HSP SUBCATEGORIES OF EILENBERG-MOORE ALGEBRAS

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Abstract. Given a triple $\mathbf{T}$ on a complete category $\mathscr{C}$ and a factorization system $\mathscr{E}/\mathscr{M}$ on the category of algebras, we show there is a 1-1 correspondence between full subcategories of the category of algebras that are closed under U-split epimorphisms, products, and $\mathscr{M}$-subobjects and triple morphisms $\mathbf{T} \longrightarrow \mathbf{S}$ for which the induced natural transformation between free functors belongs to $\mathscr{E}$.

1. Introduction

A celebrated theorem of Birkhoff's says that a subcategory of the category of algebras for a signature is given by a set of equations (which is to say, an equivalence relation on the free algebras) if and only if it is closed under homomorphic images, subobjects, and products (HSP). It is easy to generalize this to any equational category, so that a subcategory of an equational category is given by further equations if and only if it is closed under HSP.

When one tries to extend this theorem to arbitrary categories, one immediately runs into a number of difficulties. What is a homomorphic image, what is a subobject and can the theorem be proved. As we will see, the homomorphic image seems to be a bit misleading. Even Birkhoff's theorem seems to be true only modulo the axiom of choice, which is obviously not available. Then the proof itself seems to depend on the fact that the product of surjections is a surjection, which is also a form of AC. At any rate, I present here a theorem that holds in some generality and does reduce to Birkhoff's theorem over sets. It is based on the choice of a factorization system in the category of algebras, which may or may not be induced by one in the base category. As a matter of fact, even over sets, there are examples that are not examples of Birkhoff's theorem.

It would be nice if the theorem could be stated in terms of a factorization system in the base category. But over sets, for example, there is no factorization system that corresponds in any way to epimorphisms in the categories of monoids or of rings.

When I first announced these results, I received a number of comments from people that the class $\mathscr{E}$ did not have to consist entirely of epimorphisms and that completeness was unnecessary. Some of them also suggested that the triple would have to preserve the class $\mathscr{E}$ (although with the class $\mathscr{E}$ being in the category of algebras, perhaps what they meant was the cotriple preserved it). As will be seen below, the hypothesis that $\mathscr{E}$ is included in the epics is crucial to going from triples to HSP subcategories and completeness

This research has been supported by the NSERC of Canada
2000 Mathematics Subject Classification: 18C05, 18A20, 18A40.
Key words and phrases: Birkhoff subcategories, factorizations, reflective subcategories.

is crucial to going the other way. So the theorem could be stated with two different sets of hypotheses for the two directions. But this seems to miss the main the point of Birkhoff's theorem which is the 1-1 correspondence between HSP subcategories and certain kinds of morphisms of triples. Moreover, the hypothesis that every map in $\mathscr{E}$ is epic is much less exigent than that the cotriple preserve the class. The former hypothesis is normally true, while the latter is not, except in the category of sets and even there it requires either the axiom of choice or a finitary theory.

2. Factorization systems

We collect here a number of facts about factorization systems that we will need. They are all well-known. This is not the most general kind of factorization and some of the hypotheses could perhaps be relaxed a bit, but at the price of a considerable complication.

We assume a complete category $\mathscr{C}$ and two classes of morphisms $\mathscr{E}$ and $\mathscr{M}$ such that

FS1. $\mathscr{E} \cap \mathscr{M}$ consists of the isomorphisms;

FS2. $\mathscr{E}$ and $\mathscr{M}$ are each closed under composition;

FS3. Every map in $\mathscr{E}$ is an epimorphism and every map in $\mathscr{M}$ is a monomorphism;

FS4. Every map in $f \in \mathscr{C}$ factors $f = m \circ e$ with $m \in \mathscr{M}$ and $e \in \mathscr{E}$;

FS5. Given any commutative diagram

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

with $e \in \mathscr{E}$ and $m \in \mathscr{M}$ there is a unique arrow $\operatorname{cod}(e) \longrightarrow \operatorname{dom}(m)$ making both triangles commute. This is called the diagonal fill-in.

FS6. For any object C of $\mathscr{C}$, there is, up to isomorphism, only a set of arrows of $\mathscr{E}$ whose domain is C.

An epimorphism in a category is called extremal if it does not factor through any proper subobject of its codomain and dually, a monomorphism is called extremal if it does not factor through any proper epimorphism from its codomain.

2.1. Proposition. *The following facts about factorization systems are well-known and we omit the proofs.*

1. *Every extremal epimorphism is in $\mathscr{E}$ and dually, every extremal monomorphism is in $\mathscr{M}$.*

2. *If*

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "g"'] & \cdot \end{tikzcd}$$

is a pullback and $g \in \mathscr{M}$ then $f \in \mathscr{M}$; dually if the square is a pushout and $f \in \mathscr{E}$, then $g \in \mathscr{E}$.

3. *The factorization in FS–4 is unique up to isomorphism.*

4. *If m is an arrow of $\mathscr{C}$ that has the diagonal fill-in property with respect to every $e \in \mathscr{E}$ then $f \in \mathscr{M}$; dually if e has the diagonal fill-in property with respect to every $m \in \mathscr{M}$, then $e \in \mathscr{E}$.*

3. Morphism of triples

Let $\mathscr{C}$ be a category and $\mathbf{T} = (T, \eta, \mu)$ be a triple on $\mathscr{C}$. If it is necessary to distinguish η and μ from the unit and multiplication of another triple, we will write $\eta^{\mathbf{T}}$ and $\mu^{\mathbf{T}}$.

If $\mathbf{S} = (S, \eta^{\mathbf{S}}, \mu^{\mathbf{S}})$ is another triple on $\mathscr{C}$, a morphism $\alpha\colon \mathbf{T} \longrightarrow \mathbf{S}$ is a natural transformation of $T \longrightarrow S$ such that the diagrams

$$\begin{tikzcd} & \mathrm{Id} \arrow[dl, "\eta^{\mathbf{T}}"'] \arrow[dr, "\eta^{\mathbf{S}}"] & \\ T \arrow[rr, "\alpha"'] & & S \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "T\alpha"] \arrow[d, "\mu^{\mathbf{T}}"'] & TS \arrow[r, "\alpha S"] & S^2 \arrow[d, "\mu^{\mathbf{S}}"] \\ T \arrow[rr, "\alpha"'] & & S \end{tikzcd}$$

commute.

3.1. Proposition. *Suppose that $\alpha\colon \mathbf{T} \longrightarrow \mathbf{S}$ is a morphism of triples. Then α induces a functor $\mathscr{C}^{\alpha}\colon \mathscr{C}^{\mathbf{S}} \longrightarrow \mathscr{C}^{\mathbf{T}}$ that takes the $\mathbf{S}$-algebra $(X, \widehat{x})$ to the $\mathbf{T}$-algebra $(X, x = \widehat{x} \circ \alpha X)$.*

Proof. On objects, this can be read from the diagrams

$$\begin{tikzcd} T^2X \arrow[r, "T\alpha X"] \arrow[dd, "\mu^{\mathbf{T}} X"'] & TSX \arrow[r, "T\widehat{x}"] \arrow[d, "\alpha SX"'] & TX \arrow[d, "\alpha X"] \\ & S^2X \arrow[r, "S\widehat{x}"] \arrow[d, "\mu^{\mathbf{S}} X"'] & SX \arrow[d, "\widehat{x}"] \\ TX \arrow[r, "\alpha X"'] & SX \arrow[r, "\widehat{x}"'] & X \end{tikzcd} \qquad \begin{tikzcd} & TX \arrow[d, "\alpha X"] \\ X \arrow[ur, "\eta^{\mathbf{T}} X"] \arrow[r, "\eta^{\mathbf{S}} X"] \arrow[dr, "\mathrm{id}"'] & SX \arrow[d, "\widehat{x}"] \\ & X \end{tikzcd}$$

If $f\colon (X,\widehat{x}) \longrightarrow (Y,\widehat{y})$ is a morphism of $\mathbf{S}$-algebras, then the diagram

$$\begin{tikzcd} TX \arrow[r, "\alpha X"] \arrow[d, "Tf"'] & SX \arrow[r, "\widehat{x}"] \arrow[d, "Sf"] & X \arrow[d, "f"] \\ TY \arrow[r, "\alpha Y"'] & SY \arrow[r, "\widehat{y}"'] & Y \end{tikzcd}$$

commutes so that f is also a morphism of $\mathbf{T}$-algebras. ∎

Let $F^{\mathbf{T}}\colon \mathscr{C} \longrightarrow \mathscr{C}^{\mathbf{T}}$ and $F^{\mathbf{S}}\colon \mathscr{C} \longrightarrow \mathscr{C}^{\mathbf{S}}$ denote the free algebra functors given by $F^{\mathbf{T}}X = (TX, \mu^{\mathbf{T}}X)$ and $F^{\mathbf{S}}X = (SX, \mu^{\mathbf{S}}X)$. It follows from the preceding proposition that $(F^{\mathbf{S}}X, \mu^{\mathbf{S}}X \circ \alpha SX)$ is a $\mathbf{T}$-algebra for any $X \in \mathrm{Ob}(\mathscr{C})$. Since $\widehat{x}\colon (SX, \mu^{\mathbf{S}}X) \longrightarrow (X, \widehat{x})$ is a morphism of $\mathbf{S}$-algebras, it also follows that $\widehat{x}\colon (SX, \mu^{\mathbf{S}}X \circ \alpha SX) \longrightarrow (X, \widehat{x} \circ \alpha X)$ is a morphism of $\mathbf{T}$-algebras.

3.2. Proposition. *The arrow $\alpha X\colon (TX, \mu^{\mathbf{T}}X) \longrightarrow (SX, \mu^{\mathbf{S}} \circ \alpha SX)$ is a morphism of $\mathbf{T}$-algebras for any object $X \in \mathrm{Ob}(\mathscr{C})$.*

Proof. This is immediate from the commutation of

$$\begin{tikzcd} T^2 \arrow[rr, "\mu^{\mathbf{T}}"] \arrow[d, "T\alpha"'] & & T \arrow[d, "\alpha"] \\ TS \arrow[r, "\alpha S"] & S^2 \arrow[r, "\mu^{\mathbf{S}}"] & S \end{tikzcd}$$

which is part of the definition of α. ∎

3.3. Proposition. *Suppose that $\alpha\colon F^{\mathbf{T}}X \longrightarrow F^{\mathbf{S}}X$ is an epimorphism in $\mathscr{C}^{\mathbf{T}}$ for every $X \in \mathrm{Ob}(\mathscr{C})$. Then $\mathscr{C}^{\alpha}$ is full and faithful.*

Proof. If $(X,\widehat{x})$ and $(Y,\widehat{y})$ are $\mathbf{S}$-algebras and $f\colon (X, x = \widehat{x} \circ \alpha X) \longrightarrow (Y, y = \widehat{y} \circ \alpha Y)$ is a morphism of $\mathbf{T}$-algebras, then the outer and left hand squares of

$$\begin{tikzcd} F^{\mathbf{T}}X \arrow[r, "\alpha X"] \arrow[d, "F^{\mathbf{T}}f"'] & F^{\mathbf{S}}X \arrow[r, "\widehat{x}"] \arrow[d, "F^{\mathbf{S}}f"'] & (X,x) \arrow[d, "f"] \\ F^{\mathbf{T}}Y \arrow[r, "\alpha Y"'] & F^{\mathbf{S}}Y \arrow[r, "\widehat{y}"'] & (Y,y) \end{tikzcd}$$

Since αX is an epimorphism of $\mathbf{T}$-algebras, so does the right hand square. Applying the underlying functor, we conclude that

$$\begin{tikzcd} TX \arrow[r, "\widehat{x}"] \arrow[d, "Tf"'] & X \arrow[d, "f"] \\ TY \arrow[r, "\widehat{y}"'] & Y \end{tikzcd}$$

commutes. ∎

The converse of this proposition is false. For example, the category of $\mathbf{Q}$-modules (rational vector spaces) is a full subcategory of **Ab**, but the map $\mathbf{Z} \longrightarrow \mathbf{Q}$ between the free objects on one generator is certainly not epic.

The following theorem identifies those $\mathbf{S}$-algebras that are $\mathbf{T}$-algebras. It is interesting on its own and is also used in the proof of the main theorem. Since the category of $\mathbf{S}$-algebras is a full subcategory of the category of $\mathbf{T}$-algebras, it makes sense to ask if a $\mathbf{T}$-algebra is an $\mathbf{S}$-algebra.

3.4. THEOREM. *Suppose that $\alpha\colon \mathbf{T} \longrightarrow \mathbf{S}$ is a morphism of triples such that the induced $\alpha\colon F^{\mathbf{T}} \longrightarrow F^{\mathbf{S}}$ is an epimorphism in the category of $\mathbf{T}$-algebras. Then the $\mathbf{T}$-algebra (X, x) is an S-algebra if and only if x can be factored in the category of $\mathbf{T}$-algebras as $\widehat{x} \circ \alpha X$.*

PROOF. The crucial thing here is that $x\colon F^{\mathbf{T}}X \longrightarrow (X, x)$ is a morphism of $\mathbf{T}$-algebras, in contrast with the situation with n-ary operations which are not, in most cases homomorphisms of algebras. The hypothesis is that there is a morphism $\widehat{x}\colon F^{\mathbf{S}}X \longrightarrow (X, x)$ in the category of $\mathbf{T}$-algebras such that $\widehat{x} \circ \alpha X = x$. This means that the outer square of

$$\begin{tikzcd} TSX \arrow[r, "\alpha SX"] \arrow[d, "T\widehat{x}"'] & S^2X \arrow[r, "\mu^{\mathbf{S}}X"] \arrow[d, "S\widehat{x}"] & SX \arrow[d, "\widehat{x}"] \\ TX \arrow[r, "\alpha X"'] & SX \arrow[r, "\widehat{x}"'] & X \end{tikzcd}$$

commutes. This square underlies the square in $\mathscr{C}^{\mathbf{T}}$

$$\begin{tikzcd} F^{\mathbf{T}}SX \arrow[r, "\alpha SX"] \arrow[d, "F^{\mathbf{T}}\widehat{x}"'] & F^{\mathbf{S}}SX \arrow[r, "\mu^{\mathbf{S}}X"] \arrow[d, "F^{\mathbf{S}}\widehat{x}"] & F^{\mathbf{S}}X \arrow[d, "\widehat{x}"] \\ F^{\mathbf{T}}X \arrow[r, "\alpha X"'] & F^{\mathbf{S}}X \arrow[r, "\widehat{x}"'] & (X, x) \end{tikzcd}$$

which implies, given that αSX is epic in $\mathscr{C}^{\mathbf{T}}$, that the right hand square commutes. Apply the underlying functor it follows that the right hand square of the first diagram also commutes. This shows that $\widehat{x}$ commutes properly with $\mu^{\mathbf{S}}$. The unitary law follows immediately from the commutativity of

$$\begin{tikzcd}
 & TX \arrow[d, "\alpha X"] \\
X \arrow[ur, "\eta^{\mathbf{T}}X"] \arrow[r, "\eta^{\mathbf{S}}X"] \arrow[dr, "\mathrm{id}"'] & SX \arrow[d, "\widehat{x}"] \\
 & X
\end{tikzcd}$$ ■

Notice, incidentally, that if αSX should happen to be epic *in* $\mathscr{C}$, then the factorization of x need only take place in $\mathscr{C}$. This happens over sets when $\mathscr{E}$ is the class of regular epimorphisms. Of course, the existence of such a factorization of x in the underlying category will force the existence of a factorization (usually different) in $\mathscr{C}^{\mathbf{T}}$.

4. The main theorem

4.1. Theorem. *Suppose that* $\mathbf{T}$ *is a triple on the complete category* $\mathscr{C}$ *and* $\mathscr{E}/\mathscr{M}$ *a factorization system on* $\mathscr{C}^{\mathbf{T}}$ *as described in Section 2. Then there is a 1-1 correspondence between morphisms* $\alpha\colon \mathbf{T} \longrightarrow \mathbf{S}$ *of triples for which every instance of the induced* $\alpha\colon F^{\mathbf{T}} \longrightarrow F^{\mathbf{S}}$ *lies in* $\mathscr{E}$ *and full subcategories* $\mathscr{B} \subseteq \mathscr{C}^{\mathbf{T}}$ *that are closed under* U*-split quotients,* $\mathscr{M}$*-subobjects, and products.*

Proof. Suppose that $\alpha\colon \mathbf{T} \longrightarrow \mathbf{S}$ is such that every instance lies in $\mathscr{E}$. Suppose that $(X, \widehat{x})$ is an $\mathbf{S}$-algebra and that $f\colon (X, x = \widehat{X} \circ \alpha x) \longrightarrow (Y, y)$ is a U-split epimorphism $\mathbf{T}$-algebras. Let $u\colon Y \longrightarrow X$ be a right inverse of f in $\mathscr{C}$. Then $Su = F^{\mathbf{S}}u$ and $Tu = F^{\mathbf{T}}u$ are maps in $\mathscr{C}^{\mathbf{T}}$. Define $\widehat{y} = f \circ \widehat{x} \circ Su\colon SY \longrightarrow Y$. Then

$$\widehat{y} \circ \alpha Y = f \circ \widehat{x} \circ Su \circ \alpha Y = f \circ \widehat{x} \circ \alpha X \circ Tu = f \circ x \circ Tu = y \circ Tf \circ Tu = y$$

and it follows from Theorem 3.4 that $(Y, \widehat{y})$ is an $\mathbf{S}$-algebra.

Suppose that $(X, \widehat{x})$ is an $\mathbf{S}$-algebra and that (Y, y) is an $\mathscr{M}$-subalgebra of $(X, x = \alpha X \circ \widehat{x})$. The diagram

$$\begin{tikzcd}
F^{\mathbf{T}}Y \arrow[r, two heads, "\alpha Y"] \arrow[d, "y"'] & F^{\mathbf{S}}Y \arrow[d, "\widehat{x} \circ F^{\mathbf{S}}m"] \\
(Y, y) \arrow[r, tail, "m"'] & (X, x)
\end{tikzcd}$$

commutes because

$$m \circ y = x \circ F^{\mathbf{T}}m = \widehat{x} \circ \alpha X \circ F^{\mathbf{T}}m = \widehat{x} \circ F^{\mathbf{S}}m \circ \alpha Y$$

and hence the factorization system gives a map $\widehat{y}: F^{\mathbf{S}}Y \longrightarrow Y$ for which $\widehat{y} \circ \alpha Y = y$. It follows from Theorem 3.4 that $(Y, \widehat{Y})$ is an $\mathbf{S}$-algebra.

Finally, if $(X_i, \widehat{x}_i)$ is a collection of $\mathbf{S}$-algebras, then it is evident that $\prod(X_i, \widehat{x}_i)$ is also an $\mathbf{S}$-algebra since the underlying functor creates limits.

This proves the "only if" part of the statement.

Conversely, suppose that $I: \mathscr{B} \longrightarrow \mathscr{C}^{\mathbf{T}}$ is a full inclusion and that $\mathscr{B}$ is closed in $\mathscr{C}^{\mathbf{T}}$ under U-split epics, $\mathscr{M}$-subobjects, and products. Given a $\mathbf{T}$-algebra (X, x), any morphism $f: (X, x) \longrightarrow (Y, y)$ with codomain in $\mathscr{B}$ has an $\mathscr{E}/\mathscr{M}$ factorization $(X, x) \twoheadrightarrow (Z, z) \rightarrowtail (Y, y)$ and the hypothesis implies that $(Z, z) \in \mathrm{Ob}(\mathscr{B})$. Let $e_i: (X, x) \longrightarrow (Y_i, y_i)$ range over a complete set of representatives for morphisms in $\mathscr{E}$, whose codomains are objects of $\mathscr{B}$. Their product lies in $\mathscr{B}$ and so does the $\mathscr{M}$-subobject gotten from the $\mathscr{E}/\mathscr{M}$-factorization of the induced map $\langle e_i \rangle: (X, x) \longrightarrow \prod(Y_i, y_i)$. It is clear that this gives a left adjoint $\widehat{I}$ for I and then $\widehat{I} \circ F$ is a left adjoint to $U \circ I$. It is also obvious that every instance of the adjunction $\mathrm{Id} \longrightarrow I \circ \widehat{I}$ belongs to $\mathscr{E}$. The only thing left is to show that $U \circ I$ is tripleable. Since I is full and faithful and U reflects isomorphisms, so does $U \circ I$. So suppose that

$$B' \rightrightarrows B$$

is a $U \circ I$-split fork. Then with U tripleable, there is a morphism $IB \longrightarrow C$ in $\mathscr{C}^{\mathbf{T}}$ so that

$$IB' \rightrightarrows IB \longrightarrow C$$

is a U-split coequalizer diagram. But the hypothesis that $\mathscr{B}$ be closed under U-split epics implies that $C = IB''$ and it is easy to see that $B' \rightrightarrows B \longrightarrow B''$ is a coequalizer in $\mathscr{B}$. ∎

5. Examples

The most obvious examples are over sets, where, by taking $\mathscr{E}$ as the class of regular epics, which are the same as surjections, we recover Birkhoff's original theorem. But even over the category of sets, there are other possibilities. For example, if we take the epic/regular monic factorization in the category of monoids, it is easy to see that the full subcategory of groups is closed under surjective image, regular subobjects, and products and it follows that the map from a monoid to its associated group is always epic.

A similar example is that of strongly regular rings in the category of rings. A ring R is strongly regular if for all $a \in R$ there is a $b \in R$ with $a^2b = a$. It can be shown that this implies that there is a unique b such that $a^2b = aba = ba^2 = a$ and $b^2a = bab = ab^2 = b$ and this unique choice is preserved by ring homomorphisms. The full subcategory is closed under U-split coequalizers, extremal subobjects, and products. Thus it is an epi-reflective subcategory.

It is well-known that the category of (small) categories is tripleable over the category of graphs. The full subcategory of pre-orders is closed under surjections, arbitrary subobjects, and products and quotients for which the underlying graph morphism splits. It is not sufficient that the set of nodes and the set of arrows split as sets; the splitting is required to preserve domain and codomain. Here is a relevant example: the regular epimorphism in the category of categories gotten by identifying $\operatorname{cod} f$ with $\operatorname{cod} g$ in the pre-order

$$\begin{tikzcd}[row sep=tiny] & \cdot \\ \cdot \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & \cdot \end{tikzcd}$$

produces the category

$$\begin{tikzcd} \cdot \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \cdot \end{tikzcd}$$

which is not a pre-order because there are two maps between the two ends. Yet in the category of graphs, this regular epimorphism is surjective on both objects and arrows and those both are split epimorphisms in the category of sets, but not simultaneously in the category of graphs.

Let $\mathscr{C}$ be the category of completely regular topological spaces. The category of topological groups is easily shown to be tripleable over $\mathscr{C}$. There is a factorization system in which $\mathscr{E}$ consists of the maps with dense image and $\mathscr{M}$ of closed inclusions. Compact groups are clearly closed under U-split epics, $\mathscr{M}$-subobjects, and products and hence the adjoint is $\mathscr{E}$-reflective.

Another factorization system on topological groups consists of surjections and injections. The commutative topological groups are closed under U-split surjections, injective homomorphisms and products. Hence the full subcategory of topological abelian groups is a surjective-reflective subcategory of topological groups.

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