Local extension of maps

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Abstract. We continue our investigations into absolute $\mathcal{CR}$-epic spaces. Given a continuous function $f : X \longrightarrow Y$, with X absolute $\mathcal{CR}$-epic, we search for conditions which imply that Y is also absolute $\mathcal{CR}$-epic. We are particularly interested in the cases when X is a dense subset of Y and when f is a quotient mapping. To answer these questions, we consider issues of local extension of continuous functions. The results on this question are of independent interest.

Contents

1. Introduction

Unless stated otherwise, all spaces in this paper will be assumed Tychonoff, that is, completely regular and Hausdorff. In a series of papers, the current authors and others have developed at some length the notion of absolute $\mathcal{CR}$-epic spaces: a Tychonoff space X is absolute $\mathcal{CR}$-epic if for any dense embedding $X \hookrightarrow Y$ into another Tychonoff space, the induced $C(Y) \longrightarrow C(X)$ is an epimorphism in the category of commutative rings (see [Barr, *et al.* (2003), Barr, *et al.* (2005),

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Barr, *et al.* (2007b), Barr, *et al.* (2009)]). In this paper we continue these investigations. We consider a continuous map $f : X \longrightarrow Y$ and search for conditions under which the fact that X is absolute $\mathcal{CR}$-epic implies the same for Y. We are interested in two cases. In the first X is a dense subspace of Y and in the second Y is a quotient of X.

The motivation for this paper came from the study of epimorphisms of commutative rings in which we have uncovered several classes of spaces. The class of Lindelöf absolute $\mathcal{CR}$-epic spaces properly contains the class of Lindelöf CNP spaces. The latter class consists of those spaces that are P-sets in any (and therefore every) compactification. Both of these classes can be defined in terms of the extendibility of continuous functions. Lindelöf absolute $\mathcal{CR}$-epic spaces are precisely those for which continuous functions extend to neighbourhoods in arbitrary compactifications. Lindelöf CNP spaces are exactly those for which a countable sequence of functions can be extended from a point to one of its neighbourhoods in the β-compactification (Theorem 4.3). There is also a new and stronger class of spaces for which a neighbourhood works for extending all functions (the uniform property). In analyzing these classes and in examining their local-global behaviour we were led to the following definitions which discuss the extendibility of continuous functions both locally and globally. They make no reference to epimorphisms. They are general but also contain the key to studying epimorphisms induced by Lindelöf spaces. The study in the non-Lindelöf case uses different methods. See [Barr, *et al.* (2009)] for classes of punctured planks (such as the Dieudonné plank) which are absolute $\mathcal{CR}$-epic.

(1) Extension property (EP): X satisfies the EP if for every dense embedding $X \hookrightarrow Y$, every $f \in C^*(X)$ has a continuous extension to a Y-neighbourhood of X
(2) Local Extension property (LEP): X satisfies the LEP if every point of X has an X-neighbourhood that satisfies the EP.
(3) Countable extension property (CEP): X satisfies the CEP if for every dense embedding $X \hookrightarrow Y$ and every sequence $f_1, f_2, \ldots, f_n, \ldots$ of functions in $C^*(X)$, there is a single Y-neighbourhood of X to which each f_n extends.
(4) Countable local extension property (CLEP): X satisfies the CLEP if every point of X has an X-neighbourhood that satisfies the CEP.
(5) Uniform Extension Property (UEP): X satisfies the UEP if for every dense embedding $X \subseteq Y$ there is a Y-neighbourhood of X to which every $f \in C^*(X)$ extends.
(6) Uniform Local Extension Property (ULEP): X satisfies the ULEP if every point of X has an X-neighbourhood that satisfies the UEP.
(7) Sequential Bounded Property (SBP): X satisfies the SBP at a point $p \in X$ if given any sequence $f_1, f_2, \ldots, f_n, \ldots$ of functions in $C(X)$, there is an X-neighbourhood of p in which every one of the f_n is bounded. We also say that X satisfies the SBP if it satisfies it at every point.
(8) Countable neighbourhood property (CNP): X satisfies the CNP if for every sequence $U_1, U_2, \ldots, U_n, \ldots$ of βX-neighbourhoods of X, then $\bigcap U_n$ is a βX-neighbourhood of X. Topologists often say that X is a P-set in βX.

Obviously, (1) $\Rightarrow$ (2), (3) $\Rightarrow$ (4), and (5) $\Rightarrow$ (6). The converses to all three are given in 2.4. Clearly (5) $\Rightarrow$ (3) $\Rightarrow$ (1). In [Barr, *et al.* (2005), Corollary

2.13] it was shown that for Lindelöf spaces (1) $\Leftrightarrow$ absolute $\mathcal{CR}$-epic. We will see in Theorem 4.1 that for Lindelöf spaces, (7) $\Leftrightarrow$ (8) $\Rightarrow$ (3). In Example 6.5 we will see that the CEP (and even the UEP) does not imply the CNP. Combining these results, we see that a Lindelöf space that satisfies any of the eight conditions defined above is absolute $\mathcal{CR}$-epic

Remark 1.1. We will often say that "X is an EP space" (or a CEP or CNP space, etc.) as an abbreviation for "X satisfies the EP".

A **compactification** $X \subseteq K$ of X is a *dense* embedding into a compact space K. One readily sees that since every Tychonoff space can be embedded into a compact Hausdorff space, it would be sufficient, in points 1, 3, and 5 above to restrict the spaces Y to being compactifications.

When we are considering the case that X is a dense subspace of Y, it will often, but not always, be the case that Y is a subspace of βX and f is the inclusion map. A typical result is that if X is Lindelöf CNP and $A \subseteq X$, then $X \cup \mathrm{cl}_{\beta X}(A)$ is Lindelöf CNP (Theorem 4.6). Another result is that if X is Lindelöf absolute $\mathcal{CR}$-epic and A a zeroset in βX, then $X \cup A$ is absolute $\mathcal{CR}$-epic (Corollary 2.9).

Until this paper, we knew only that a Lindelöf locally compact subset of βX satisfied the UEP, but that is an immediate consequence of the fact that a locally compact space is open in its β-compactification. In Theorem 6.2 we will find other examples, which will allow us to resolve negatively a question raised by Ronnie Levy in [Levy (1980)], see 6.1.

For quotients, we had previously seen that a perfect image of an Lindelöf CNP space was Lindelöf CNP, [Barr, *et al.* (2007b), Theorem 3.5.5]. Here we extend this result to open images, Theorem 4.2, and closed images, Theorem 4.7. We also show that a quotient of a countable sum of compact spaces is CNP, Theorem 4.17. The last result will allow us to answer positively a question we have previously raised and show that the rational numbers with the "Egyptian topology" (induced by the representation of rational numbers as the sum of reciprocals of distinct integers) is CNP, hence absolute $\mathcal{CR}$-epic. One interesting thing about this result is that the rationals with the usual topology is not absolute $\mathcal{CR}$-epic.

Another theme that has arisen is encompassed in several theorems that say that if X satisfies one of the map extension conditions and $A \subseteq \beta X$ is a subspace that satisfies some subsidiary condition then $X \cup A$ or $X \cup \mathrm{cl}_{\beta X}(A)$ satisfies the same extension condition as X (Theorem 2.8, Corollary 2.9, Theorem 4.6, Theorem 4.12, Theorem 4.13, Theorem 5.4).

Notation and Conventions. If $\varphi : X \longrightarrow Y$ is a map and $A \subseteq X$ we denote by $\varphi_\#(A)$ the "universal image" of A in Y. To be precise $\varphi_\#(A)$ consists of all $y \in Y$ for which $\varphi^{-1}(y) \subseteq A$. Another way of defining it is by the formula $\varphi_\#(A) = Y - \varphi(X - A)$. The properties of $\varphi_\#$ are given in detail in [Barr, *et al.* (2007b), 2.2]. An important property—evident from the second definition above—is that when φ is a closed mapping, $\varphi_\#$ takes open sets to open sets. If E is an equivalence relation on a space X and $A \subseteq X$ we say that E **is A-admissible** or A **is E-compatible** if $(A \times X) \cap E = \Delta_A$. This means that no point of A is E-equivalent to any point of X but itself (See [Barr, *et al.* (2007b), Proposition 2.5]). In such a case, the map A to its image in X/E is a homeomorphism and we will usually treat A as a subspace of that quotient.

2. Preliminary results

We begin with a pair of known results that we will be using. See [Barr, *et al.* (2005), 2.13 and 2.14] for the proofs. See also [Engelking (1989), Problems 1.7.6 and 3.12.25].

Proposition 2.1. [Smirnov's Theorem] *A space X is Lindelöf if and only if in any compactification K of X, every K-neighbourhood of X contains a cozeroset containing X.*

Proposition 2.2. *A Lindelöf space satisfies the EP if and only for every dense embedding $X \subseteq Y$ and every point $p \in X$ every function in $C^*(X)$ extends to a neighbourhood of p in Y.*

The following is proved in detail in [Barr, *et al.* (2009), 3.1–3.3] and is central to much of this paper. The limit used in the statement is taken over the directed set of all pairs (W, x) for which $x \in X \cap W$ with $(W, x) \le (W', x')$ if and only if $W \supseteq W'$. This is, of course, only a directed preorder, but that is all that is required (see [Kelley (1955), Page 65]).

Theorem 2.3. *Suppose $X \hookrightarrow Y$ is a dense embedding and $f \in C(X)$. Then f can be extended continuously to a point $p \in Y$ if and only if*

$$\lim\{f(x) \mid x \in X \cap W \text{ and } W \text{ is a neighbourhood of } p\}$$

exists and that limit is the value of the extension to p. Moreover, the extension of f to all such points is continuous.

Theorem 2.4. *LEP (respectively CLEP, ULEP) implies EP (respectively, CEP, UEP).*

Proof. Suppose X satisfies the LEP and is densely embedded in Y. Suppose $f \in C^*(X)$. For each $x \in X$, there is an X-neighbourhood $U(x)$ of x that satisfies the EP. Let $V(x) = \mathrm{int}_Y(\mathrm{cl}_Y(U(x)))$. Then $V(x)$ is a Y-open set that contains $U(x)$ and in which $U(x)$ is dense. It follows from the EP applied to $U(x)$ that $f|U(x)$ extends to a $V(x)$-open set $W(x)$, which is evidently also Y-open. The preceding theorem implies that f extends to $\bigcup_{x\in X} W(x)$ which is a Y-open set containing X.

The arguments for the CLEP and ULEP are similar and we omit them. $\square$

Lemma 2.5. *Suppose $A = \mathbf{Z}(f)$ is a zeroset in βX disjoint from X and E a closed, A-admissible equivalence relation on βX. Then there is an $\epsilon > 0$ such that whenever $(p, q) \in E$ with $p \ne q$, $|f(p)| \wedge |f(q)| > \epsilon$.*

Proof. We may assume, without loss of generality, that $f : \beta X \longrightarrow [0, 1]$. If the conclusion fails there is a sequence of points $(p_n, q_n) \in E$ such that $p_n \ne q_n$ and for which the sequence $f(p_n) \wedge f(q_n)$ is not bounded away from 0. Admissibility implies that $f(p_n) \wedge f(q_n)$ is never actually 0. By choosing a subsequence and interchanging p_n with q_n, if necessary, we can suppose that the sequence of $f(p_n)$ is not bounded away from 0. There are two cases, depending on whether the sequence of $f(q_n)$ is bounded away from 0 or not. In the former case, any limit point (p, q) of the sequence (p_n, q_n) belongs to E since E is closed. Clearly $p \in A$ and $q \notin A$, which contradicts the A-admissibility of E.

If neither of the sequences $f(p_n)$, $f(q_n)$ is bounded away from 0, then, by appropriate choice of subsequences, we can suppose that $f(p_n) \vee f(q_n) < f(p_{n-1}) \wedge$

$f(q_{n-1})$. Let $B_n = f^{-1}[f(p_n) \wedge f(q_n), 1]$. Then $B_n \subseteq B_{n+1}$ for all n, both p_n and q_n belong to B_n, while neither p_{n+1} nor q_{n+1} does. Suppose, by induction, that we have found, for all $m < n$, continuous functions $g_m : B_m \longrightarrow [0,1]$ such that $g_m(p_m) = 0$, $g_m(q_m) = 1$ and $g_m|B_k = g_k$ for all $k < m$. Now construct $g_n : B_n \longrightarrow [0,1]$ as follows. Since p_n and q_n lie outside the closed set B_{n-1}, we can extend g_{n-1} to $\widehat{g}_{n-1}$ on $B_{n-1} \cup \{p_n, q_n\}$ by letting $\widehat{g}_{n-1}(p_n) = 0$ and $\widehat{g}_{n-1}(q_n) = 1$. The extended domain is a closed subspace of the compact set B_n and so $\widehat{g}_{n-1}$ can be extended to a continuous function $g_n : B_n \longrightarrow [0,1]$ as desired. We let g be the function defined on $B = \mathrm{coz}(f) = \bigcup B_n$ whose restriction to B_n is g_n. To see that g is continuous, we note that $g^{-1}(f(p_n) \wedge f(q_n), 1]$ is open, contained in B_n, and contains B_{n-1} so that $B_{n-1} \subseteq \mathrm{int}(B_n)$. Thus $B = \bigcup \mathrm{int}(B_n)$. Since $g|\mathrm{int}(B_n) = g_n|\mathrm{int}(B_n)$ is continuous, it follows that g is continuous on B. Since $X \subseteq B \subseteq \beta X$, we see that $\beta X = \beta B$ and g extends to βX. But any limit point (p,q) of the sequence (p_n, q_n) lies in $E \cap (A \times A) = \Delta_A$, which is impossible since $p = q$ while $g(p) = 0$ and $g(q) = 1$. □

Corollary 2.6. *Under the same hypotheses, there is an open set $U \supseteq A$ such that E is U-compatible.*

Proof. Just take $U = f^{-1}[0, \epsilon)$. □

Theorem 2.7. *Suppose that $\{A_\alpha\}$ is a family of zerosets in βX, all disjoint from X and $A = \bigcup A_\alpha$. Then for every A-compatible equivalence relation E on βX, there is a βX-open set $U \supseteq A$ such that U is E-compatible.*

Proof. An A-compatible equivalence relation E is also A_α-compatible and so there is an open $U_\alpha \supseteq A_\alpha$ such that E is also U_α-compatible. Set $U = \bigcup U_\alpha$ and then $A \subseteq U$ and E is U-compatible. □

Theorem 2.8. *Suppose X is Lindelöf and satisfies the EP and $A = \bigcup A_\alpha$ is a union of zerosets in βX. Then $X \cup A$ satisfies the EP.*

Proof. Let K be a compactification of $X \cup A$ and hence of X since X is dense in $X \cup A$. Let $f \in C^*(X \cup A)$. Since X is absolute $\mathcal{CR}$-epic, there is a K-neighbourhood U of X to which f extends. Since X is Lindelöf we can assume that U is a cozeroset by Smirnov's Theorem. Let $\theta : \beta(X \cup A) = \beta X \longrightarrow K$ be the canonical map and let

$$E = \{(u,v) \in \beta X \times \beta X \mid \theta(u) = \theta(v)\}$$

(called its kernel pair). Then $V = \theta^{-1}(U)$ is a cozeroset of $\beta(X \cup A)$ containing X. The difference of a zeroset and a cozeroset is also a zeroset. Thus $A - V = \bigcup(A_\alpha - V)$ is a union of zerosets disjoint from X. The previous theorem implies there is an E-compatible $\beta(X \cup A)$-neighbourhood W of $A - V$ and then f extends to $U \cup \theta(W)$. But $\theta(W) = \theta_\#(W)$ is a neighbourhood of $A - U = \theta(A - V)$ and hence $U \cup \theta(W)$ is a neighbourhood of $X \cup \theta(A - V) = X \cup A$. □

Corollary 2.9. *Suppose that X is Lindelöf absolute $\mathcal{CR}$-epic and $A = \bigcup A_n$ is a union of at most countably many zerosets in βX. Then $X \cup A$ is Lindelöf absolute $\mathcal{CR}$-epic.*

Proof. A zeroset is Lindelöf and so is the union of countably many of them. □

3. A topological interlude

Lemma 3.1. *Suppose X and Y are spaces and K and L are compactifications of X and Y, respectively. Suppose*

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, two heads, "\theta"'] & K \arrow[d, "\varphi"] \\ Y \arrow[r, hook] & L \end{tikzcd}$$

is a commutative square with θ a closed surjection. Then for any $p \in K$ for which $y = \varphi(p) \in Y$, we have $p \in \mathrm{cl}_K(\theta^{-1}(y))$.

Proof. Let $A = \theta^{-1}(y)$. If the conclusion fails, there is a closed K-neighbourhood W of p disjoint from A. Since X is dense in K, we can suppose that $W = \mathrm{cl}_K(X \cap W)$. Then $U = X - W$ is an X-open subset of X such that $A \subseteq U$ and $U \cap W = \emptyset$. Thus $\theta_\#(U)$ is an open neighbourhood of $\varphi(p) = \theta_\#(A)$ in Y. Since $\theta_\#(U) \cap \theta(W \cap X) = \emptyset$, we have

$$\varphi(p) \notin \mathrm{cl}_L(\theta(W \cap X)) = \mathrm{cl}_L(\varphi(W \cap X)) \supseteq \varphi(\mathrm{cl}_K(W \cap X)) = \varphi(W)$$

which is a contradiction. $\square$

The following is well known when A and B are disjoint and, in fact, characterizes normal spaces. This more general case must be known, but we have not found it in standard references.

Lemma 3.2. *Let X be a normal space and A and B two closed subsets of X. Then $\mathrm{cl}_{\beta X}(A \cap B) = \mathrm{cl}_{\beta X}(A) \cap \mathrm{cl}_{\beta X}(B)$.*

Proof. Clearly $\mathrm{cl}_{\beta X}(A \cap B) \subseteq \mathrm{cl}_{\beta X}(A) \cap \mathrm{cl}_{\beta X}(B)$. So let $p \in \mathrm{cl}_{\beta X}(A) \cap \mathrm{cl}_{\beta X}(B)$ and suppose that $p \notin \mathrm{cl}_{\beta X}(A \cap B)$. Then there is a closed neighbourhood U of p in βX such that $U \cap A \cap B = \emptyset$. Obviously $p \in \mathrm{cl}_{\beta X}(U \cap A)$ and similarly $p \in \mathrm{cl}_{\beta X}(U \cap B)$ which contradicts the special case of disjoint closed sets. $\square$

The following is true for abstract sets. We omit the easy proof.

Lemma 3.3. *Suppose $\theta : T \longrightarrow S$ is a function, $A \subseteq S$, and $B \subseteq T$. Then $\theta(B \cap \theta^{-1}(A)) = \theta(B) \cap A$.* $\square$

Lemma 3.4. *Let $\varphi : K \longrightarrow L$ be a continuous map of compact spaces. Let $Y \subseteq L$ and $X = \varphi^{-1}(Y)$. Then $\theta = \varphi|X$ is closed.*

Proof. Suppose that A is a closed subset of X. Then $A = \mathrm{cl}_K(A) \cap X$ and then $\theta(A) = \varphi(A) = \varphi(\mathrm{cl}_K A \cap \varphi^{-1}(Y)) = \varphi(\mathrm{cl}_K(A)) \cap Y$, which is the intersection of a compact set with Y and is therefore closed in Y. $\square$

A commutative square

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d, "\tau"'] & Y \arrow[d, "\theta"] \\ Z \arrow[r, "\rho"'] & T \end{tikzcd}$$

of not necessarily Tychonoff spaces is called a *pushout* in $\mathcal{T}op$, provided that, given any space W and continuous maps $\mu : Y \longrightarrow W$ and $\nu : Z \longrightarrow W$ such that $\mu\sigma = \nu\tau$, there is a unique $\omega : T \longrightarrow Z$ such that $\omega\theta = \mu$ and $\omega\rho = \nu$. Even if σ and τ are subspace inclusions, θ and ρ need not be. If all four maps are subspace inclusions, then a necessary and sufficient condition that the square be a pushout is that $X = Y \cap Z$ and that a subset of T is closed if and only its intersection with each of Y and Z is. In general, if X, Y, and Z are Tychonoff spaces, it does not follow that T is. However in the following lemma, the corresponding space is a given as a subspace of a Tychonoff space and therefore is one as well.

Lemma 3.5. *Suppose X is normal and $A \subseteq X$. Then the square*

$$\begin{tikzcd} X \cap \mathrm{cl}_{\beta X}(A) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{cl}_{\beta X}(A) \arrow[r] & \mathrm{cl}_{\beta X}(A) \cup X \end{tikzcd}$$

is a pushout of topological spaces.

Proof. We may as well suppose that A is closed in X in which case we can identity $\mathrm{cl}_{\beta X}(A)$ with βA and $X \cap \mathrm{cl}_{\beta X}(A) = \mathrm{cl}_X(A) = A$. Let $Y = \beta A \cup X$. Then the square in question is

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ \beta A \arrow[r] & Y \end{tikzcd}$$

From the remarks preceding, it suffices to show that for any $B \subseteq Y$, if $B \cap \beta A$ is compact and $B \cap X$ is closed in X, then B is closed in Y. So let B be such a set. We have that $\mathrm{cl}_Y(B) = \mathrm{cl}_Y((B \cap \beta A) \cup (B \cap X)) = \mathrm{cl}_Y(B \cap \beta A) \cup \mathrm{cl}_Y(B \cap X)$. Since $B \cap \beta A$ is compact, the first term is just $B \cap \beta A \subseteq B$. As for the second term, we have that $X \cap \mathrm{cl}_Y(B \cap X) = \mathrm{cl}_X(B \cap X) = B \cap X \subseteq B$, while by 3.2,

$$\begin{aligned} \beta A \cap \mathrm{cl}_Y(B \cap X) &= \mathrm{cl}_{\beta X}(A) \cap \mathrm{cl}_{\beta X}(B \cap X) \cap Y = \mathrm{cl}_{\beta X}(A \cap B \cap X) \cap Y \\ &= \mathrm{cl}_{\beta X}(A \cap B) = \mathrm{cl}_{\beta A}(A \cap B) \subseteq \mathrm{cl}_{\beta A}(\beta A \cap B) = \beta A \cap B \subseteq B \end{aligned}$$

and so $\mathrm{cl}_Y(B) \subseteq B$. □

4. Spaces satisfying the CEP and CNP

Theorem 4.1. *For any Lindelöf space X, the conditions SBP and CNP are equivalent and imply the CEP.*

Proof. We showed in [Barr, *et al.* (2009), Theorem 7.4] that X satisfies CNP if and only if it satisfies the SBP at every point. Here we will show that CNP $\Rightarrow$ CEP. Suppose that X satisfies the CNP and is densely embedded in a space Y. Let $f_1, f_2, \ldots, f_n, \ldots$ be a sequence of functions in $C^*(X)$. The CNP implies that each f_n extends to a Y-neighbourhood U_n of X, [Barr, *et al.* (2007b), Corollary 3.4]. The CNP also implies that $U = \bigcap U_n$ is a Y-neighbourhood of X (and hence of each of its points) to which each f_n extends. □

The following is an application of the equivalence of the CNP and SBP.

Theorem 4.2. *An open image of a Lindelöf CNP space is also Lindelöf CNP.*

Proof. We will show that the SBP is preserved under open surjections. Suppose $\theta : X \longrightarrow Y$ is an open surjection and X satisfies the SBP. Given any sequence f_1, $f_2, \ldots, f_n, \ldots \in C(Y)$ and any $y \in Y$, let $x \in \theta^{-1}(y)$. Since X satisfies SBP, there is an open neighbourhood U of x on which each term of the sequence $f_1\theta$, $f_2\theta, \ldots$, $f_n\theta, \ldots$ is bounded. Then $\theta(U)$ is the desired open neighbourhood of y. □

See Example 6.5 below for a space that satisfies the CEP (in fact, the UEP) but not the CNP. Here is a result that highlights the difference between them. The equivalence of the CEP and CLEP implies that when the CEP is satisfied by a space X then for each countable sequence $f_1, f_2, \ldots, f_n, \ldots$ in $C^*(X)$ and each point $p \in X$ there is a βX-neighbourhood of p to which each f_n extends. If we replace $C^*(X)$ by $C(X)$ we get the following:

Theorem 4.3. *A Lindelöf space X satisfies CNP if and only if there is a compactification K of X with the property that for each countable sequence $f_1, f_2, \ldots$, $f_n, \ldots$ of functions in $C(X)$ and each point $x \in X$ there is a K-open set containing X to which each f_n extends. Moreover, if this condition is satisfied by any compactification of X it is satisfied by all of them.*

Proof. We know from Theorem 4.1 that CNP implies CEP, which implies the extendability of a sequence of bounded functions to some open subset of K that contains X. If $f : X \longrightarrow \mathbf{R}$ is continuous, then $f/(1 + |f|)$ is bounded and if it extends to an open set $U \supseteq X$, then the only obstacle to extending f is that it might take on infinite values. Thus f extends to the one-point compactification $\mathbf{R} \cup \{\infty\}$ of $\mathbf{R}$ and, since $\mathbf{R}$ is open in its one-point compactification, we see that f extends to an $\mathbf{R}$-valued function on an open set. Since X is a P-set in K, the conclusion follows for a countable sequence of functions.

For the other direction, suppose $\{V_n\}$ is countable family of K-open sets containing X. Since X is Lindelöf, each V_n contains a cozeroset $\mathrm{coz}(f_n)$ that contains X. We can suppose that $f_n : \beta X \longrightarrow [0, 1]$. Since $1/f_n$ is continuous on $\mathrm{coz}(f_n)$, there is, for each $x \in X$, a K-neighbourhood $U(x)$ to which every $1/f_n$ extends. This implies that for all $n \in \mathbf{N}$, $U(x) \subseteq \mathrm{coz}(f_n) \subseteq V_n$, whence $U(x) \subseteq \bigcap_{n\in\mathbf{N}} V_n$. Then $\bigcup_{x\in X} U(x) \subseteq \bigcap_{n\in\mathbf{N}} V_n$ and the former is a K-open set containing X. Thus X is a P-set in K, which implies the CNP and that X is a P-set in any compactification ([Barr, *et al.* (2007b), Theorem 3.3]). □

Corollary 4.4. *Suppose that $\{X_n\}$ is a finite or countable family of subsets of the compact set K such that each X_n is dense in K and is Lindelöf CNP. Then $\bigcup X_n$ is Lindelöf CNP.*

Proof. For any sequence of functions $f_1, f_2, \ldots, f_m, \ldots$, there is, for each n, a K-open set U_n containing X_n to which each f_m extends. But then $U = \bigcup U_n$ is a K-open set containing $\bigcup X_n$ to which each f_m extends. □

The following is an obvious reformulation of the CNP.

Proposition 4.5. *A space X has CNP if and only if for every countable sequence $K_1, K_2, \ldots, K_n, \ldots$ of compact sets in $\beta X - X$, we have $\mathrm{cl}_{\beta X}(\bigcup K_n) \subseteq \beta X - X$.* □

The proof of the following theorem is a substantial simplification of our original one and we thank Ronnie Levy for suggesting it.

Theorem 4.6. *If X is Lindelöf CNP and $A \subseteq X$, then $X \cup \mathrm{cl}_{\beta X}(A)$ is also Lindelöf CNP.*

Proof. Since $\mathrm{cl}_{\beta X}(A) = \mathrm{cl}_{\beta X}(\mathrm{cl}_X(A))$, we can suppose, without loss of generality, that A is closed in X and therefore Lindelöf. Let $\{K_n\}$ be a sequence of compact subsets of $\beta X - X - \mathrm{cl}_{\beta X}(A)$ and $B = \bigcup K_n$. Then $Y = X \cup B$ is Lindelöf and hence normal ([Kelley (1955), Lemma 4.1]). Since B is disjoint from $\mathrm{cl}_{\beta X}(A)$ and A is closed in X, it follows that A is closed in Y. Since B is a countable union of closed sets disjoint from X, the CNP hypothesis implies that $\mathrm{cl}_Y(B)$ is also disjoint from X. Hence A and $\mathrm{cl}_Y(B)$ are disjoint closed sets in a normal space and therefore $\mathrm{cl}_{\beta X}(A)$ is disjoint from $\mathrm{cl}_{\beta X}(B)$ and hence so is $X \cup \mathrm{cl}_{\beta X}(A)$. □

Theorem 4.7. *A closed image of a Lindelöf CNP space is also Lindelöf CNP.*

Proof. Let $\theta : X \longrightarrow Y$ be a closed surjection and suppose X is Lindelöf CNP. The space Y is clearly Lindelöf. Write $\varphi = \beta(\theta) : \beta X \longrightarrow \beta Y$. Let $\{L_n\}$ be a sequence of compact subsets of $\beta Y - Y$. Let $K_n = \varphi^{-1}(L_n)$ and $K = \mathrm{cl}_{\beta X}\,(\bigcup K_n)$. We claim that $L = \varphi(K)$ is contained in $\beta Y - Y$. If not, there exists $p \in K$ for which $y = \varphi(p) \in Y$ so, by Lemma 3.1, we see $p \in \mathrm{cl}_{\beta X}(\theta^{-1}(y))$. By Theorem 4.6, $X \cup \mathrm{cl}_{\beta X}(\theta^{-1}(p))$ satisfies the CNP and hence $\mathrm{cl}_{\beta X}(\theta^{-1}(y))$ is disjoint from $\mathrm{cl}_{\beta X}(\bigcup K_n)$, which is a contradiction. □

Compare this with [Barr, *et al.* (2007b), 3.5.5] where we require a perfect surjection to draw that conclusion.

It is an easy consequence that if you form a quotient space of a Lindelöf (hence normal) CNP space by collapsing a closed subspace to a point (or by collapsing any finite number of disjoint closed subspaces to points) the resultant space is Lindelöf CNP. As an example of how that can be applied, we have:

Corollary 4.8. *Let X be Lindelöf CNP and $A \subseteq X$ be a subspace. If $f_1, f_2, \ldots,$ $f_n, \ldots$ is a sequence of real-valued functions on X, each one bounded on A, then there is a single neighbourhood of A on which each one is bounded.*

Proof. By replacing A by $\mathrm{cl}_X A$ (on which each f_n will continue to be bounded), we can suppose that A is closed. By replacing each f_n by $|f_n|$ we can suppose that the values of the f_n are non-negative. If b_n is an upper bound of f_n on A, then we can replace f_n by $f_n \vee b_n$ and suppose that each f_n is constant on A. The quotient mapping $\theta : X \longrightarrow X/A$, gotten by identifying the points of A to a single point, is closed and Theorem 4.7 implies that X/A is CNP. Evidently all the functions f_n, being constant on A, descend to the quotient. Thus there is a single neighbourhood U of the point $\{A\}$ on which each f_n is bounded and then $\theta^{-1}(U)$ is the required neighbourhood of A. □

The following strengthens one of the cases of [Barr, *et al.* (2009), Lemma 6.9].

Theorem 4.9. *Suppose X and Y are Tychonoff spaces with a common subspace A. Suppose that A is closed in X and that X is normal. Then the amalgamated sum $X +_A Y$ is Tychonoff.*

Proof. The amalgamated sum is the pushout (for the definition of pushout, see the proof of 3.4) in the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X +_A Y \end{tikzcd}$$

A subset $B \subseteq X +_A Y$ is closed if and only if $B \cap X$ and $B \cap Y$ are closed in X and Y, respectively. Points clearly have that property, so we need only show that the amalgamated space is completely regular. So let B be closed and $p \notin B$. We first consider the case that $p \in X - A$. Then $p \notin A \cup (B \cap X)$ and the latter is a closed subset of X. There is a continuous function $f : X \longrightarrow [0, 1]$ for which $f(p) = 1$, while f vanishes on $A \cup (B \cap X)$. Let g be the constant function 0 on Y. Then $f|A = g|A$ and therefore there is an $h : X +_A Y \longrightarrow [0, 1]$ whose restrictions to X and Y are f and g, respectively. Evidently, $h(p) = 1$, while h vanishes on B.

For the case that $p \in Y$ begin with a function $g : Y \longrightarrow [0, 1]$ such that $g(p) = 1$, while g vanishes on $B \cap Y$. The function that is g on A and 0 on $B \cap X$ is continuous on $A \cup (B \cap X)$ since it is continuous on each of two closed subsets of X and agrees on the overlap. This function then extends, by normality, to a continuous function $f : X \longrightarrow [0, 1]$. Since $f|A = g|A$ we get the function h as required. $\square$

Theorem 4.10. *Suppose that X and Y are CNP spaces with a common subspace A that is closed in each and that at least one of X and Y is normal. Then the amalgamated sum $X +_A Y$ also satisfies the CNP.*

Proof. The canonical map $\theta : X + Y \longrightarrow X +_A Y$ is a closed (even perfect) surjection since if B is closed in X then $\theta^{-1}\theta(B) = B + (B \cap A)$. $\square$

Theorem 4.11. *Let X be normal CNP, $A \subseteq X$ be a closed subspace and K be a compactification of A. Then $X +_A K$ is CNP.*

Proof. Since A is closed in X, it is known that βA is embedded in βX. Let $K = (\beta A)/E$ and $F = E \cup \Delta_{\beta X}$, which is a compact X-compatible equivalence relation on βX. We thus get a map $\varphi : \beta X \longrightarrow \beta X/F$ for which $\varphi^{-1}(X) = X$ and $\varphi^{-1}(K) = \mathrm{cl}_{\beta X}(A)$. The fact (from Lemma 3.5) that

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{cl}_{\beta X}(A) \arrow[r] & X \cup \mathrm{cl}_{\beta X}(A) \end{tikzcd}$$

is a pushout implies that $\theta : X \cup \mathrm{cl}_{\beta X}(A) \longrightarrow X \cup K$ is continuous. The fact that $X \cup K$ has the pushout topology embeds it into $\beta X/F$. Then 3.4 implies that θ is closed and then the result follows from Theorem 4.7. $\square$

Here is an application of Theorem 4.11. By contrast, see 4.14.

Theorem 4.12. *Suppose that X is normal CNP and that $A \subseteq \beta X$ is a locally compact set such that $A \subseteq \mathrm{cl}_{\beta X}(A \cap X)$. Then $X \cup A$ is CNP.*

Proof. Let $K = \mathrm{cl}_{\beta X}(A)$. Since A is locally compact it is open in K. Since $X \cap A$ is dense in A by hypothesis and A is evidently dense in K we can conclude that K is a compactification of $X \cap K$. In particular, for any K-open set $U \subseteq K$ we have that $U \subseteq \mathrm{cl}_K(X \cap U)$. By hypothesis, every point $p \in A$ has a compact K-neighbourhood V_p. Let $U_p = \mathrm{int}_K V_p$ so that U_p is a K-open set containing p. Clearly, $\mathrm{cl}_K U_p \subseteq V_p$ and is a compact neighbourhood of p. Thus we may replace V_p by $\mathrm{cl}_K U_p$ and assume that V_p is a regular closed set. Since $U_p \subseteq \mathrm{cl}_K(X \cap V_p) \subseteq V_p$ it follows immediately that $V_p = \mathrm{cl}_K(X \cap V_p)$. The preceding theorem implies that $X +_{X \cap V_p} V_p$ is CNP and it follows from Lemma 3.5 that that space can be identified with $X \cup V_p$. Suppose now that $\{W_n\}$ is a countable sequence of open sets of βX, all containing $X \cup A$. Then $\mathrm{int}_{\beta X}\left(\bigcap W_n\right)$ contains each $X \cup V_p$ and hence contains their union, which is $X \cup A$. □

Theorem 4.13. *Let X be Lindelöf CNP and $A \subseteq \beta X$ such that A itself is Lindelöf CNP and $A \subseteq \mathrm{cl}_{\beta X}(A \cap X)$. Then $X \cup A$ is Lindelöf CNP.*

Proof. Let $K = \mathrm{cl}_{\beta X} A$. As above K is a compactification of both $X \cap A$ and $X \cap K$. Let $f_1, f_2, \ldots, f_n, \ldots$ be a sequence of functions in $C(X \cup A)$. Since A is CNP and K is a compactification of A, it follows from Theorem 4.3 that there is a K-open set $U \subseteq K$ that includes A and to which each f_n extends. Since U is open in K, it follows that $U \subseteq \mathrm{cl}_K(X \cap U) = \mathrm{cl}_{\beta X}(X \cap U)$ and is also locally compact so that by 4.12 $X \cup U$ is CNP. It follows from Theorem 4.3 that there is a βX-neighbourhood of $X \cup U$ to which each f_n extends. The conclusion now follows from the converse part of the same theorem. □

Example 4.14. We give an example of a compactification K of an Lindelöf CNP space X and a closed subspace $A \subseteq X$ for which $X \cup \mathrm{cl}_K(A)$ is not absolute $\mathcal{CR}$-epic.

We take $X = \mathbf{N}$. It is known that there is a compactification K of $\mathbf{N}$ for which $K - \mathbf{N}$ is the unit interval $I = [0, 1]$, see [Chandler (1976), Theorem 7.8]. Take an open interval $U \subseteq I$ and a point $p \in I$ not in $\mathrm{cl}_I(U)$. Since $\mathrm{cl}_I(U)$ is compact, it is also closed in K. Thus there is a function $f : K \longrightarrow [0, 1]$ such that $f(p) = 1$, while f vanishes on $\mathrm{cl}(U)$. The set $V = f^{-1}[0, 1/2)$ is open and contains $\mathrm{cl}(U)$. If $A = V \cap \mathbf{N}$, then $U \subseteq V \subseteq \mathrm{cl}_K(A) \subseteq f^{-1}[0, 1/2]$. It follows that $p \notin \mathrm{cl}_K(A)$ so that $\mathrm{cl}_K(A) \cap I$ is a closed subset that is neither empty nor all of I. We claim that $B = \mathbf{N} \cup \mathrm{cl}_K(A)$ does not satisfy CNP. In fact, $K - B$ is an open subset of I and therefore a countable union of open intervals, each of which is σ-compact and hence $K - B$ is also σ-compact. It is, in particular, an F_σ and its complement B is a G_δ. A G_δ that satisfies the CNP is open. But if B were open, $B \cap I$ would be clopen in I and different from $\emptyset$ and I. Thus B does not satisfy the CNP.

To see that B is not even absolute $\mathcal{CR}$-epic, we begin by observing that every point of K is a G_δ. For $\mathbf{N}$ this is obvious. If $p \in K - \mathbf{N}$, there is a function $f : K - \mathbf{N} \longrightarrow [0, 1]$ that vanishes only at p. This function extends to K. Since $\mathbf{N}$ is Lindelöf and open in K, there is a function $g : L \longrightarrow [0, 1]$ with $\mathrm{coz}(g) = \mathbf{N}$ and then we see that $f + g$ vanishes only at p, which is thereby a G_δ. Since B is Lindelöf and not CNP, it is not locally compact. It follows from [Barr, *et al.* (2005), Theorem 4.2] that K is first countable at every point. Then the condition 2 of [Barr, *et al.* (2005), Theorem 4.3] applies, from which we conclude that B is not absolute $\mathcal{CR}$-epic.

Definition 4.15. Let $\theta : X \longrightarrow Y$ be continuous. We say that θ **has local sections** if for all $p \in Y$ there is a neighbourhood U of p and a map $\varphi : U \longrightarrow X$ such that $\theta\varphi$ is the inclusion $U \hookrightarrow Y$.

Theorem 4.16. *Suppose that $\theta : X \longrightarrow Y$ has local sections. If X is CNP so is Y.*

Proof. Suppose that $p \in Y$. Choose a neighbourhood U of p on which there is a section φ. We can choose U open, in which case $\theta^{-1}(U)$ will also be open and hence satisfy the CNP (see [Barr, *et al.* (2007b), Theorem 3.5.4]). Clearly the image of φ is contained in $\theta^{-1}(U)$ and hence U is a retract of $\theta^{-1}(U)$. A retract in a Hausdorff space is closed and hence $\varphi(U)$ is also CNP (Theorem 3.4.1, op. cit.). Since $\varphi(U) \equiv U$, it follows that each point of Y has a CNP neighbourhood and thus Y is CNP (Theorem 3.4.2, op.cit.). $\square$

4.1. The Egyptian topology on $\mathbf{Q}$. There is a topology on the rational numbers $\mathbf{Q}$ that is derived from the representation of rationals as Egyptian fractions. The easiest way to describe this is to let $\mathbf{N}^{+*}$ denote the one point compactification of the positive integers and let $X = \sum_{k=1}^{\infty}(\mathbf{N}^{+*})^k$. Map $f : \{0,1\} \times X \longrightarrow \mathbf{Q}$ by letting

$$f(n_0, n_1, n_2, \ldots, n_k) = (-1)^{n_0}\left(\frac{1}{n_1} + \frac{1}{n_1+n_2} + \cdots + \frac{1}{n_1+\cdots+n_k}\right)$$

This is surjective since every rational number has at least one (actually infinitely many) representations as a sum of distinct unit—or Egyptian—fractions. Of course, any term with ∞ in the denominator is 0. The **Egyptian topology** on $\mathbf{Q}$ is the quotient topology induced by f. The resultant space is obviously Hausdorff since the topology is finer than the usual topology on $\mathbf{Q}$.

For several years we have been wondering whether $\mathbf{Q}$ with the Egyptian topology was absolute $\mathcal{CR}$-epic. The following theorem gives a positive answer to the question. One of the implications is that the Egyptian topology on $\mathbf{Q}$ is finer than the usual topology since the latter is not absolute $\mathcal{CR}$-epic ([Barr, *et al.* (2005), Example 1.3.12]).

Theorem 4.17. *Suppose the Hausdorff space $X = \bigcup_{n\in\mathbf{N}} K_n$ is a union of compact sets and has the quotient topology from $\sum_{n\in\mathbf{N}} K_n$. Then X is Tychonoff and Lindelöf CNP.*

Proof. We begin by showing X is Tychonoff. The inverse image of a point in $\bigcup K_n$ consists of at most one point in each summand and is therefore closed. We can assume, without loss of generality that $K_1 \subseteq K_2 \subseteq \cdots$. Let $A \subseteq X$ be closed and $p \notin A$. We will construct a series of continuous functions $f_n : K_n \longrightarrow [0,1]$ each extending the previous one and let $f : X \longrightarrow [0,1]$ be the unique function whose restriction to K_n is f_n. The quotient topology is such that a function is continuous if its restriction to each K_n is. We may assume without loss of generality that $p \in K_1$. Begin by letting $f_1 : K_1 \longrightarrow [0,1]$ be any function for which $f_1(p) = 0$ and $f_1(A \cap K_1) = 1$. First extend this to $K_1 \cup (A \cap K_2)$ by letting the extended function be 1 on $A \cap K_2$. Since $K_1 \cup (A \cap K_2)$ is compact, it is C-embedded in K_2 and hence may be extended to a function $f_2 : K_2 \longrightarrow [0,1]$. Continue the obvious induction to get the required function f.

From Lemma 4.5, to show CNP, it is (necessary and) sufficient to show that if $L_1, L_2, \ldots$ is a countable family of compact subsets of $\beta X - X$, and $L = \bigcup L_n$, then $\mathrm{cl}_{\beta X} L$ is disjoint from X. So assume we are given such a family and assume that $p \in X$. We will show that there is a neighbourhood V of L and a function $f : X \longrightarrow [0,1]$ such that $f(p) = 0$ and $f(V \cap X) = 1$. Finding such a V and f will suffice since from $L \subseteq \mathrm{int}_{\beta X}(V) \subseteq \mathrm{cl}_{\beta X}(\mathrm{int}_{\beta X}(V) \cap X)$, it will follow that $f(p) = 0$, while $f(L) = 1$ and hence that $p \notin \mathrm{cl}_{\beta X}(L)$. Since p was an arbitrary point of X, it therefore follows that $X \cap \mathrm{cl}_{\beta X}(L) = \emptyset$. To construct this function, we again suppose that the K_n are nested and that $p \in K_1$. Begin by choosing a closed (hence compact) neighbourhood V_1 of L_1 that misses p. There is a function $f_1 : K_1 \longrightarrow [0,1]$ with $f_1(p) = 0$ and $f_1(V_1 \cap K_1) = 1$. The next step is to choose a closed neighbourhood V_2 of L_2 that is disjoint from K_1. This is possible because K_1 is a compact set inside X and L_2 is a compact set disjoint from X. First extend f_1 to the set $K_1 \cup ((V_1 \cup V_2) \cap K_2)$ by letting it be 1 on $(V_1 \cup V_2) \cap K_2$. This works because $(V_1 \cup V_2) \cap K_2 \cap K_1 = (V_1 \cup V_2) \cap K_1 = V_1 \cap K_1$ since V_2 is disjoint from K_1. Then let f_2 be a further extension of f_1 to all of K_2, with $f_2 = 0$ on $(V_1 \cup V_2) \cap K_2$. Continue by induction to finally get f. Again, the restriction to each K_n is continuous and therefore f is continuous in the quotient topology. That X is Lindelöf is obvious. □

Example 4.18. Let $\mathbf{N}^*$ denote the one-point compactification of $\mathbf{N}$. Map the space $\mathbf{N} \times \mathbf{N}^* \xrightarrow{\theta} \mathbf{Q}$ by enumerating the rationals in a sequence $q_1, q_2, \ldots, q_n, \ldots$ and defining $\theta(k,m) = q_k + 1/m$ while $\theta(k,\infty) = q_k$. This is obviously continuous, but cannot be a quotient mapping since $\mathbf{Q}$ is not even absolute $\mathcal{CR}$-epic, let alone CNP. We conclude that there must exist a discontinuous function $f : \mathbf{Q} \longrightarrow \mathbf{R}$ that nonetheless satisfies $\lim_{m \to \infty} f(q+1/m) = f(q)$ for all $q \in \mathbf{Q}$. After we mentioned the existence of such a function to Alan Dow, he sent us a simple construction of an explicit one that is nowhere continuous, in fact, unbounded in every interval of rationals.

Example 4.19. On the other hand, the countability in Theorem 4.17 is crucial. Let $S \subseteq \mathbf{Q}^{\mathbf{N}^*}$ denote the set of convergent sequences with their limits. Give S the discrete topology and consider the evaluation map $S \times \mathbf{N}^* \longrightarrow \mathbf{Q}$. This is clearly a quotient mapping since a map on $\mathbf{Q}$ that preserves the limits of all convergent sequences is continuous. But $\mathbf{Q}$ does not satisfy the CNP; it is not even absolute $\mathcal{CR}$-epic.

5. Subspaces and extensions of EP spaces

Theorem 5.1. *A closed C^*-embedded subspace of an EP space is also an EP space.*

Proof. Let X satisfy the EP and A be a closed C^*-embedded subspace. It is sufficient to show that in any compactification K of A, every function in $C^*(A)$ extends to a K-neighbourhood of A. In [Barr, *et al.* (2007b), 6.1–6.3], we showed that the amalgamated sum $X +_A K$ has enough real-valued functions to separate points and thus maps injectively to its associated Tychonoff space that we will denote Z. Thus the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "\theta"] \\ K \arrow[r, "\varphi"'] & Z \end{tikzcd}$$

is a pushout in the category of Tychonoff spaces (although not necessarily in $\mathcal{T}op$). We also showed that θ is a topological embedding. We claim that θ is a dense embedding. In fact, if $W = \mathrm{cl}_Z(\theta(X))$, then $\varphi^{-1}(W)$ is a closed subspace of K containing A, which means that $\varphi^{-1}(W) = K$. But then φ and θ both factor through W, which is impossible for $W \neq Z$. Then any $f \in C^*(A)$ extends to X and then to a Z-neighbourhood U of $\theta(X)$. It follows that $\varphi^{-1}(U)$ is a K-neighbourhood of A. □

Theorem 5.2. *An open subspace of a normal EP space is also an EP space.*

Proof. Let X be a normal EP space and U an open subset. For each $x \in X$ there is a closed neighbourhood $V(x)$ of x contained in U. Since $V(x)$ is closed and embedded in the normal space X, it is C^*-embedded. By the preceding theorem, $V(x)$ is an EP space. Now apply Theorem 2.4. □

Theorem 5.3. *Suppose X is a Lindelöf EP space and $A \subseteq X$. Then $X \cup \mathrm{cl}_{\beta X}(A)$ is also Lindelöf and EP (and is therefore absolute $\mathcal{CR}$-epic).*

Proof. Let $Y = X \cup \mathrm{cl}_{\beta X}(A)$. We must show that whenever K is a compactification of Y (and hence of X, since X is dense in Y), then every $f \in C^*(Y)$ extends to a neighbourhood of Y in K. Let f be such a function. Since X is absolute $\mathcal{CR}$-epic, the maximum extension of $f|X$ in K (implicit in 2.3) includes a cozeroset $U \supseteq X$ and also includes $\mathrm{cl}_{\beta X}(A)$, so that it includes $W = U \cup \mathrm{cl}_{\beta X}(A)$. Let $\theta : \beta X \longrightarrow K$ be the canonical map and $V = \theta^{-1}(U)$. Clearly V is a cozeroset in βX that contains X. A cozeroset in a compact space is locally compact and Lindelöf and hence CNP ([Barr, *et al.* (2007b), Example 5.1]).

Now let $Z = V \cup \mathrm{cl}_{\beta X}(A)$. Since $A \subseteq X \subseteq V$, it follows from Theorem 4.6 that Z is Lindelöf CNP. Since K is a compactification of $X \cup \mathrm{cl}_{\beta X}(A)$, it follows that θ maps $\mathrm{cl}_{\beta X}(A)$ homeomorphically on its image in K. Thus $\theta^{-1}(U \cup \mathrm{cl}_{\beta X}(A)) = V \cup \mathrm{cl}_{\beta X}(A)$ and it follows from 3.4 that $\theta|Z$ is closed. From 4.7, we see that $W = \theta(Z)$ is CNP. Since it is obviously Lindelöf, it is absolute $\mathcal{CR}$-epic and the map f that we started with extends to an open subset of K that contains W and, *a fortiori*, $X \cup \mathrm{cl}_{\beta X}(A)$. □

Theorem 5.4. *Suppose X is Lindelöf absolute $\mathcal{CR}$-epic and $A \subseteq \beta X$ is either a zero-set or a cozero-set. Then $X \cup A$ is Lindelöf absolute $\mathcal{CR}$-epic.*

Proof. The case of a zeroset is already covered in 2.9. A cozero-set is the union of zero-sets and the result follows from 2.8. □

6. Extensions that satisfy the UEP

Lemma 6.1. *A space X satisfies the UEP if and only if for every closed X-admissible equivalence relation E on βX, there is an E-compatible βX-neighbourhood of X.*

Proof. Suppose X satisfies the UEP. Suppose that E is a closed X-admissible equivalence relation on βX and $K = \beta X/E$ with canonical map $\theta : \beta X \longrightarrow K$. Let V be a K-neighbourhood of X such that every $f \in C^*(X)$ extends to V. Clearly $U = \theta^{-1}(V)$ is a neighbourhood of X. Suppose E is not U-admissible. Then there is a point $p \in U$ and a point $q \in \beta X$ (the latter might or might not belong to U)

such that $p \neq q$, but $(p,q) \in E$. There is an $f \in C(\beta X)$ such that $f(p) = 0$ and $f(q) = 1$. Then it is obvious that $f|X$ has no extension to U.

For the converse, let K be a compactification of X and let $\theta : \beta X \twoheadrightarrow K$ be the canonical map. If E is the kernel pair of θ, then E is X-admissible, hence there is an E-compatible $U \supseteq X$. Since every $f \in C^*(X)$ extends to βX, it follows immediately that every such f extends to U. □

Theorem 6.2. *Suppose X satisfies the UEP and $\{A_\alpha\}$ is a family of zerosets in βX, each disjoint from X. Then if $A = \bigcup A_\alpha$, $X \cup A$ satisfies the UEP.*

Proof. Let K be a compactification of $X \cup A$, and E be the kernel pair of the canonical map $\beta X \twoheadrightarrow K$. Since X satisfies the UEP, there is a βX-open set U containing X such that E is also U-admissible. Theorem 2.7 supplies an open set $V \supseteq A$ such that every A-admissible equivalence relation is V-admissible and hence every $(X \cup A)$-admissible equivalence relation is also $(U \cup V)$-admissible. The conclusion now follows from Lemma 6.1. □

Corollary 6.3. *Suppose X is Lindelöf. Then the conclusion of the preceding theorem is true without the assumption that the A_α are disjoint from X.*

Proof. Let K be a compactification of $X \cup A$. Let E be the kernel pair of the canonical map $\theta : \beta X \twoheadrightarrow K$ and let U be an E-admissible βX-neighbourhood of X. Since X is Lindelöf, Smirnov's Theorem implies that we may take U to be a cozeroset. But then U is locally compact Lindelöf and has the UEP. For each α, $A_\alpha - U$ is then a zeroset and the preceding theorem implies that $U \cup (A - U) = U \cup A$ satisfies the UEP. Thus there is an E-admissible open set $V \supseteq A$. It follows that $U \cup V$ is an E-admissible open set containing $X \cup A$. □

Corollary 6.4. *If X is Lindelöf and satisfies the UEP and U is an open set in βX, then $X \cup U$ satisfies the UEP.*

Proof. An open set in a Tychonoff space is a union of cozerosets and every cozeroset is a (countable) union of zerosets. □

6.1. Levy's question. Ronnie Levy has shown that any Tychonoff space X that is not pseudocompact can be densely and properly embedded into a space Y with the property that for any $p \in Y - X$ there is an $f \in C^*(X)$ that cannot be extended to p, [Levy (1980), Corollary 6.2]. He further raised the question of whether Y could be taken as pseudocompact or even compact. Here we use the preceding corollary to provide a negative answer to this question.

Our counter-example uses the space X of [Barr, *et al.* (2009), 6.1–6.4]. Begin with a countable family $\{X_n\}$. For this example, we will suppose the X_n are locally compact, non-compact, and Lindelöf (and therefore not pseudocompact). The space $\beta(\sum X_n)$ can be viewed as the union of three disjoint subsets: $A =$

$\beta(\sum X_n) - \sum(\beta X_n)$, $B = \sum(X_n)$, and $C = \sum(\beta X_n - X_n)$, which can be pictured:

<table>
<tr><td colspan="2">$A = \beta(\sum X_n) - \sum \beta X_n$</td></tr>
<tr><td>$B = \sum X_n$</td><td>$C = \sum(\beta X_n - X_n)$</td></tr>
</table>

We let $f : \beta(\sum X_n) \longrightarrow [0,1]$ be the continuous extension of the map on $\sum X_n$ that takes every element of X_n to $1/n$. It is clear that f is identically $1/n$ on all of βX_n and hence the zeroset of f is exactly A. We let $X = A \cup B$, which is Lindelöf. Now let $X \longrightarrow Y$ be a dense embedding of X into a pseudocompact space. Then $K = \beta Y$ is a compactification of X which means it is quotient of βX. For each n, the set X_n is clopen in B. If it were clopen in Y, Y could not be pseudocompact. In fact, let $\theta : A \cup B \cup C \longrightarrow K$ be the canonical surjection and $Z = \theta^{-1}(Y)$. A clopen set is C-embedded and, since X_n is not pseudocompact, there is an unbounded function on X_n. Since X_n is open, it cannot be closed, which means that some point of its frontier $\beta X_n - X_n$ meets Z. But for sufficiently large n, every bounded function on $A \cup B$ extends to all of $\beta X_n - X_n$. This is inconsistent with a positive answer to Levy's question.

We note that the same argument shows that we cannot ensure that Levy's space is connected since if there were no point of $\beta X_n - X_n$ in Y, X_n would be a clopen subset of Y.

Example 6.5. We have just seen that the space X above satisfies the UEP (and therefore the CEP), but it was shown in [Barr, *et al.* (2009), 6.1–6.4] that it did not satisfy the CNP. This demonstrates that CNP is stronger than the CEP (and independent of the UEP). *Added in proof:* The one-point Lindelöfization of an uncountable discrete space provides an example of a space that satisfies the CNP and not the UEP.

6.2. Open questions.

1. Are finite products of Lindelöf CNP spaces either Lindelöf or CNP?
2. Since both open and closed images of Lindelöf CNP spaces are Lindelöf CNP, it is natural to ask about general quotient maps. Note that if $X \longrightarrow Y$ is a quotient map that factors into a finite sequence of alternately open and closed maps and X is Lindelöf CNP, so is Y.
3. The results on the Egyptian topology on Q, suggests the question of which countable spaces are absolute $\mathcal{CR}$-epic. Such spaces, if Tychonoff, are necessarily totally disconnected, and we can start by considering the extremely disconnected case. (There is a countable extremally disconnected space in $\beta\mathbf{Q} - \mathbf{Q}$ that has no isolated points and is not absolute $\mathcal{CR}$-epic (see [Barr, *et al.* (2005), 4.4]. On the other hand assuming that there are P-points in $\beta\mathbf{N} - \mathbf{N}$, there is a countable extremally disconnected Lindelöf CNP space without isolated points constructed in [Dow, etal (1988)].)

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