tf.\mu.\mu^{-1} \subset y.Tf.$

Hence $f.\bar{x} \subset y.Tf$ from which the result easily follows.

Proposition 2.4. The obvious functor $\underline{S} \to \underline{S}^{R(\mathbb{T})}$ has a left adjoint.

Proof: This proof differs from the preceding in two respects. First the adjoint does not preserve the underlying set (by example). Second, the proof is nonconstructive, being an application of the adjoint functor theorem. It is clear from the description of limits in $\underline{S}^{P(\mathbb{T})}$ that the functor $\underline{S}^{\mathbb{T}} \to \underline{S}^{P(\mathbb{T})}$ preserves them. Also $\underline{S}^{R(\mathbb{T})}$ being a coreflective subcategory of $\underline{S}^{P(\mathbb{T})}$ computes its limits in that category so that $\underline{S}^{\mathbb{T}} \to \underline{S}^{R(\mathbb{T})}$ preserves limits. So it is only necessary to find a solution set. Given (X,x), consider all algebras of the form $(TX, \mu X)/\sim$ for which the composite $X \xrightarrow{\eta X} TX \longrightarrow TX/\sim$ is a mapping of relational algebras. This is clearly small. If f: $(X,x) \to (Y,y)$ is a mapping of X into a $\mathbb{T}$-algebra, y.Tf: $TX \to Y$ is a function and $y.Tf\mu X = y.\mu Y.T^2f = y.T(y.Tf)$ so y.Tf: $(TX, \mu X) \to (Y,y)$ is a map of algebras. As usual, it factors as $(TX,\mu X) \xrightarrow{b} (Y_o, y_o) \xrightarrow{a} (Y,y)$ where (Y_o, y_o) is a factor algebra of $(TX, \mu X)$ and a is mono. Then

$a.b.\eta X = y.Tf.\eta X = y.\eta Y.f = f$ and

$a.b.\eta X.x = f.x \subset y.Tf = y.Ta.T(b.\eta X) = a.y_o.T(b.\eta X)$ and a is mono so $(b.\eta X).x \subset y_o.T(b.\eta X)$. Thus $b.\eta X: (X,x) \to (Y_o, y_o)$ factors f and the codomain is a member of the given set. We note in passing that this gives a new proof of the cocompleteness of $\underline{S}^{\mathbb{T}}$.

3. Topological spaces

Theorem 3.1. There is a natural equivalence J between the category $\underline{T}$ of topological spaces and continuous maps and the category $\underline{S}^{R(\mathbb{T})}$ where $\mathbb{T} = (T, \eta, \mu)$ is the compact hausdorff spaces triple.

The proof is given by a series of propositions. We let $\underline{u}$ denote an ultrafilter on X.

Proposition 3.2. If C is a topological space, X = UC is its underlying set and x: $TX \to X$ is given by taking $x(\underline{u})$ to be the set of limits of $\underline{u}$, then $x.\eta \supset X$ and $x.Tx \subset x.\mu$.

Proof: The first statement is trivial, being merely the statement that if $\underline{p}$ is the principal ultrafilter at the point $p \in X$, then $p \in x(\underline{p})$ which is certainly true in any topological space. We can factor x as $TX \xleftarrow{j} DX \xleftarrow{d'x} \Gamma_x \xrightarrow{cx} X$ where j is mono, and d'x and cx are epi (the latter because of principal ultrafilters). T preserves both monos and epis so that

$$T^2X \xleftarrow{Tj} TDX \xleftarrow{Td'x} T\Gamma_x \xrightarrow{Tcx} TX$$

is the same type of decomposition. The mapping Tj consists of taking an ultrafilter on DX and using it as a filter base on TX. The filter generated is an ultrafilter. Thus if $\underline{a} \in T^2X$, $Tx(\underline{a}) \neq \emptyset$ iff given $A \in \underline{a}$, $DA = A \cap DX \in \underline{a}$ also. Now suppose that $\underline{a} \in T^2X$, $Tx(\underline{a}) = \underline{u}$ and $x(\underline{u}) = p$. Suppose that $\mu(\underline{a}) = \underline{v}$. Then we must show that $x(\underline{v}) = p$ as well. This means showing that every open neighborhood U of p is in $\underline{v}$ or, from the definition of μ, that $\{\underline{w} | U \in \underline{w}\} \in \underline{a}$. Now U is open which means that $x(\underline{w}) \cap U \neq \emptyset$. Suppose we had $A = \{\underline{w} | U \notin \underline{w}\} \in \underline{a}$. Now there is an ultrafilter $\underline{b}$ on $T\Gamma_x$ whose projections are $\underline{a}$ and $\underline{u}$ and there would have to be a $B \in \underline{b}$ whose projections were A on the one hand and some U_1 on the other. If $(\underline{w}, q) \in B$, then $U \notin \underline{w}$ but $q \in x(\underline{w})$. As noted above, $x(\underline{w}) \cap U = \emptyset$ and so $q \notin U$. Therefore, $U \cap U_1 = \emptyset$ which is a contradiction. Hence $A \notin \underline{a}$ and since $\underline{a}$ is an ultrafilter, its complement, $\{\underline{w} | U \in \underline{w}\} \in \underline{a}$. This completes the proof.

Proposition 3.3. If C and D are topological spaces with underlying sets X and Y and convergence mappings x and y respectively, then

$f: X \to Y$ underlies a continuous mapping iff $f.x \subset x.Tf$.

Proof: This is nothing but a translation of a well known theorem which states that f is continuous iff whenever $\underline{u}$ is an ultrafilter on X and $\underline{u}$ converges to p then $Tf(\underline{u})$ converges to fp (see [E-G]).

Corollary 3.4. There is a natural $J: \underline{T} \to \underline{S}^{R(\mathbb{T})}$ which is full and faithful.

Proof: Of course J is defined by JC = (UC, convergence) as above and the preceding two propositions state that J is a well defined functor and is full. It is clearly faithful since U is.

Now suppose (X,x) is a relational algebra. We must show it is J of something. If $e: U \to X$ is a subset inclusion we define a new subset $\bar{e}: \bar{U} \to X$ by the following diagram

$$\begin{array}{ccccc} TX & \xleftarrow{dx} & \Gamma_X & \xrightarrow{cx} & X \\ {\scriptstyle Te}\uparrow & & \uparrow{\scriptstyle g} & & \uparrow{\scriptstyle \bar{e}} \\ TU & \xleftarrow{dU} & \Gamma_U & \xrightarrow{cU} & \bar{U} \end{array} \qquad (3.5)$$

in which Γ_U is the pullback of Te and dx and $\bar{e}.cU$ is the mono/epi factorization of $cx.\gamma$. In words, $\bar{U}$ is the set of "limits" of all ultrafilters containing U.

Proposition 3.6. $U \mapsto \bar{U}$ is a closure operator.

Proof: We must show a) $\bar{\emptyset} = \emptyset$, b) $U \subset \bar{U}$, c) $\overline{U_1 \cup U_2} = \bar{U}_1 \cup \bar{U}_2$ and d) $\bar{U} = \bar{\bar{U}}$. Part a) is trivial since $T\emptyset = \emptyset$ and a pullback along $\emptyset$ is empty. b) is also easy for $p \in U$; the principal ultrafilter $\underline{p}$ contains U and converges to p. This implies that $\overline{U_1 \cup U_2} \supset \bar{U}_1 \cup \bar{U}_2$

so showing c) requires only the reverse inclusion. But if an ultrafilter $\underline{u}$ contains $U_1 \cup U_2$ then it must contain one of them and so we get the reverse inclusion. d) is harder. Again, we already have one inclusion. $\bar{U} \subset \bar{\bar{U}}$. It is easily seen to be sufficient to find a relation $t: \bar{\bar{U}} \to \bar{U}$ such that $\bar{\bar{e}} \subset \bar{e}.t$. For then image($\bar{\bar{e}}$) $\subset$ image($\bar{e}.t$) $\subset$ image($\bar{e}$) and the result is proved. We define $t = cU.dU^{-1}.\mu U.dU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1}$. In words, take a point of $\bar{\bar{U}}$ and find an ultrafilter on $\bar{U}$ converging to this. Represent that in turn by an ultrafilter on TU. Converge that under μ to an ultrafilter on U and converge that under x. Now recalling that $g.dU^{-1} = dx^{-1}.Te$ since Γ_U is a pullback and $f.f^{-1} \supset$ identity when f is epi, we have:

$$
\begin{aligned}
\bar{e}.t &= \bar{e}.cU.dU^{-1}.\mu U.TdU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1} \\
&= cx.g.dU^{-1}.\mu U.TdU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1} \\
&= cx.dx^{-1}.Te.\mu U.TdU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1} \\
&= x.\mu X.T^2e.TdU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1} \\
&\supset x.Tx.T^2e.TdU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1} \\
&= x.Tcx.Tdx^{-1}.T^2e.TdU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1} \\
&= x.Tcx.Tg.TdU^{-1}.TdU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1} \\
&\supset x.T\bar{e}.TcU.TcU^{-1}.d\bar{U}.c\bar{U}^{-1} \\
&= x.T\bar{e}.d\bar{U}.c\bar{U}^{-1} \\
&= cx.dx^{-1}.dx.\bar{g}.c\bar{U}^{-1} \\
&\supset cx.\bar{g}.c\bar{U}^{-1} = \bar{\bar{e}}.c\bar{U}.c\bar{U}^{-1} = \bar{\bar{e}}
\end{aligned}
$$

It is well known that such a closure operator induces a unique topology on X. There is now only one thing left to complete the proof of Theorem 3.1.

Proposition 3.7. Suppose $(X,x) \in \underline{S}^{R(\mathbb{T})}$ and the closure is defined as above. Then for any $\underline{u} \in TX$, $p \in x(\underline{u})$ iff $p \in \bigcap \{\overline{U} | U \in \underline{a}\}$.

Remark. Since this latter is the formula for ultrafilter convergence under a closure operator, this means that x is convergence (as described in the introduction) under that topology.

Proof: One way is clear. For if $p \in x(\underline{u})$ and $U \in \underline{u}$, $\underline{u}$ is an ultrafilter containing U and p is one of its limits under x so $p \in \overline{U}$. To go the other way we suppose that $U \in \underline{u}$ converges to $p \in \overline{U}$. We want to show that $p \in x(\underline{u})$. The idea of the proof is to find an $\underline{a} \in T^2X$ with $\mu(\underline{a}) = \underline{u}$ while $Tx(\underline{a}) = \underline{p}$, the principal ultrafilter at p. Then $p \in x(\underline{p}) \subset x.\mu(\underline{a}) = x(\underline{u})$. We begin by constructing an ultrafilter $\underline{b}$ on Γ_x. Let $B = \{(\underline{v},p) | p \in x(\underline{v})\} \subset \Gamma_x$ and for each $U \in \underline{u}$ let $B_U = \{(\underline{v},q) | U \in \underline{v}$ and $q \in x(\underline{v})\}$. Then for $U_1, U_2 \in U$, $B_{U_1} \cap B_{U_2} \supset B_{U_1 \cap U_2} \neq \emptyset$ since $U_1 \cap U_2 \neq \emptyset$. Moreover, since $p \in \overline{U}$ for each $U \in \underline{u}$, we can find a $\underline{v}$ such that $U \in \underline{U}$ and $p \in x(\underline{v})$. Then $(\underline{v},p) \in B \cap B_U$. Thus B together with $\{B_U\}$ generates a proper filter which is contained in an ultrafilter $\underline{b}$ on Γ_x. Clearly one projection is $\underline{p}$ since B projects to $\{p\}$. The other projection is $\underline{a}$ and it is clear that for all $U \in \underline{u}$, $\{\underline{v} | U \in v\} \supset dx(B_U)$ and thus is in $\underline{a}$. Hence $U \in \mu(\underline{a})$ for all $U \in \underline{u}$ and so $\underline{u} = \mu(\underline{a})$.

4. Other examples.

In this section we consider a few other examples of relational algebras although in no case are the results as striking as the preceding. The simplest example is the identity triple $\mathbb{I} = (I,1,1)$. Here the laws simply reduce to $x: X \twoheadrightarrow X$ subject to $x.x \subset x$ and $x \supset X$. Thus an algebra for this triple is a set with a transitive, reflexive relation, usually called a preordered set. It is simple to check that mappings are exactly the order preserving functions. It is perhaps interesting to note in this connection that the algebras, $\underline{S}$, are a coreflexive (hence tripleable) subcategory of preordered sets.

The next example is the triple $\mathbb{T} = (T,\eta,\mu)$ where $TX = X+1$, where + denotes the coproduct. The category of algebras is the category of pointed sets. A relational algebra is a set X with a relation $x: X+1 \twoheadrightarrow X$. Now X+1 is a coproduct and so x is determined by its restrictions $x|X$ and $x|1$. The first restriction is a preorder, as before, and the second is just a subset $X_o \subset X$. The unitary law is automatic (as soon as $x|X$ is a preorder) while the associative law requires that X_o be a ray. That is, if $p \in X_o$, $q \leq p$ (or $q \in x(p)$) then $q \in X_o$ also. Thus the algebras are pairs $(X, X_o \subset X)$ where X is a preordered set and X_o is a ray. Morphisms are maps of pairs which preserve the order as well.

Similar considerations apply to the category of relational models of the triple where TX is the set of ultrafilters on X+1.

The algebras are pairs (X, X_o) where X is a topological space and $X_o \subset X$ is a closed subspace. Mappings are continuous maps of pairs.

No other interesting examples are known to us. It seems fairly clear that Cat, for example, is a full subcategory of the relational models of the theory of monoids (however the nullary operation 1 becomes the set of objects of a category!) but just being a full subcategory is not very informative. (E.g. the recent work of the "Čech school" seems to show that practically everything is fully embedded in the category of semigroups).

References

[ZTB] - B. Eckmann, ed., "Seminar on Triples and Algebraic Homology Theory", Lecture Notes in Mathematics, no. 80, Springer Verlag, 1969.

[Ma] - E. Manes, A triple theoretic construction of compact algebras, ZTB, 91-118.

[E-G] - R. Ellis and W. H. Gottschalk, Homomorphisms of transformation groups, Trans. Amer. Math. Soc. 94 (1960), 258-271.

[Mac] - S. Mac Lane, An algebra of additive relations, Proc. NAS-USA 47, (1961), 1043-1051.