Oriented singular homology

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Abstract

We formulate three slightly different notions of oriented singular homology and show that all three are homotopic to ordinary singular homology.

1 Introduction

A standard fact about simplicial homology theory on simplicial complexes is that the oriented chain group is homotopic to the ordered (that might more evocatively be called unoriented) chain group. This was first proved in [Eilenberg, 1944] as motivation for his new definition of ordered singular homology to replace the oriented singular homology as defined, for example, in [Lefschetz, 1942]. Curiously, Eilenberg does not seem to have raised, let alone answered, the question as to whether his definition led to the same homology groups as did Lefschetz'. For simplicial complexes, the two definitions coincide, since they give the same groups as simplicial homology, but nothing was said about what happens beyond that domain.

Eilenberg's work was at least partly a response to a criticism by Čech of Lefschetz' definition. Čech pointed out that the chain group used by Lefschetz (it is the group that will be denoted C/U below) has elements of order 2 and is thus not truly free. It is not altogether clear, at this remove, why this was considered such a disadvantage, but it evidently was (see [Eilenberg, Steenrod, 1952], page 206). Eilenberg observed, for example, that this torsion in the chain groups could cause difficulty in defining morphisms on the chains.

Eilenberg also pointed out in his 1944 paper that, although he had proved that oriented simplicial homology of simplicial complexes was isomorphic to ordered singular

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homology, even naturally so, he could not exhibit an arrow in either direction between the chain groups that induced this isomorphism. Instead he gave natural maps from the ordered simplicial chain groups to each of the other two that induced the isomorphism. It is an interesting consequence of the results here that we find natural maps in both directions between the ordered and oriented singular groups. One direction can be composed with the obvious natural map from the oriented simplicial to the oriented singular chain groups to give a natural map that is a homotopy equivalence from the oriented simplicial chain group to the ordered singular chains.

The present paper was originally motivated by a question raised by Robert Milson about the equivalence of oriented and ordered singular homology, in connection with his research into cohomology rings. I am indebted to Jon Beck, who led me to the discussion of these definitions by Eilenberg and Steenrod in their book, which is turn led me to the papers of Eilenberg and Lefschetz cited above.

As usual, we identify the standard n-simplex Δ_n as

$$\{(t_0, \ldots, t_n) \in \mathbf{R}^{n+1} | t_i \geq 0 \text{ and } t_0 + \cdots + t_n = 1\}$$

For a topological space X, a singular n-chain in X is a continuous map $\sigma\colon \Delta_n \longrightarrow X$. Incidentally, one does not realize until reading the old papers what a great simplification was the use of a standard simplex to define chains, instead of allowing all possible homeomorphs of the n simplex modulo an equivalence relation.

It is clear that the permutation group Σ_{n+1} acts on the set of singular n chains by the formula

$$(p\sigma)(t_0, \ldots, t_n) = \sigma \circ p^{-1}(t_0, \ldots, t_n) = \sigma(t_{p^{-1}0}, \ldots, t_{p^{-1}n})$$

for $p \in \Sigma_{n+1}$ and $\sigma\colon \Delta_n \longrightarrow X$. The singular n-chain functor C_n assigns to each space X the free abelian group generated by the singular n-chains and is a module over Σ_{n+1} as well. One way of defining an oriented chain group, the one that gives Lefschetz' definition, is to factor out of C_n the subgroup U_n generated by all chains of the form $p\sigma - \mathrm{sgn}(p)\sigma$ where σ is an n-simplex and $p \in \Sigma_{n+1}$. A second way is to factor U_n by the subgroup V_n generated by U_n and all singular n-simplexes σ such that $p\sigma = \sigma$ for some transposition $p \in \Sigma$. Of course, if σ is such a simplex, then $2\sigma \in U_n$ so the only difference from the first definition is some 2-torsion. Finally, one might note that there remain other 2-torsion simplexes mod U_n, namely all those σ for which $p\sigma = \sigma$ for some odd permutation σ. We let W_n denote the subgroup generated by these together with U_n. There is no analog to these in the simplicial theory, since one easily sees that in that theory, a simplex that is fixed under any permutation is fixed under some transposition.

One might reasonably call any of C_n/U_n, C_n/V_n or C_n/W_n the oriented singular complex functor. It follows from the theorem below that you can use whichever of the definitions is convenient.

1.1 Theorem. *Each of the maps* $C \longrightarrow C/U \longrightarrow C/V \longrightarrow C/W$ *is a homotopy equivalence.*

This theorem states that this equivalence is in the functor category, which means that both the homotopy inverses and the homotopies themselves are natural transformations.

2 The oriented singular complex

2.1 FDP complexes. Our first job is to show that the three subfunctors U_n, V_n and W_n are the nth components of subcomplexes of C_n. We begin with the observation that any map (not just an order preserving one) from the finite ordinals $f: m+1 \longrightarrow n+1$ induces a map of singular chain functors $C_n \longrightarrow C_m$. Then if σ is such a singular n-simplex, $\sigma \circ f: \Delta_m \longrightarrow X$ is a singular m-simplex.

It is clear that any function $m+1 \longrightarrow n+1$ can be represented as an order preserving function followed by a permutation of $n+1$. Since the order-preserving maps are composites of faces and degeneracies, it follows that the faces, degeneracies and permutations generate the action of all the maps. Accordingly, we define an *FDP complex* to be a functor on the category of finite non-empty sets and an *augmented FDP complex* to be a functor on the category of finite sets. This is equivalent to an (augmented) simplicial set $\{X_n \| n \geq 0\}$ (resp. $n \geq -1$), face operators $d^i = d^i_n: X_n \longrightarrow X_{n-1}$ and $s^i = s^i_n: X_n \longrightarrow X_{n+1}$ for $0 \leq i \leq n$ subject to the usual simplicial identities. In addition, there are involutions $p^i = p^i_n: X_n \longrightarrow X_n$ for $0 \leq i \leq n-1$. These correspond to transposition of adjacent elements and are known to generate the symmetric group ([Sayers, 1934]).

They satisfy the equations $p^i p^{i+1} p^i = p^{i+1} p^i p^{i+1}$ as well as the following commutations with respect to the face and degeneracy operators:

$$d^i p^j = \begin{cases} p^{j-1} d^i & \text{if } i < j \\ d^{i+1} & \text{if } i = j \\ d^{i-1} & \text{if } i = j+1 \\ p^j d^i & \text{if } i > j+1 \end{cases}$$

$$s^i p^j = \begin{cases} p^{j+1} s^i & \text{if } i < j \\ p^{i+1} p^i s^{i+1} & \text{if } i = j \\ p^{i-1} p^i s^{i-1} & \text{if } i = j+1 \\ p^j s^i & \text{if } i > j+1 \end{cases}$$

2.2 Proposition. *The subfunctors* U_n, V_n *and* W_n *of* C_n *are the* n*th components of subcomplexes* U, V *and* W, *resp. of* C.

Proof. Case of U_n. U_n is generated by the images of the $1+p^j$, $j=0,\ldots,n-1$. We have

$$\begin{aligned}
d\circ(1+p^j) &= \sum_{i=0}^{n}(-1)^i d^i\circ(1+p^j)\\
&= \sum_{i=0}^{j-1}(-1)^i d^i\circ(1+p^j) + (-1)^j d^j(1+p^j)\\
&\qquad + (-1)^{j+1}d^{j+1}(1+p^j) + \sum_{i=j+2}^{n}(-1)^i d^i\circ(1+p^j)\\
&= \sum_{i=0}^{j-1}(-1)^i(1+p^{j-1})\circ d^i + (-1)^j(d^j+d^{j+1})\\
&\qquad + (-1)^{j+1}(d^{j+1}+d^j) + \sum_{i=j+2}^{n}(-1)^i(1+p^j)\circ d^i\\
&= (1+p^{j-1})\circ\sum_{i=0}^{j-1}(-1)^i d^i + i(1+p^j)\sum_{i=j+2}^{n}(-1)^i\circ d^i\\
&\subseteq U_{n-1}
\end{aligned}$$

Case of V_n. First I claim that V_n is generated by U_n and adjacent transpositions. Temporarily let V_n be the subgroup of C_n generated by U_n and adjacent transpositions. Suppose that σ is a simplex such that $\sigma=\sigma\circ(\,i\quad j\,)$ for some $i<j$. If $j-i=1$, then $\sigma\in V_n$ by definition. Otherwise, make the inductive hypothesis that $\tau\in V_n$ whenever $\tau=\tau\circ(\,i+1\quad j\,)$. Then

$$\sigma\circ(\,i\quad i+1\,)(\,i+1\quad j\,)=\sigma\circ(\,i\quad j\,)(\,i\quad i+1\,)=\sigma\circ(\,i\quad i+1\,)$$

so that $\sigma\circ(\,i\quad i+1\,)\in V_n$. But $\sigma+\sigma\circ(\,i\quad i+1\,)\in U_n\subseteq V_n$ and hence $\sigma\in V_n$.

Since $d(U_n)\subseteq U_{n_1}$ we need only show that if $\sigma=p^i\sigma$, then $d(\sigma)\in V_{n-1}$. But $d^i\sigma=d^ip^i\sigma=d^{i+1}\sigma$ so that the two terms $(-1)^id^i\sigma$ and $(-1)^{i+1}d^{i+1}\sigma$ cancel. The remaining terms are of the form $d^j\sigma$ for $j<i$ or $j>i+1$. In the first case, we have $d^j\sigma=d^jp^i\sigma=p^{i-1}d^j\sigma$ so that $d^j\sigma$ is fixed by p^{i-1} or by p^i and hence belongs to V_{n-1} and therefore so does $d\sigma$.

Case of W_n. What we will establish in this case is that C_n/W_n is the torsion-free quotient of C_n/U_n. Since the torsion subgroup is invariant under any homomorphism, it follows that any homomorphism that takes U_n to U_{n-1} also takes W_n to W_{n-1}.

Now let S be a set on which Σ_m acts for some m and suppose that C is the free abelian group generated by S and that U and W are subgroups of C defined as follows. U is generated by all $ps - \mathrm{sgn}(p)s$ for $s \in S$ and $p \in \Sigma_m$ and W is the subgroup generated by U as well as all $s \in S$ such that $ps = s$ for some odd permutation $p \in \Sigma_m$. I claim that under those circumstances, W/U is the torsion subgroup of C/U. It is sufficient to show that every element of W is torsion mod U, which is obvious as noted above, and that C/W is torsion free. It is sufficient to do this when S is transitive, since C, U and W all split into direct sums corresponding to orbits. In that case, C/U is generated by the class containing any element of S (which I suppose is non-empty). If $s \in S$ is fixed by an odd permutation, then $2s \in U$ and $C/U \cong \mathbf{Z}_2$, while $W = C$. If not, then the isotropy group of S is included in the alternating group and then $C/U = C/W = \mathbf{Z}$. In each case, W/U is the torsion subgroup of C/U. □

It follows that we have chain complex functors C/U, C/V and C/W.

3 Acyclic models

Our main tool is the theorem on acyclic models which was formulated and proved in slightly different language in [Barr, Beck, 1966].

Let $G\colon \mathscr{X} \longrightarrow \mathscr{X}$ be a functor and $\epsilon\colon G \longrightarrow \mathrm{Id}$ a natural transformation. We say that an augmented chain complex functor K is ϵ-presentable if for each $n \geq 0$, there is a natural transformation $\theta_n\colon K_n \longrightarrow K_n G$ such that $K_n\epsilon \circ \theta_n = \mathrm{id}$. We say that the augmented chain complex functor $L \longrightarrow L_{-1} \longrightarrow 0$ is G-contractible if $LG \longrightarrow L_{-1}G \longrightarrow 0$ has a natural contracting homotopy.

3.1 Theorem. *Suppose $\mathscr{X}$ is a category and $K \longrightarrow K_{-1} \longrightarrow 0$ and $L \longrightarrow L_{-1} \longrightarrow 0$ are augmented chain complex functors from X to an abelian category. Suppose G is an endofunctor on $\mathscr{X}$ and $\epsilon\colon G \longrightarrow \mathrm{Id}$ a natural transformation. If K is ϵ-presentable and $L \longrightarrow L_{-1} \longrightarrow 0$ is G-contractible, then any natural transformation $K_{-1} \longrightarrow L_{-1}$ can be extended to a natural chain map $K \longrightarrow L$ and any two such extensions are naturally homotopic.*

Of course, it follows that if both K and L are ϵ-presentable and both $K \longrightarrow K_{-1} \longrightarrow 0$ and $L \longrightarrow L_{-1} \longrightarrow 0$ G-contractible, then an isomorphism $K_{-1} \longrightarrow L_{-1}$ extends to a homotopy equivalence $K \longrightarrow L$.

We will be applying this theorem with the following functor G which was called C and used for similar purposes in [Kleisli, 1974] where all details may be found.

For a space X and element $x \in X$, let $I \underset{x}{\multimap} X$ denote the space of paths $\pi\colon I \longrightarrow X$ such that $\pi(0) = x$, topologized with the compact/open topology. Define $GX = \sum_{x \in X} I \underset{x}{\multimap} X$. Of course, the point set of GX is just the set of paths in X, but the topology is not that of the path space, since paths starting at distinct points are in different components. We define $\epsilon X\colon GX \longrightarrow X$ as evaluation at 1.

4 Proof of 1.1

In order to understand the next theorem, we will think of Σ_{n+1} as embedded as the subgroup of Σ_{n+2} consisting of those permutations of $\{0,\ldots,n+1\}$ that fix $n+1$.

4.1 Proposition. *There is a natural chain contraction s on $CG \longrightarrow C_{-1} \longrightarrow 0$ such that for an n-simplex σ and $p \in \Sigma_{n+1}$, $s(p\sigma) = ps(\sigma)$.*

Proof. Let X be any space and suppose that $\sigma\colon \Delta_n \longrightarrow GX$ is a singular n-simplex. Then σ is a continuous function $\Delta_n \longrightarrow I \underset{x}{\multimap} X$ for some $x \in X$ since Δ_n is connected and GX is the union of those connected components. We can think of this as a continuous map we still call $\sigma\colon \Delta_n \times I \longrightarrow X$ and write $\sigma(t_0,\ldots,t_n;u)$ for $(t_0,\ldots,t_n) \in \Delta_n$ and $u \in I$ such that $\sigma(t_0,\ldots,t_n;0) = x$ for all $(t_0,\ldots,t_n) \in \Delta_n$. First we consider the case of dimensions -1 and 0. A path component (-1-simplex) of GX is determined by an $x \in X$ and we define $sx(u) = x$, the constant path at x. A singular 0-simplex is just a path in X. However, to be consistent with our previous notation, we write $\sigma(t;u)$ with $t = 1$ the only allowed value for t. Then define $(s\sigma)(t_0,t_1;u) = \sigma(\frac{t_0}{1-t_1};(1-t_1)u)$. Of course, $t_0 + t_1 = 1$, so this is really $\sigma(1;t_0u)$.

$$(d^1 s\sigma)(1;u) = (s\sigma)(1,0;u) = \sigma(1;u)$$

while

$$(d^0 s\sigma)(1;u) = (s\sigma)(0,1;u) = \sigma(1;0)$$

while $sd^0\sigma(1;u)$ is the constant path at $d^0\sigma$. Now $d^0\colon C_0(GX) \longrightarrow H_0(GX)$ assigns to each path its path component and the path components of GX are in one-one correspondence with the elements of X by the map that takes π to $\pi(0)$. Thus $sd^0\sigma$ is the constant path at $\sigma(1;0)$ and so we have $d^1 s\sigma = \sigma$ and $d^0 s\sigma = sd^0\sigma$. Next we turn to dimension $n > 0$. Define

$$(s\sigma)(t_0,\ldots,t_{n+1};u) = \begin{cases} \sigma(\frac{t_0}{1-t_{n+1}},\ldots,\frac{t_n}{1-t_{n+1}};(1-t_{n+1})u) & \text{if } t_{n+1} \neq 1 \\ x & \text{if } t_{n+1} = 1 \end{cases}$$

We see that $s\sigma$ is an $n+1$-simplex in GX and that

$$\begin{aligned} d^{n+1}s\sigma(t_0,\ldots,t_n;u) &= (s\sigma)(t_0,\ldots,t_n,0;u) \\ &= \sigma(t_0,\ldots,t_n;u) \end{aligned}$$

while for $i \le n$,

$$\begin{aligned} d^i(s\sigma)(t_0,\ldots,t_n;u) &= s\sigma(t_0,\ldots,t_{i-1},0,t_i,\ldots,t_n;u) \\ &= \sigma\left(\frac{t_0}{1-t_{n+1}},\ldots,\frac{t_{i-1}}{1-t_{n+1}},0,\frac{t_i}{1-t_{n+1}},\ldots,\frac{t_{n-1}}{1-t_{n+1}};u\right) \end{aligned}$$

and

$$s(d^i\sigma)(t_0,\dots,t_n;u) = d^i\sigma\left(\frac{t_0}{1-t_{n+1}},\dots,\frac{t_{n-1}}{1-t_{n+1}};u\right)$$
$$= \sigma\left(\frac{t_0}{1-t_{n+1}},\dots,\frac{t_{i-1}}{1-t_{n+1}},0,\frac{t_i}{1-t_{n+1}},\dots,\frac{t_{n-1}}{1-t_{n+1}};u\right)$$

Thus $d^{n+1}s = \mathrm{id}$ and for $i \le n$, $d^i s = sd^i$ and we have a contraction at the simplicial level, and it is standard that the sequence of $(-1)^n s_n$ is a contraction in the associated chain complex. It is evident that for any permutation $p \in \Sigma_{n+1}$, $s(\sigma\circ p) = (s\sigma)\circ p$. Since $p\sigma = \sigma\circ p^{-1}$, it follows that $s(p\sigma) = ps(\sigma)$. □

4.2 Proposition. *For all $n \ge 0$, there is a natural transformation $\theta_n\colon C_n \longrightarrow C_nG$ such that $C_n\epsilon\circ\theta_n = \mathrm{id}$ and such that for each $p\in\Sigma_{n+1}$, $\theta_n(p\sigma) = p\theta_n(\sigma)$.*

Proof. Let X be a topological space. Define $\theta_n X\colon C_nX \longrightarrow C_nGX$ by

$$\theta_n(\sigma)(t_0,\dots,t_n)(u) = \sigma\left(ut_0+\frac{1-u}{n+1},\dots,ut_n+\frac{1-u}{n+1}\right)$$

which is a simplex in the component of GX based at $\sigma(\frac{1-u}{n+1},\dots,\frac{1-u}{n+1})$. It is clear that $\theta_n(\sigma)(t_0,\dots,t_n)(1) = \sigma(t_0,\dots,t_n)$ and that $\theta(\sigma\circ p) = \theta(\sigma)\circ p$, from which the second claim follows. □

Now we can apply acyclic models to C as well as C/U, C/V and C/W. We have shown that C is ϵ-presentable and that $C \longrightarrow C_{-1} \longrightarrow 0$ is G-contractible and both natural transformations commute with the action of the symmetric groups. This implies that $s(U_n)\subseteq U_{n+1}$, $s(V_n)\subseteq V_{n+1}$ and $s(W_n)\subseteq W_{n+1}$ and similarly that $\theta_n(U_n)\subseteq U_nG$, $\theta_n(V_n)\subseteq V_nG$ and $\theta_n(W_n)\subseteq W_nG$ and so C/U, C/V and C/W are also ϵ-presentable and the complexes augmented over $C_{-1}\longrightarrow 0$ are G-contractible. Since all four complexes have the same augmentation term, it follows that all the maps and composites in $C\longrightarrow C/U\longrightarrow C/V\longrightarrow C/W$ are homotopy equivalences. □

5 An explicit computation

The acyclic models theorem gives an explicit construction and you can work out what the homotopy inverse to the projection $C\longrightarrow C/W$ is in low dimensions. Basically, what is needed is a map $f_n\colon C_n\longrightarrow C_n$ such that for each transposition $p\in\Sigma_{n+1}$ and singular n-simplex σ, $f_n(\sigma\circ p) = -f_n(\sigma)$ and if p is an arbitrary odd permutation such that $\sigma\circ p = \sigma$, then $f_n(\sigma) = 0$. If you follow the construction given in the proof, here is what happens in dimension 1. Suppose $[a_0,a_1]$ is a 1-simplex. This means there is a simplex $\sigma\colon\Delta_1\longrightarrow X$ such that $\sigma(0) = a_0$ and $\sigma(1) = a_1$. Let $a_{01} = \sigma(1/2)$ be the barycenter. Then $f_1([a_0,a_1]) = [a_0,a_{01}] - [a_1,a_{01}]$. Both simplexes on the left use σ to interpolate between the given endpoints. Compare this with the simplicial subdivision map given in [Rotman, 1988], p. 113 or [Dold, 1980], p. 40, where $\mathrm{Sd}([a_0, a_1]) = [a_{01}, a_1] - [a_{01}, a_0]$. In degree 2, following analogous conventions, we get the following map:

$$\begin{aligned} f_2([a_0, a_1, a_2]) = {} & [a_0, a_{01}, a_{012}] - [a_1, a_{01}, a_{012}] + [a_1, a_{12}, a_{012}] \\ & - [a_2, a_{12}, a_{012}] + [a_2, a_{02}, a_{012}] + [a_0, a_{02}, a_{012}] \end{aligned}$$

It seems clear what the general pattern is, but I have not actually verified this in any higher dimension.

6 Appendix: some historical notes

I had never understood the reason for the names "oriented" and "ordered" in connection with the chain groups nor could recall which was which. After looking at the references [Eilenberg] and [Lefschetz], it became clear. Originally, all chain groups were oriented, that is based on oriented simplexes. An oriented chain is simply a chain, simplicial or singular, based on an oriented simplex, that is a simplex with a chosen orientation. If you changed the orientation, you changed the sign of the chain. This was necessary to make $d^2 = 0$ and doubtless went back to the earliest days of algebraic topology. They didn't use standard simplexes, but any oriented simplex and identified two simplexes if they differed by an orientation preserving "barycentric map", meaning one that was the linear extension of a map that takes vertices to vertices. An orientation reversing barycentric map just changed the sign of the chain. What Eilenberg did was to replace the oriented simplex by an ordered simplex, that is one with a linear ordering of its vertices. Now in identifying two chains, only order preserving maps were permitted. This gave a larger chain group (fewer identifications could be made) since now, for example, the 1-simplex $[v_0, v_1]$ is not related to the simplex $[v_1, v_0]$ since the barycentric map that mediates between them does not preserve the order on the vertices. In this way, the group of chains based on ordered simplexes became the ordered chain group.

I had believed that singular homology was defined in order to prove the topological invariance of homology. This is apparently incorrect. Lefschetz gives a direct proof of the invariance and develops singular homology only later. It seems to have been Eilenberg's idea, and only after he had defined the ordered singular chain group, to use singular homology to prove invariance.

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