COHOMOLOGY AND OBSTRUCTIONS: COMMUTATIVE ALGEBRAS

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Introduction

Associated with each of the classical cohomology theories in algebra has been a theory relating H^2 (H^3 as classically numbered) to obstructions to non-singular extensions and H^1 with coefficients in a "center" to the non-singular extension theory (see [Eilenberg & Mac Lane (1947), Hochschild (1947), Hochschild (1954), Mac Lane (1958), Shukla (1961), Harrison (1962)]). In this paper we carry out the entire process using triple cohomology. Because of the special constructions which arise, we do not know how to do this in any generality. Here we restrict attention to the category of commutative (associative) algebras. It will be clear how to make the theory work for groups, associative algebras and Lie algebras. My student, Grace Orzech, is studying more general situations at present. I would like to thank her for her careful reading of the first draft of this paper.

The triple cohomology is described at length elsewhere in this volume [Barr & Beck (1969)]. We use the adjoint pair

$$\begin{array}{ccc} \mathbf{CommAlg} & \xrightarrow{\;G\;} & \mathbf{CommAlg} \\ & {\scriptstyle U}\searrow \quad \nearrow{\scriptstyle F} & \\ & \mathbf{Sets} & \end{array}$$

for our cotriple $\mathbf{G} = (G, \varepsilon, \delta)$. We let

$$\begin{aligned} \varepsilon^i &= G^i \varepsilon G^{n-i} \colon G^{n+1} \longrightarrow G^n, \\ \delta^i &= G^i \delta G^{n-i} \colon G^{n+1} \longrightarrow G^{n+2} \qquad \text{and} \\ \varepsilon &= \Sigma(-1)^i \varepsilon^i \colon G^{n+1} \longrightarrow G^n. \end{aligned}$$

It is shown in [Barr & Beck (1969)] that the associated chain complex

$$\cdots \xrightarrow{\varepsilon} G^{n+1}R \xrightarrow{\varepsilon} \cdots \xrightarrow{\varepsilon} G^2R \xrightarrow{\varepsilon} GR \xrightarrow{\varepsilon} R \longrightarrow 0$$

is exact. This fact will be needed below.

This work was partially supported by NSF grant GP-5478

More generally we will have occasion to consider simplicial objects (or at least the first few terms thereof)

$$X\colon \cdots \; \vdots \; X_n \; \vdots \; \cdots \; X_2 \; X_1 \; X_0$$

with face maps $d^i\colon X_n \longrightarrow X_{n-1}$, $0 \le i \le n$, and degeneracies $s^i\colon X_{n-1} \longrightarrow X_n$, $0 \le i \le n-1$ subject to the usual identities (see [Huber (1961)]). The simplicial normalization theorem[1], which we will have occasion to use many times, states that the three complexes K_*X, T_*X and N_*X defined by

$$K_nX = \bigcap_{i=1}^{n} \ker(d^i\colon X_n \longrightarrow X_{n-1})$$

with boundary d induced by d^0,

$$T_nX = X_n,$$

with boundary $d = \sum_{i=0}^{n}(-1)^i d^i$, and

$$N_nX = X_n \Big/ \sum_{i=0}^{n} \operatorname{im}(s^i\colon X_{n-1} \longrightarrow X_n)$$

with boundary d induced by $\sum_{i=0}^{n}(-1)^i d^i$ are all homotopic and in fact the natural inclusions $K_*X \subseteq T_*X$ and projections $T_*X \longrightarrow N_*X$ have homotopy inverses. In our context the X_n will be algebras and the d^i will be algebra homomorphisms, but of course d is merely an additive map.

We deliberately refrain from saying whether or not the algebras are required to have a unit. The algebras Z, A, $Z(T, A)$, ZA are proper ideals (notation $A < T$) of other algebras and the theory becomes vacuous if they are required to be unitary. On the other hand the algebras labeled B, E, M, P, R, T can be required or not required to have a unit, as the reader desires. There is no effect on the cohomology (although G changes slightly, being in one case the polynomial algebra cotriple and in the other case the subalgebra of polynomials with 0 constant term). The reader may choose for himself between having a unit or having all the algebras considered in the same category. Adjunction of an identity is an exact functor which takes the one projective class on to the other (see [Barr & Beck (1969), Theorem 5.2], for the significance of that remark). (Also, see [Barr (1968a), Section 3])

Underlying everything is a commutative ring which everything is assumed to be an algebra over. It plays no role once it has been used to define G. By specializing it to the ring of integers we recover a theory for commutative rings.

[1] (see [Dold & Puppe (1961)])

1. The class $\mathbf{E}$

Let A be a commutative algebra. If $A < T$, let $Z(A,T) = \{ t \in T \mid tA = 0 \}$. Then $Z(A,T)$ is an ideal of T. In particular $ZA = Z(A,A)$ is an ideal of A. Note that Z is not functorial in A (although $Z(A,\text{-})$ is functorial on the category of algebras under A). It is clear that $ZA = A \cap Z(A,T)$. Let $\mathbf{E} = \mathbf{E}A$ denote the equivalence classes of exact sequences of algebras

$$0 \longrightarrow ZA \longrightarrow A \longrightarrow T/Z(A,T) \longrightarrow T/(A + Z(A,T)) \longrightarrow 0$$

for $A < T$. Equivalence is by isomorphisms which fix ZA and A. (*A priori* it is not a set; this possibility will disappear below.)

Let $\mathbf{E}'$ denote the set of $\lambda\colon A \longrightarrow E$ where E is a subalgebra of $\mathrm{Hom}_A(A,A)$ which contains all multiplications $\lambda a\colon A \longrightarrow A$, given by $(\lambda a)(a') = aa'$.

PROPOSITION 1.1. *There is a natural* 1-1 *correspondence* $\mathbf{E} \cong \mathbf{E}'$.

PROOF. Given $A < T$, let E be the algebra of multiplications on A by elements of T. There is a natural map $T \longrightarrow E$ and its kernel is evidently $Z(A,T)$. If $T > A < T'$, then T and T' induce the same endomorphism of A if and only if $T/Z(A,T) = T'/Z(A,T')$ by an isomorphism which fixes A and induces $T/(A + Z(A,T)) \cong T'/(A + Z(A,T'))$.

To go the other way, given $\lambda\colon A \longrightarrow E \in \mathbf{E}'$, let P be the algebra whose module structure is $E \times A$ and multiplication is given by $(e,a)(e',a') = (ee', ea' + e'a + aa')$. ($ea$ is defined as the value of the endomorphism e.) $A \longrightarrow P$ is the coordinate mapping and embeds A as an ideal of P with $Z(A,P) = \{ (-\lambda a, a) \mid a \in A \}$. The associated sequence is easily seen to be

$$0 \longrightarrow ZA \longrightarrow A \xrightarrow{\lambda} E \xrightarrow{\pi} M \longrightarrow 0$$

where π is cokerλ. ∎

From now on we will identify $\mathbf{E}$ with $\mathbf{E}'$ and call it $\mathbf{E}$.

Notice that we have constructed a natural representative $P = PE$ in each class of $\mathbf{E}$. It comes equipped with maps $d^0, d^1\colon P \longrightarrow E$ where $d^0(e,a) = e + \lambda a$ and $d^1(e,a) = e$. Note that $A = \ker d^1$ and $Z(A,P) = \ker d^0$. In particular $\ker d^0 \cdot \ker d^1 = 0$.

$P = P(T/Z(A,T))$ can be described directly as follows. Let $K \rightrightarrows T$ be the kernel pair of $T \longrightarrow T/A$. This means that

$$\begin{array}{ccc} K & \longrightarrow & T \\ \downarrow & & \downarrow \\ T & \longrightarrow & T/A \end{array}$$

is a pullback. Equivalently $K = \{ (t,t') \in T \times T \mid t + A = t' + A \}$, the two maps being the restrictions of the coordinate projections. It is easily seen that $\Delta_Z = \{ (z,z) \mid z \in Z(A,T) \} < K$ and that $K/\Delta_Z \cong P$.

Let $d^0, d^1, d^2 \colon B \longrightarrow P$ be the kernel triple of $d^0, d^1 \colon P \longrightarrow A$. This means that $d^0d^0 = d^0d^1$, $d^1d^1 = d^1d^2$, $d^0d^2 = d^1d^0$, and B is universal with respect to these identities. Explicitly B is the set of all triples $(p, p', p'') \in P \times P \times P$ with $d^0p = d^0p'$, $d^1p' = d^1p''$, $d^0p'' = d^1p$, the maps being the coordinate projections.

Proposition 1.2. *The "truncated simplicial algebra",*

$$0 \longrightarrow B \rightrightarrows\!\!\!\rightarrow P \rightrightarrows E \longrightarrow M \longrightarrow 0$$

is exact in the sense that the associated (normalized) chain complex

$$0 \longrightarrow \ker d^1 \cap \ker d^2 \longrightarrow \ker d^1 \longrightarrow E \longrightarrow M \longrightarrow 0$$

is exact. (The maps are those induced by restricting d^0 as in K_*.)

The proof is an elementary computation and is omitted. ■

Note that we are thinking of this as a simplicial algebra even though the degeneracies have not been described. They easily can be, but we have need only for $s^0 \colon E \longrightarrow P$, which is the coordinate injection, $s^0e = (e, 0)$. Recall that $d \colon B \longrightarrow P$ is the additive map $d^0 - d^1 + d^2$. The simplicial identities imply that $d^0d = d^0(d^0 - d^1 + d^2) = d^0d^2 = d^1d^0 = d^1(d^0 - d^1 + d^2) = d^1d$.

Finally note that ZA is a module over M, since it is an E-module on which the image of λ acts trivially. This implies that it is a module over B, P and E and that each face operator preserves the structure.

Proposition 1.3. *There is a derivation $\partial \colon B \longrightarrow ZA$ given by the formula*

$$\partial x = (1 - s^0d^0)dx = (1 - s^0d^1)dx$$

Proof. First we see that $\partial x \in ZA = \ker d^0 \cap \ker d^1$, since $d^i\partial x = d^i(1 - s^0d^i)dx(d^i - d^i)dx = 0$ for $i = 0, 1$. To show that it is a derivation, first recall that $\ker d^0 \cdot \ker d^1 = Z(A, P) \cdot A = 0$. Then for $b_1, b_2 \in B$,

$$\begin{aligned}\partial b_1 \cdot b_2 + b_1 \cdot \partial b_2 &= (1 - s^0d^0)db_1 \cdot d^0b_2 + d^1b_1 \cdot (1 - s^0d^0)db_2 \\ &= d^0b_1 \cdot d^0b_2 - d^1b_1 \cdot d^0b_2 + d^2b_1 \cdot d^0b_2 - s^0d^0d^2b_1 \cdot d^0b_2 \\ &\quad + d^1b_1 \cdot d^0b_2 - d^1b_1 \cdot d^1b_2 + d^1b_1 \cdot d^2b_2 - d^1b_1 \cdot s^0d^0d^2b_2\end{aligned}$$

To this we add $(d^2b_1 - d^1b_1)(d^2b_2 - s^0d^0d^2b_2)$ and $(s^0d^0d^2b_1 - d^2b_1)(d^0b_2 - s^0d^0d^2b_2)$, each easily seen to be in $\ker d^0 \cdot \ker d^1 = 0$, and get

$$\begin{aligned}&d^0b_1 \cdot d^0b_2 - d^1b_1 \cdot d^1b_2 + d^2b_1 \cdot d^2b_2 - s^0d^0d^2b_1 \cdot s^0d^0d^2b_2 \\ &= d^0(b_1b_2) - d^1(b_1b_2) + d^2(b_1b_2) - s^0d^0d^2(b_1b_2) \\ &= (1 - s^0d^0)d(b_1b_2) = \partial(b_1b_2)\end{aligned}$$

■

2. The obstruction to a morphism

We consider an algebra R and are interested in extensions

$$0 \longrightarrow A \longrightarrow T \longrightarrow R \longrightarrow 0$$

In the singular case, $A^2 = 0$, such an extension leads to an R-module structure on A. This comes about from a surjection $T \longrightarrow E$ where $E \in \mathbf{E}$, and, since A is annihilated, we get a surjection $R \longrightarrow E$ by which R operates on A. In general we can only map $R \longrightarrow M$. Obstruction theory is concerned with the following question. Given a surjection $p: R \longrightarrow M$, classify all extensions which induce the given map. The first problem is to discover whether or not there are any. (Note: in a general category, "surjection" should probably be used to describe a map which has a kernel pair and is the coequalizer of them.) Since GR is projective in the category, we can find $p_0: GR \longrightarrow E$ with $\pi p_0 = p\varepsilon$. If $\widetilde{d}^0, \widetilde{d}^1: \widetilde{P} \longrightarrow E$ is the kernel pair of π, then there is an induced map $u: P \longrightarrow \widetilde{P}$ such that $d^i u = \widetilde{d}^i$, $i = 0, 1$ which is easily seen to be onto. The universal property of $\widetilde{P}$ guarantees the existence of a map $\widetilde{p}_1: G^2R \longrightarrow \widetilde{P}$ with $\widetilde{d}^i \widetilde{p}_1 = p_0 \varepsilon^i$, $i = 0, 1$. Projectivity of G^2R and the fact that u is onto guarantee the existence of $p_1: G^2R \longrightarrow P$ with $up_1 = \widetilde{p}_1$, and then $d^i p_1 = d^i u \widetilde{p}_1 = \widetilde{d}^i \widetilde{p}_1 = p_0 \varepsilon^i$, $i = 0, 1$. Finally, the universal property of B implies the existence of $p_2: G^3R \longrightarrow B$ with $d^i p_2 = p_1 \varepsilon^i$, $i = 0, 1, 2$. Then $\partial p_2: G^3R \longrightarrow ZA$ is a derivation and $\partial p_2 \varepsilon = (1 - s^0 d^0) d p_2 \varepsilon = (1 - s^0 d^0) p_1 \varepsilon\varepsilon = 0$. Thus ∂p_2 is a cocycle in $\operatorname{Der}(G^3R, ZA)$.

PROPOSITION 2.1. *The homology class of ∂p_2 in* $\operatorname{Der}(G^3R, ZA)$ *does not depend on the choices of p_0, p_1 and p_2 made. (p_2 actually is not an arbitrary choice.)*

PROOF. $\partial p_2 = (1 - s^0 d^0) p_1 \varepsilon$ and so doesn't depend on p_2 at all. Now let σ_0, σ_1 be new choices of p_0, p_1. Since $\pi p_0 = \varepsilon p = \pi p_1$, there is an $\widetilde{h}^0: GR \longrightarrow \widetilde{P}$ with $\widetilde{d}^0 \widetilde{h}^0 = p_0$, $\widetilde{d}^1 \widetilde{h}^0 = \sigma_0$. Again, since u is onto, there exists $h^0: GR \longrightarrow P$ with $uh^0 = \widetilde{h}^0$, and then $d^0 h^0 = p_0$, $d^1 h^0 = \sigma_0$. Also $\pi d^0 p_1 = \pi p_0 \varepsilon^0 = \pi \sigma_0 \varepsilon^0 = \pi d^0 \sigma_1 = \pi d^1 \sigma_1$ and by a similar argument we can find $v: G^2R \longrightarrow P$ with $d^0 v = d^0 p_1$ and $d^1 v = d^1 \sigma_1$. Now consider the three maps $p_1, v, h^0 \varepsilon^1: G^2R \longrightarrow P$. $d^0 p_1 = d^0 v$, $d^1 v = d^1 \sigma_1 = \sigma_0 \varepsilon^1 = d^1 h^0 \varepsilon^1$ and $d^0 h^0 \varepsilon^1 = p_0 \varepsilon^1 = d^1 p_1$, so by the universal mapping property of B, there is $h^0: G^2R \longrightarrow B$ with $d^0 h^0 = p_1$, $d^1 h^0 = v$, $d^2 h^0 = h^0 \varepsilon^1$. By a similar consideration of $h^0 \varepsilon^0, v, \sigma_1: G^2R \longrightarrow P$ we deduce the existence of $h^1: G^2R \longrightarrow B$ such that $d^0 h^1 = h^0 d^0$, $d^1 h^1 = v$, $d^2 h^1 = \sigma_1$. The reader will recognize the construction of a simplicial homotopy between the p_i and the σ_i. We have

$$\begin{aligned}(\partial h^0 - \partial h^1)\varepsilon &= (1 - s^0 d^0) d(h^0 - h^1)\varepsilon \\ &= (1 - s^0 d^0)(d^0 h^0 - d^1 h^0 + d^2 h^0 - d^0 h^1 + d^1 h^1 - d^2 h^1)\varepsilon \\ &= (1 - s^0 d^0)(d^0 h^0 - d^2 h^1 + h^0 \varepsilon^1 - h^0 \varepsilon^0)\varepsilon \\ &= (1 - s^0 d^0)(p_1 - \sigma_1 + h^0 \varepsilon)\varepsilon = (1 - s^0 d^0)(p_1 - \sigma_1)\varepsilon \\ &= (1 - s^0 d^0) d(p_2 - \sigma_2) = \partial p_2 - \partial \sigma_2\end{aligned}$$

This shows that ∂p_2 and $\partial\sigma_2$ are in the same cohomology class in $\mathrm{Der}(G^3R, ZA)$, which class we denote by $[p]$ and which is called the *obstruction* of p. We say that p is *unobstructed* provided $[p] = 0$. ∎

THEOREM 2.2. *A surjection $p\colon R \longrightarrow M$ arises from an extension if and only if p is unobstructed.*

PROOF. Suppose p arises from

$$0 \longrightarrow A \longrightarrow T \longrightarrow R \longrightarrow 0$$

Then we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & K & \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} & T & \longrightarrow & R & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \nu_0} & & \downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & B & \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} & P & \overset{\longrightarrow}{\underset{\longrightarrow}{\longleftarrow}} & E & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where $e^0, e^1\colon K \rightrightarrows T$ is the kernel pair of $T \longrightarrow R$ and $t^0\colon T \longrightarrow K$ is the diagonal map. Commutativity of the leftmost square means that each of three distinct squares commutes, i.e. with the upper, middle or lower arrows. Recalling that $E = T/Z(A,T)$ and $P = K/\Delta_Z$ we see that the vertical arrows are onto. Then there is a $\sigma_0\colon GR \longrightarrow T$ with $\nu_0\sigma_0 = p_0$. Since K is the kernel pair, we have $\sigma_1\colon G^2R \longrightarrow K$ with $e^i\sigma_1 = \sigma_0\varepsilon^i$, $i = 0, 1$. Then $\nu_1\sigma_1$ is a possible choice for p_1 and we will assume $p_1 = \nu_1\sigma_1$. Then $\partial p_2 = (1 - s^0d^0)p_1\varepsilon = (1 - s^0d^0)\nu_1\sigma_1\varepsilon = \nu_1(1 - t^0e^0)\sigma_1\varepsilon$. But $e^0(1 - t^0e^0)\sigma_1\varepsilon = 0$ and $e^1(1 - t^0e^0)\sigma_1\varepsilon = (e^1 - e^0)\sigma_1\varepsilon = \sigma_0(\varepsilon^1 - \varepsilon^0)\varepsilon = \sigma_0\varepsilon\varepsilon = 0$, and since e^0, e^1 are jointly monic, i.e. define a monic $K \longrightarrow T \times T$, this implies that $\nu_1(1 - t^0e^0)\sigma_1\varepsilon = 0$.

Conversely, suppose p, p_0, p_1, p_2 are given and there is a derivation $\tau\colon G^2R \longrightarrow ZA$ such that $\partial p_2 = \tau\varepsilon$. Let $\widetilde{p}_1\colon G^2R \longrightarrow P$ be $p_1 - \tau$ where we abuse language and think of τ as taking values in $P \supseteq ZA$. Then $\widetilde{p}_1$ can be easily shown to be an algebra homomorphism above p_0. Choosing $\widetilde{p}_2$ above $\widetilde{p}_1$ we have new choices p, p_0, $\widetilde{p}_1$, $\widetilde{p}_2$ and

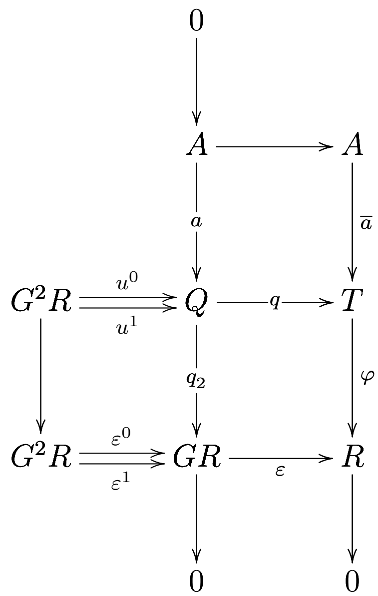
$$\begin{aligned}
\partial\widetilde{p}_2 &= (1 - s^0d^0)d\widetilde{p}_2\varepsilon = (1 - s^0d^0)\widetilde{p}_1\varepsilon = (1 - s^0d^0)(p_1 - \tau)\varepsilon \\
&= (1 - s^0d^0)p_1\varepsilon - (1 - s^0d^0)\tau\varepsilon = \partial p_2 - \tau\varepsilon = 0,
\end{aligned}$$

since $(1 - s^0d^0)$ is the identity when restricted to $ZA = \ker d^0 \cap \ker d^1$. Thus we can assume that p_0, p_1, p_2 has been chosen so that $\partial p_2 = 0$ already.

Let

$$\begin{array}{ccc}
Q & \xrightarrow{q_1} & P \\
{\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle d^1} \\
GR & \xrightarrow[p_0]{} & E
\end{array}$$

be a pullback. Since the pullback is computed in the underlying module category, d^1 is onto so q_2 is onto. Also the induced map $\ker q_2 \longrightarrow \ker d^1 = A$ is an isomorphism (this is true in an arbitrary pointed category) and we will identify $\ker q_2$ with a map $a\colon A \longrightarrow Q$ such that $q_1 a = \ker d^1$. Now let $u^0, u^1\colon G^2R \longrightarrow Q$ be defined by the conditions $q_1 u^0 = s^0 d^0 p_1$, $q_2 u^0 = \varepsilon^0$, $q_1 u^1 = p_1$, $q_2 u^1 = \varepsilon^1$. In the commutative diagram



the rows are coequalizers and the columns are exact. The exactness of the right column follows from the commutativity of colimits. We claim that the map $\bar{a}$ is 1-1.

This requires showing that $\operatorname{im} a \cap \ker q = 0$. $\ker q$ is the ideal generated by the image of $u = u^0 - u^1$. Also $\operatorname{im} a = \ker q_2$. Consequently the result will follow from

Proposition 2.3. *The image of u is an ideal and* $\operatorname{im} u \cap \ker q_2 = 0$.

Proof. If $x \in G^2R$, $y \in Q$, let $x' = \delta q_2 y$. We claim that $u(xx') = ux \cdot y$. To prove this it suffices to show that $q_i u(xx') = q_i(ux \cdot y)$ for $i = 1, 2$ (because of the definition of pullback). But

$$\begin{aligned} q_2 u(xx') &= \varepsilon(xx') = \varepsilon^0 x \cdot \varepsilon^0 x' - \varepsilon^1 x \cdot \varepsilon^1 x' \\ &= \varepsilon^0 x \cdot q_2 y - \varepsilon^1 x \cdot q_2 y = q_2(u^0 x \cdot y) - q_2(u^1 x \cdot y) \\ &= q_2(ux \cdot y) \end{aligned}$$

Next observe that our assumption is that $(1 - s^0 d^1) p_1$ is zero on $\operatorname{im} \varepsilon = \ker \varepsilon$. In particular, $(s^0 d^1 - 1) p_1 \delta = 0$. ($\varepsilon\delta = \varepsilon^0\delta - \varepsilon^1\delta = 0$.) Also $(s^0 d^0 - 1) p_1 x \cdot (s^0 d^1 - 1) q_1 y \in \ker d^0 \cdot \ker d^1 = 0$.

Then we have,

$$\begin{aligned}
q_1u(xx') &= (q_1u^0 - q_1u^1)(xx') = (s^0d^0p_1 - p_1)(xx')\\
&= s^0d^0p_1x \cdot s^0d^0x' - p_1x \cdot p_1x'\\
&= (s^0d^0p_1x - p_1x)s^0d^0p_1x' + p_1x \cdot (s^0d^0p_1x' - p_1x')\\
&= (s^0d^0 - 1)p_1x \cdot s^0d^0p_1\delta q_2y + p_1x \cdot (s^0d^0 - 1)p_1\delta q_2y\\
&= (s^0d^0 - 1)p_1x \cdot s^0p_0\varepsilon^0\delta q_2y\\
&= (s^0d^0 - 1)p_1x \cdot s^0p_0q_2y = (s^0d^0 - 1)p_1x \cdot s^0d^1q_1y\\
&= (s^0d^0 - 1)p_1x \cdot q_1y + (s^0d^0 - 1)p_1x \cdot (s^0d^1 - 1)q_1y\\
&= (s^0d^0p_1x - p_1x)q_1y = q_1ux \cdot q_1y = q_1(ux \cdot y)
\end{aligned}$$

Now if $ux \in \ker q_2$, then $0 = q_2ux = \varepsilon x$, $x \in \ker\varepsilon = \operatorname{im}\varepsilon$, and $0 = (s^0d^0 - 1)p_1x = q_1ux$. But then $ux = 0$. ∎

Now to complete the proof of 2.2 we show

PROPOSITION 2.4. *There is a* $\tau: T \longrightarrow E$ *which is onto, whose kernel is* $Z(A,T)$ *and such that* $p\varphi = \pi\tau$.

PROOF. Let τ be defined as the unique map for which $\tau q = d^0q_1$. This defines a map, for $d^0q_1u^0 = d^0s^0d^0p_1 = d^0p_1 = d^0q_1u'$. τ is seen to be onto by applying the 5-lemma to the diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & T & \longrightarrow & R & \longrightarrow & 0\\
 & & \Big\downarrow{=} & & \Big\downarrow{\tau} & & \Big\downarrow{p} & & \\
 & & A & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

since p is assumed onto. $\pi\tau q = \pi d^0q_1 = \pi d^1q_1 = \pi p_0q_2 = p\varepsilon q_2 = p\varphi q$ and q is onto, so $\pi\tau = p\varphi$. Now if we represent elements of Q as pairs $(x,\rho) \in GX \times P$ subject to $p_0x = d^1\rho$, $\tau(x,\rho) = d^0\rho$. Then $\ker\tau = \{\,(x,\rho) \mid d^0\rho = 0\,\}$. That is,

$$\begin{array}{ccc}
\ker\tau & \longrightarrow & \ker d^0\\
\Big\downarrow & & \Big\downarrow{d^1|\ker d^0}\\
GX & \xrightarrow[p_0]{} & E
\end{array}$$

is a pullback. A is represented as $\{\,(0,\rho') \mid d^1\rho' = 0\,\}$. Now

$$\begin{aligned}
Z(A,T) &= \{\,(x,\rho) \in Q \mid d^1\rho' = 0 \implies \rho\rho' = 0\,\}\\
&= \{\,(x,\rho) \in Q \mid \rho \in Z(A,P)\,\}
\end{aligned}$$

It was observed in Section 1 that $Z(A,P) = \ker d^1$. Thus $Z(A,T) = \{\,(x,\rho) \in Q \mid \rho \in \ker d^1\,\} = \ker \tau$. ■

3. The action of H^1

This section is devoted to proving the following.

THEOREM 3.1. *Let $p\colon R \longrightarrow M$ be unobstructed. Let $\mathbf{\Sigma} = \mathbf{\Sigma} p$ denote the equivalence classes of of extensions*

$$0 \longrightarrow A \longrightarrow T \longrightarrow R \longrightarrow 0$$

which induce p. Then the group $H^1(R, ZA)$ acts on $\mathbf{\Sigma} p$ as a principal homogeneous representation. (This means that for any $\Sigma \in \mathbf{\Sigma}$, multiplication by Σ is a 1-1 correspondence $H^1(R, ZA) \cong \mathbf{\Sigma}$.)

PROOF. Let $\mathbf{\Lambda}$ denote the equivalence classes of singular extensions

$$0 \longrightarrow ZA \longrightarrow U \longrightarrow R \longrightarrow 0$$

which induce the same module structure on ZA as that given by p (recalling that ZA is always an M-module). Then $\mathbf{\Lambda} \cong H^1(R, ZA)$ where the addition in $\mathbf{\Lambda}$ is by Baer sum and is denoted by $\Lambda_1 + \Lambda_2$, $\Lambda_1, \Lambda_2 \in \mathbf{\Lambda}$. We will describe operations $\mathbf{\Lambda} \times \mathbf{\Sigma} \longrightarrow \mathbf{\Sigma}$, denoted by $(\Lambda, \Sigma) \mapsto \Lambda + \Sigma$, and $\mathbf{\Sigma} \times \mathbf{\Sigma} \longrightarrow \mathbf{\Lambda}$, denoted by $(\Sigma, \Sigma') \mapsto \Sigma - \Sigma'$, such that

a) $(\Lambda_1 + \Lambda_2) + \Sigma = \Lambda_1 + (\Lambda_2 + \Sigma)$

b) $(\Sigma_1 - \Sigma_2) + \Sigma_2 = \Sigma_1$

c) $(\Lambda + \Sigma) - \Sigma = \Lambda$

for $\Lambda, \Lambda_1, \Lambda_2 \in \mathbf{\Lambda}$, $\Sigma, \Sigma_1, \Sigma_2 \in \mathbf{\Sigma}$ (Proposition 3.2). This will clearly prove Theorem 3.1.

We describe $\Lambda + \Sigma$ as follows. Let

$$0 \longrightarrow ZA \longrightarrow U \xrightarrow{\psi} R \longrightarrow 0 \in \Lambda$$

$$0 \longrightarrow A \longrightarrow T \xrightarrow{\varphi} R \longrightarrow 0 \in \Sigma$$

(Here we mean representatives of equivalence classes.) To simplify notation we assume $ZA < U$ and $A < T$. Let

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & R \end{array}$$

be a pullback. This means $V = \{\,(t,u) \in T \times U \mid \varphi t = \psi u\,\}$. Then $I = \{\,(z, -z) \mid z \in ZA\,\} < V$. Let $T' = V/I$. Map $A \longrightarrow T'$ by $a \mapsto (a,0) + I$. Map $T' \longrightarrow R$ by $(t,u) + I \mapsto \varphi t =$

ψu. This is clearly well defined modulo I. Clearly $0 \longrightarrow A \longrightarrow T' \xrightarrow{\varphi'} R \longrightarrow 0$ is a complex and φ' is onto. It is exact since $\ker(V \longrightarrow R) = \ker(T \longrightarrow R) \times 0 + 0 \times \ker(U \longrightarrow R) = A \times 0 + 0 \times ZA = A \times 0 + I$ (since $ZA \subseteq A$). $Z(T', A) = \{\, (t,u) + I \in V/I \mid t \in Z(T, A) \,\}$. Map $T' \longrightarrow T/Z(T, A)$ by $(t, u) + I \mapsto t + Z(T, A)$. This is well defined modulo I and its kernel is $Z(T', A)$. Since $U \longrightarrow R$ is onto, so is $V \longrightarrow T$, and hence $T' \longrightarrow T/Z(T, A)$ is also. Thus $T'/Z(T', A) \cong T/Z(T, A)$ and the isomorphism is coherent with φ and φ' and with the maps $T \longleftarrow A \longrightarrow T'$. Thus

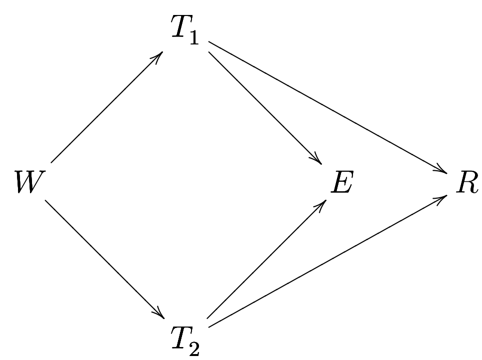
$$0 \longrightarrow A \longrightarrow T' \xrightarrow{\varphi'} R \longrightarrow 0 \in\in \mathbf{\Sigma}$$

(This notation means the sequence belongs to some $\Sigma' \in \mathbf{\Sigma}$.)

To define $\Sigma_1 - \Sigma_2$ let Σ_i be represented by the sequence

$$0 \longrightarrow A \longrightarrow T_i \xrightarrow{\varphi_i} R \longrightarrow 0, \qquad i = 1, 2$$

where we again suppose $A < T_i$. We may also suppose $T_1/Z(A, T_1) = E = T_2/Z(A, T_2)$ and $T_1 \xrightarrow{\tau_1} E \xleftarrow{\tau_2} T_2$ are the projections. Let



be a limit. This means $W = \{\, (t_1, t_2) \in T_1 \times T_2 \mid \tau_1 t_1 = \tau_2 t_2 \text{ and } \varphi_1 t_1 = \varphi_2 t_2 \,\}$. Then $J = \{\, (a, a) \mid a \in A \,\} < W$. Map $ZA \longrightarrow W/J$ by $z \mapsto (z, 0) + J$ and $\varphi\colon W/J \longrightarrow R$ by $(t_1, t_2) + J \mapsto \varphi_1 t_1 = \varphi_2 t_2$. If $(t_1, t_2) + J \in \ker \varphi$, then $\varphi_1 t_1 = 0 = \varphi_2 t_2$, so $t_1, t_2 \in A$. Then $(t_1, t_2) = (t_1 - t_2, 0) + (t_2, t_2)$. But then $\tau_1(t_1 - t_2) = 0$, so $t_1 - t_2 \in A \cap Z(A, T_1) = ZA$. Thus $ZA \subseteq \ker \varphi$, and clearly $\ker \varphi \subseteq ZA$. Now given $r \in R$, we can find $t_i \in T_i$ with $\varphi_i t_i = r$, $i = 1, 2$. Then $\pi(\tau_1 t_1 - \tau_2 t_2) = \pi\tau_1 t_1 - \pi\tau_2 t_2 = p\varphi_1 t_1 - p\varphi_2 t_2 = 0$, so $\tau_1 t_1 - \tau_2 t_2 = \lambda a$ for some $a \in A$. (Recall $\lambda\colon A \longrightarrow E$ is the multiplication map.) But then $\tau_1 t_1 = \tau_2(t_2 + a)$ and $\varphi_1 t_1 = \varphi_2(t_2 + a)$, so $(t_1, t_2 + a) + J \in W/J$ and $\varphi(t_1, t_2 + a) = r$. Thus φ is onto and

$$0 \longrightarrow ZA \longrightarrow W/J \longrightarrow R \longrightarrow 0 \in\in \mathbf{\Lambda}$$

Note that the correct R-module structure is induced on ZA because p is the same.

Proposition 3.2. *For any* $\Lambda, \Lambda_1, \Lambda_2 \in \mathbf{\Lambda}$, $\Sigma, \Sigma_1, \Sigma_2 \in \mathbf{\Sigma}$,

a) $(\Lambda_1 + \Lambda_2) + \Sigma = \Lambda_1 + (\Lambda_2 + \Sigma)$

b) $(\Sigma_1 - \Sigma_2) + \Sigma_2 = \Sigma_1$

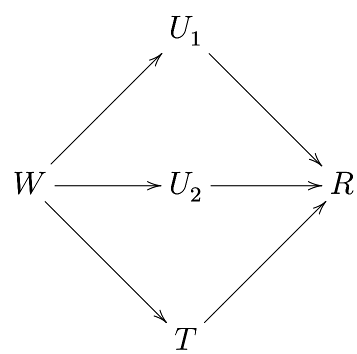
c) $(\Lambda + \Sigma) - \Sigma = \Lambda$

Proof. a) Let

$$0 \longrightarrow ZA \longrightarrow U_i \xrightarrow{\psi_i} R \longrightarrow 0, \qquad i = 1, 2,$$

$$0 \longrightarrow Z \longrightarrow T \xrightarrow{\varphi} R \longrightarrow 0$$

represent Λ_1, Λ_2, Σ respectively. An element of $(\Lambda_1 + \Lambda_2) + \Sigma$ is represented by a triple (u_1, u_2, t) such that $\psi(u_1, u_2) = \varphi t$ where $\psi(u_1, u_2) = \psi_1 u_1 = \psi_2 u_2$. An element of $\Lambda_1 + (\Lambda_2 + \Sigma)$ is represented by a triple (u_1, u_2, t) where $\psi_1 u_1 = \varphi'(u_2, t)$ and $\varphi'(u_2, t) = \psi_2 u_2 = \varphi t$. Thus each of them is the limit



modulo a certain ideal which is easily seen to be the same in each case, namely $\{ (z_1, z_2, z_3) \mid z_i \in Z \text{ and } z_1 + z_2 + z_3 = 0 \}$.

b) Let Σ_1 and Σ_2 be represented by sequences $0 \longrightarrow A \longrightarrow T_i \xrightarrow{\varphi_i} R \longrightarrow 0$. Let $\tau_i : T_i \longrightarrow E$ as above for $i = 1, 2$. Let $(\Sigma_1 - \Sigma_2) + \Sigma_2$ be represented by

$$0 \longrightarrow A \longrightarrow T \longrightarrow R \longrightarrow 0$$

Then an element of T can be represented as a triple (t_1, t_2, t'_2) subject to the condition $\tau_1 t_1 = \tau_2 t_2$, $\varphi_1 t_1 = \varphi_2 t_2 = \varphi_2 t'_2$. These conditions imply that $t'_2 - t_2 \in A$ and we can map $\sigma : T \longrightarrow T_2$ by $\sigma(t_1, t_2, t'_2) = t_1 + (t'_2 - t_2)$. To show that σ is an algebra homomorphism, recall that $\tau_1 t_1 = \tau_2 t_2$ implies that t_1 and t_2 act the same on A. Now if $(t_1, t_2, t'_2), (s_1, s_2, s'_2) \in T$,

$$\begin{aligned}
\sigma(t_1, t_2, t'_2) \cdot \sigma(s_1, s_2, s'_2) &= (t_1 + (t'_2 - t_2))(s_1 + (s'_2 - s_2)) \\
&= t_1 s_1 + t_1(s'_2 - s_2) + (t'_2 - t_2)s_1 + (t'_2 - t_2)(s'_2 - s_2) \\
&= t_1 s_1 + t_2(s'_2 - s_2) + (t'_2 - t_2)s_2 + (t'_2 - t_2)(s'_2 - s_2) \\
&= t_1 s_1 + t'_2 s'_2 - t_2 s_2 = \sigma(t_1 s_1, t_2 s_2, t'_2 s'_2) \\
&= \sigma((t_1, t_2, t'_2)(s_1, s_2, s'_2))
\end{aligned}$$

Also the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & T & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & T_2 & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

commutes and the sequences are equivalent.

c) Let Λ and Σ and $(\Lambda + \Sigma) - \Sigma$ be represented by sequences

$$0 \longrightarrow ZA \longrightarrow U \xrightarrow{\psi} R \longrightarrow 0$$

$$0 \longrightarrow A \longrightarrow T \xrightarrow{\varphi} R \longrightarrow 0$$

$$0 \longrightarrow ZA \longrightarrow U' \xrightarrow{\psi'} R \longrightarrow 0,$$

respectively. An element of U' is represented by a triple (t, u, t') subject to $\varphi t = \psi u = \varphi t'$ and $\tau t = \tau t'$. The equivalence relation is generated by all $(z, a-z, a)$, $a \in A$, $z \in ZA$. The relations imply that $t - t' \in ZA$, so the map $\sigma: U' \longrightarrow U$ which takes $(t, u, t') \mapsto u + (t - t')$ makes sense and is easily seen to be well defined. For $s, s', t, t' \in T$, $u, v \in U$, we have

$$\begin{aligned} \sigma(t, u, t')\sigma(s, v, s') &= (u + t - t')(v + s - s') \\ &= uv + u(s - s') + (t - t')v + (t - t')(s - s') \\ &= uv + t(s - s') + (t - t')s' \\ &= uv + ts - t's' = \sigma(ts, uv, t's') \\ &= \sigma((t, u, t')(s, v, s')) \end{aligned}$$

Since $ZA \longrightarrow U'$ takes $z \mapsto (z, 0, 0)$ and $\psi'(t, u, t') = \psi u = \psi u + \psi(t - t')$, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & ZA & \longrightarrow & U' & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & ZA & \longrightarrow & U & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

commutes and gives the equivalence. ∎

4. Every element of H^2 is an obstruction

The title of this section means the following. Given an R-module Z and a class $\xi \in H^2(R, Z)$, it is possible to find an algebra A and an $E \in EA$ of the form

$$0 \longrightarrow ZA \longrightarrow A \longrightarrow E \longrightarrow M \longrightarrow 0$$

and a surjection $p: R \longrightarrow M$ such that $Z \cong ZA$ as an R-module (via p) and $[p] = \xi$. It is clear that this statement together with Theorem 2.2 characterizes H^2 completely. No smaller group contains all obstructions and no factor group is fine enough to test whether a p comes from an extension. In particular, this shows that in degrees 1 and 2 these groups must coincide with those of Harrison (renumbered) (see [Harrison (1962)]) and Lichtenbaum and Schlessinger (see [Lichtenbaum & Schlessinger (1967)]). In particular those coincide. See also Gerstenhaber ([Gerstenhaber (1966), Gerstenhaber (1967)]) and Barr ([Barr (1968a)]).

THEOREM 4.1. *Every element of H^2 is an obstruction.*

PROOF. Represent ξ by a derivation $\rho: G^3R \longrightarrow Z$. This derivation has the property that $\rho\varepsilon = 0$ and by the simplicial normalization theorem we may also suppose $\rho\delta^0 = p\delta^1 = 0$. Let $V = \{ (x, z) \in G^2R \times Z \mid \varepsilon^1 x = 0 \}$. (Here Z is given trivial multiplication.) Let $I = \{ (\varepsilon^0 y, -\rho y) \mid y \in G^3R, \varepsilon^1 y = \varepsilon^2 y = 0 \}$. $I \subseteq V$ for $\varepsilon^1\varepsilon^0 y = \varepsilon^0\varepsilon^2 y = 0$. I claim that $I < V$. In fact for $(x, z) \in V$, $(\varepsilon^0 y, -\rho y) \in I$, $(x, z)(\varepsilon^0 y, -\rho y) = (x \cdot \varepsilon^0 y, 0)$. Now $\delta^0 x \cdot y \in G^3R$ satisfies $\varepsilon^0(\delta^0 x \cdot y) = x \cdot \varepsilon^0 y$, $\varepsilon^i(\delta^0 x \cdot y) = \varepsilon^i\delta^0 x\varepsilon^i y = 0$, $i = 1, 2$. Moreover $\rho(\delta^0 x \cdot y) = \rho\delta^0 x \cdot y + \delta^0 x \cdot \rho y$. Now $\rho\delta^0 = 0$ by assumption and the action of G^3R on Z is obtained by applying face operators into R (any composite of them is the same) and then multiplying. In particular, $\delta^0 x \cdot \rho y = \varepsilon^1\varepsilon^1\delta^0 x \cdot \rho y = \varepsilon^1 x \cdot \rho y = 0$, since $\varepsilon^1 x = 0$. Thus $(x, z)(\varepsilon^0 y, -\rho y) = (\varepsilon^0(\delta^0 x \cdot y), -\rho(\delta^0 x \cdot y))$ and I is an ideal. Let $A = V/I$. I claim that the composite $Z \longrightarrow V \longrightarrow V/I$ is 1-1 and embeds Z as ZA. For if $(0, z) = (\varepsilon^0 y, -\rho y)$, then $\varepsilon^0 y = \varepsilon^1 y = \varepsilon^2 y = 0$ so that y is a cycle and hence a boundary, $y = \varepsilon z$. But then $\rho y = \rho\varepsilon z = 0$. This shows that $Z \cap I = 0$. If $(x, z) + I \in ZA$, $(x, z)(x', z') = (xx', 0) \in I$ for all $(x', z') \in V$. In particular $\varepsilon(xx') = \varepsilon^0(xx') = \varepsilon^0 x \cdot \varepsilon^0 x' = 0$ for all x' with $\varepsilon^1 x' = 0$. By the simplicial normalization theorem this mean $\varepsilon^0 x \cdot \ker\varepsilon = 0$. Let $w \in GR$ be the basis element corresponding to $0 \in R$. Then w is not a zero divisor, but $w \in \ker\varepsilon$. Hence $\varepsilon^0 x = 0$ and $x = \varepsilon y$ and by the normalization theorem we may suppose $\varepsilon^1 y = \varepsilon^2 y = 0$. Therefore $(x, z) = (\varepsilon^0 y, -\rho y) + (0, z + \rho y) \equiv (0, z + \rho y) \pmod{I}$. On the other hand $Z + I \subseteq ZA$.

Let GR operate on V by $y(x, z) = (\delta y \cdot x, yz)$ where GR operates on Z via $p\varepsilon$. I is a GR-submodule for $y'(\varepsilon^0 y, -\rho y) = (\delta y' \cdot \varepsilon^0 y, -y' \cdot \rho y) = (\varepsilon^0(\delta\delta y' \cdot y), -\rho(\delta\delta y' \cdot y))$, since $\rho(\delta\delta y' \cdot y) = \delta\delta y' \cdot \rho y + \rho\delta\delta y' \cdot y = y' \cdot \rho y$. Hence A is a GR-algebra.

Let E be the algebra of endomorphisms of A which is generated by the multiplications from GR and the inner multiplications. Let $p_0: GR \longrightarrow E$ and $\lambda: A \longrightarrow E$ be the indicated maps. Then $E = \operatorname{im} p_0 + \operatorname{im} \lambda$. This implies that πp_0 is onto where $\pi: E \longrightarrow M$ is cokerλ.

Now we wish to map $p: R \longrightarrow M$ such that $p\varepsilon = \pi p_0$. In order to do this we must show that for $x \in G^2R$, $p_0\varepsilon^0 x$ and $p_0\varepsilon^1 x$ differ by an inner multiplication. First we show that if $(x', z) \in V$, then $(x \cdot x' - \delta\varepsilon^0 x \cdot x', 0) \in I$. In fact let $y = (1 - \delta^0\varepsilon^1)(\delta^1 y \cdot \delta^0 x)$. Then $\varepsilon^1 y = 0$ and $\varepsilon^2 y = 0$ also, since $\varepsilon^2\delta^0 x' = \delta\varepsilon^1 x' = 0$. $\varepsilon^0 y = (\varepsilon^0 - \varepsilon^1)(\delta^1 x \cdot \delta^0 x') = \delta\varepsilon^0 y \cdot x - x \cdot x'$.

Finally $\rho y = 0$ because of the assumption we made that $\rho\delta^i = 0$. Now

$$\begin{aligned}(p_0\varepsilon^0 x - p_0\varepsilon^1 x)(x', z) &= ((\delta\varepsilon^0 x - \delta\varepsilon^1 x)x', xz - xz)\\ &= ((x - \delta\varepsilon^1 x)x', 0) \pmod{I}\\ &= (x - \delta\varepsilon^1 x, 0)(x', z)\end{aligned}$$

where $(x - \delta\varepsilon^1 x, 0) \in V$. Thus we have shown

LEMMA 4.2. *$p_0\varepsilon^0 x - p_0\varepsilon^1 x$ is the inner multiplication $\lambda((x - \delta\varepsilon^1 x, 0) + I)$.*

Then map $p: R \longrightarrow M$ as indicated. Now $\pi p_0 = p\varepsilon$ is a surjection and so is p.

P is constructed as pairs (e, a), $e \in E$, $a \in A$ with multplication $(e, a)(e', a') = (ee', ea' + e'a + aa')$. Map $p_1: G^2R \longrightarrow E$ by $p_1 x = (p_0\varepsilon^1 x, (x - \delta^0\varepsilon^1 x, 0) + I)$. Then $d^0 p_1 x = p_0\varepsilon^1 x + \lambda((x - \delta^0\varepsilon^1, 0) + I) = p_0\varepsilon^1 x + p_0\varepsilon^0 x - p_0\varepsilon^1 x = p_0\varepsilon^0 x$ by Lemma 4.2. Also $d^1 p_1 x = p_0\varepsilon^1 x$ and thus p_1 is a suitable map. If $p_2: G^3R \longrightarrow B$ is chosen as prescribed, then for any $x \in G^3R$,

$$\begin{aligned}(1 - s^0 d^1)dp_2 x &= (1 - s^0 d^1)p_1\varepsilon x\\ &= (1 - s^0 d^1)(p_0\varepsilon^1\varepsilon x, (\varepsilon x - \delta^0\varepsilon^1\varepsilon x, 0) + I)\\ &= (p_0\varepsilon^1\varepsilon x, (\varepsilon x - \delta^0\varepsilon^1\varepsilon x, 0) + I) - (p_0\varepsilon^1\varepsilon x, 0)\\ &= (0, (\varepsilon x - \delta^0\varepsilon^1\varepsilon x, 0) + I)\end{aligned}$$

The proof is completed by showing that $(\varepsilon x - \delta^0\varepsilon^1\varepsilon x, 0) \equiv (0, \rho x) \pmod{I}$. Let $y = (1 - \delta^0\varepsilon^1)(1 - \delta^1\varepsilon^2)x$. Then $\varepsilon^1 y = \varepsilon^2 y = 0$ clearly and $\varepsilon^0 y = (\varepsilon^0 - \varepsilon^1)(1 - \delta^1\varepsilon^2)x = (\varepsilon^0 - \varepsilon^1 + \varepsilon^2 - \delta^0\varepsilon^1\varepsilon^0)x = (\varepsilon^0 - \varepsilon^1 + \varepsilon^2 - \delta^0\varepsilon^1(\varepsilon^0 - \varepsilon^1 + \varepsilon^2))x = (\varepsilon - \delta^0\varepsilon^1\varepsilon)x$, while $\rho y = \rho x$, since we have assumed that $\rho\delta^i = 0$. Thus $\partial p = \rho$ and $[p] = \xi$. This completes the proof. ■

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