

## THE EXISTENCE OF INJECTIVE EFFACEMENTS

BY  
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**Introduction.** The main result of this paper is little more than a juxtaposition of a remark in [Leroux] (§5, part A), with a theorem of [Banaschewski]. However, the results are individually little known and their juxtaposition not at all.

More explicitly, Leroux remarks, in somewhat different form,

**THEOREM A.** *Let the category  $\mathbf{X}$  be complete and cocomplete, and have a set of cogenerators. Then a necessary and sufficient condition that there exist injective effacements (with respect to the class of regular monomorphisms) is that  $\mathbf{X}$  be coregular and satisfy, in addition, condition COEX 4\*.*

Banaschewski has proved, again in somewhat different form,

**THEOREM B.** *Let the coregular category  $\mathbf{X}$  be complete and cocomplete, satisfy the condition COEX 5\*, and be co-well-powered. Then every object of  $\mathbf{X}$  has an injective hull (with respect to the class of regular monomorphisms) if and only if there is a bound for each  $X \in \mathbf{X}$  on the length of essential sequences emanating from  $X$ .*

These terms are all defined in §1. We remark here that in the abelian case, COEX 4\* and COEX 5\* reduce to AB4 and AB5, respectively.

Since injective effacements will allow us to bound essential sequences and COEX 5\*  $\Rightarrow$  COEX 4\*, we are led easily to the following.

**THEOREM C.** *Let the coregular category  $\mathbf{X}$  be complete and cocomplete, have cogenerators, and satisfy condition COEX 5\*. Then every object of  $\mathbf{X}$  has an injective hull (with respect to the class of regular monomorphisms).*

Note that Theorem C resembles Grothendieck's theorem for abelian categories. The only difference is that Grothendieck was able to bound essential sequences in the presence of generators, whereas here we require cogenerators. If it could be done using only generators, a very straightforward proof of the embedding theorems in [Barr] would result.

We will consistently use the notation  $\mapsto$  and  $\twoheadrightarrow$  to denote regular monomorphisms and epimorphisms, respectively.

We use  $X +_Z Y$  to denote the pushout of the square

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \downarrow & \\ & X & \end{array}$$

and  $\Pi_X Y_i$  to denote the fibred coproduct of a family  $\{X \rightarrow Y_i\}$ .

1. **Definitions.** A map in a category is called a *regular monomorphism* if it is the equalizer of a pair of maps. If its cokernel pair exists, it is the equalizer of them. A map is called a *universal regular monomorphism* if its pushout with every map with the same domain exists and if the map across from it in such a pushout square is always a regular monomorphism. A category is called *coregular* if every regular monomorphism is universal. In that case (and provided equalizers and cokernel pairs exist), every map has a canonical factorization as an epimorphism followed by a regular monomorphism (see [Barr], I.(2.3)). A coregular category is said to satisfy condition *COEX 4\** provided all coproducts exist and if, given a family of regular monomorphisms  $\{X_i \rightarrow Y_i\}$ , the induced map  $\Pi X_i \rightarrow \Pi Y_i$  is one as well. The category  $\mathbf{X}$  is said to satisfy condition *COEX 5\** if colimits exist and if, given any filtered category  $\mathbf{I}$ , functors  $F, G: \mathbf{I} \rightarrow \mathbf{X}$ , and a natural transformation  $F \rightarrow G$  such that  $F_i \rightarrow G_i$  is a regular monomorphism for all  $i \in \mathbf{I}$ , then the induced map  $\text{colim } F \rightarrow \text{colim } G$  is one as well. A regular monomorphism  $X \rightarrow I$  is called an *injective effacement* if any diagram

$$\begin{array}{ccc} X & \longrightarrow & I \\ \uparrow & & \\ Y & \longrightarrow & Z \end{array}$$

can be completed to a commutative square by a map  $Z \rightarrow I$ . An object  $I$  is easily seen to be *injective* if and only if its identity map is an injective effacement. A regular monomorphism  $X \rightarrow Y$  is called *essential* if for any  $Y \rightarrow Z$ , if the composite  $X \rightarrow Y \rightarrow Z$  is a regular monomorphism, so is  $Y \rightarrow Z$ . If  $X \rightarrow I$  is essential and  $I$  is injective, then it is well known (and obvious) that  $I$  is an *injective hull* of  $X$ , meaning that  $I$  is a regular subobject of every injective container of  $X$ . A *cogenerating set* in  $\mathbf{X}$  is a set  $\Gamma$  of objects in  $\mathbf{X}$  such that for every epimorphism  $X \twoheadrightarrow Y$ , not an isomorphism, there is a  $G \in \Gamma$  and a map  $X \rightarrow G$  which does not factor through  $Y$ .

2. **Proof of Theorem A.** Throughout this section  $\mathbf{X}$  denotes a category which satisfies the hypotheses of Theorem A.

**PROPOSITION 1.** *Let  $\mathbf{X}$  be coregular and satisfy COEX 4\* and let  $X \rightarrow X_s$ ,  $s \in S$  be a family of regular monomorphisms of  $\mathbf{X}$ . Then the canonical map  $X \rightarrow \prod_{s \in S} X_s$  is a regular monomorphism as well.*

**Proof.** First, by COEX 4\*,

$$\prod X \mapsto \prod X_s.$$

Second, the square

$$\begin{array}{ccc} \prod X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \prod X_s & \longrightarrow & \prod_X X_s \end{array}$$

in which the top horizontal map is the codiagonal is a pushout and the result follows from coregularity.

Now we can prove the forward implication of Theorem A. Consider the set  $R$  of all possible irredundant families

$$\{f_s: X \rightarrow G_s \mid s \in S_r, r \in R, G_s \in \Gamma\},$$

for which the induced map

$$\langle f_s \rangle: X \rightarrow \prod_{s \in S_r} G_s = H_r$$

is a regular monomorphism. Up to change of index sets there is only a set of such families—and a set of such  $H_r$ —since each is determined by choosing a subset  $\Sigma \subset \Gamma$  and for each  $G \in \Sigma$  a map  $X \rightarrow G$ . Finally, we let

$$I = \prod_X H_r.$$

From the preceding proposition we see that the natural map  $X \mapsto I$  is a regular monomorphism. Now let  $X \mapsto Y$  be any regular monomorphism. Choose a regular monomorphism  $Y \mapsto \prod_{t \in T} G_t$  with  $G_t \in \Gamma$  for all  $t \in T$ . Then evidently  $X \mapsto \prod_{t \in T} G_t$ , but the family of  $X \rightarrow G_t$  might be redundant. Then choose a subset  $T_0 \subset T$  so that  $\{X \rightarrow G_t\}, t \in T_0$  is irredundant but representative of all the maps  $X \rightarrow Y \rightarrow G_t$  and then  $T_0$  is one of the  $S_r$ . Thus we have the diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ I & \longrightarrow & \prod_{t \in T_0} G_t \end{array}$$

$\prod_{t \in T} G_t$

in which the bottom map is the coproduct injection. Thus any  $X \mapsto Y$  gives a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow & \swarrow \\ & I & \end{array}$$

If, more generally, we have a diagram

$$\begin{array}{ccc} X & \longrightarrow & I \\ \uparrow & & \\ Y & \longrightarrow & Z \end{array}$$

we complete it by applying the above argument to  $X \mapsto X +_Y Z$ .

The converse part of Theorem A is easy and is left to the reader. Note that in any commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow & \swarrow \\ & & I \end{array}$$

the map  $X \mapsto Y$  will have to be a regular monomorphism.

**3. Proof of Theorem B.** Throughout this section we suppose that  $X$  satisfies the hypotheses of Theorem B.

**PROPOSITION 2.** *Given any regular monomorphism  $X \mapsto Y$ , there is an epimorphism  $Y \twoheadrightarrow Z$  such that the composite  $X \rightarrow Z$  is an essential monomorphism.*

**Proof.** If  $X \mapsto Y$  is already essential, we are done. Otherwise there is a map  $Y \rightarrow Z_1$ , not a regular monomorphism, for which the composition  $X \mapsto Y_1$  is one. Let  $Y \twoheadrightarrow Y_1 \mapsto Z_1$  be the epimorphism, regular monomorphism factorization of this map. Then clearly we have

$$\begin{array}{ccc} X & \longrightarrow & Y \\ & \searrow & \swarrow \\ & & Y_1 \end{array}$$

If  $X \mapsto Y_1$  is still not essential, we can continue to have  $Y_1 \twoheadrightarrow Y_2$  with  $X \mapsto Y_2$ . Either the process terminates at a finite ordinal or we get a sequence

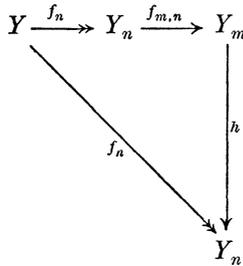
$$X \mapsto Y \twoheadrightarrow Y_1 \twoheadrightarrow Y_2 \twoheadrightarrow \cdots \twoheadrightarrow Y_n \twoheadrightarrow \cdots$$

such that every  $X \mapsto Y_n$ . Then by COEX 5\*, we have

$$X \mapsto Y_\omega = \text{colim } Y_n.$$

On the other hand, by general principles,  $Y_n \twoheadrightarrow Y_\omega$  for all  $n \in \omega$ . By continuing this way, we either find an ordinal  $n$  such that  $X \mapsto Y_n$  is essential or else continue to define for all ordinals  $n$  objects  $Y_n$  and maps  $f_n: Y \twoheadrightarrow Y_n, f_{m,n}: Y_n \twoheadrightarrow Y_m$  for  $m \geq n$  satisfying the obvious coherence. Since  $Y$  is co-well-powered there must exist  $m > n$  such that  $f_n$  and  $f_m$  represent the same quotient object of  $Y$ , this means we

have a commutative diagram



in which  $h$  is an isomorphism. But the epimorphism  $f_n$  can be cancelled from  $f_n = hf_{m,n}f_n$  to give  $hf_{m,n} = Y_n$  and  $f_{m,n} = h^{-1}$  is an isomorphism. But then  $f_{m,n} = f_{m,n+1} \cdot f_{n+1,n}$  is a regular monomorphism which implies (given the uniqueness of the factorization) that  $f_{n+1,n}$  is and hence  $f_{n+1,n}$  is an isomorphism also which contradicts its construction.

REMARKS. This amounts to verifying the dual of (P3) in [Banaschewski]. I am indebted to the referee for pointing out a gap in my original proof of this proposition.

Now we are ready to prove the forward implication in Theorem B. Let  $X$  be an object. If  $X$  has an essential extension, choose one, say  $X \mapsto X_1$ . If  $X_1$  has an essential extension, choose one  $X_1 \mapsto X_2$ . Either the process terminates or we get a sequence, defined for all finite ordinals,

$$X \mapsto X_1 \mapsto X_2 \mapsto \dots \mapsto X_n \mapsto \dots$$

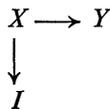
of essential extensions. Then by COEX 5\* the induced map

$$X \mapsto \text{colim } X_n$$

is a regular monomorphism. Apply the preceding proposition to get

$$X \mapsto \text{colim } X \twoheadrightarrow X_\omega,$$

for which the composite is a regular monomorphism. In this way we build a transfinite sequence of essential extensions which, by hypothesis, must terminate in an object  $I$  which has no essential extension. Let  $I \mapsto Y$  be a regular monomorphism. Apply the proposition to get  $I \mapsto Y \twoheadrightarrow Z$  with  $I \mapsto Z$  an essential extension. But then  $I \mapsto Z$  is an isomorphism. Hence every regular monomorphism with domain  $I$  is split. Now consider a diagram



Then  $I \mapsto Y +_X I$  is a split monomorphism and the composite  $Y \mapsto Y +_X I \mapsto I$  by the splitting map gives a map  $Y \mapsto I$  which makes the triangle commute. Thus

every object has an essential monomorphism into an injective, evidently an injective envelope.

**4. Proof of Theorem C.** Throughout this section we suppose that  $X$  satisfies the hypotheses of Theorem C.

**PROPOSITION 3.** *The category  $\mathbf{X}$  is co-well-powered.*

**Proof.** The quotients  $X \twoheadrightarrow Y$  of  $X$  are determined by those maps  $X \rightarrow G$ ,  $G \in \Gamma$  which factor through  $Y$  (see [Barr], II. (2.1)).

**PROPOSITION 4.** *Each object  $X \in \mathbf{X}$  has only a set of regular subobjects.*

**Proof.** A regular subobject  $Y \rightarrow X$  is determined by its cokernel pair  $X +_Y X$ . This is a quotient of  $X + X$  and there are only a set of those.

Now we can prove that a category which satisfies the hypotheses of Theorem C satisfies those of Theorem B. The only thing to worry about is the possibility of a sequence of proper essential extensions

$$X \mapsto X_1 \mapsto X_2 \mapsto \cdots \mapsto X_n \mapsto \cdots$$

extending through all ordinals. But in that case, choose an injective effacement  $X \mapsto I$ . Let  $N$  be the cardinal of the set of regular subobjects of  $I$  and  $M$  an ordinal of cardinality  $> N$ . Then  $X_M$  has more than  $N$  regular subobjects—all the  $X_n$ ,  $n < M$ , at least. Now there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad} & X_M \\ & \searrow & \swarrow \\ & & I \end{array}$$

and since  $X \mapsto X_M$  is essential, the map  $X_M \mapsto I$  is a monomorphism as well. But then  $I$  will have more than  $N$  subobjects, a contradiction.

**REMARK.** The referee has observed that in order to find an injective hull of a single  $X$ , it is not necessary to suppose all coproducts exist; only those up to  $\sum_{G \in \Gamma} \#(X, G)$  are required.

#### REFERENCES

1. B. Banaschewski, *Projective covers in categories of topological spaces and topological algebras*, "Proceedings of the Topology Conference", Kanpur, 1968, (to appear).
2. M. Barr, *Exact categories*, in "Exact Categories and Categories of Sheaves", Lecture Notes in Mathematics **236** (1971), 1–120.
3. P. Leroux, *Sur les structures d'effacement*, Math. Z. **121** (1971), 329–340.

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