Coalgebras Over a Commutative Ring

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INTRODUCTION

By a *coalgebra* over the commutative ring K or a *K-coalgebra*, we understand a cocommutative, coassociative K-coalgebra with counit. More explicitly we mean a K-module C equipped with maps

$$\delta: C \to C \otimes_K C,$$
$$\epsilon: C \to K,$$

subject to the requirement (where we write $\otimes$ for $\otimes_K$) that the following diagrams commute:

$$C \xrightarrow{\ \delta\ } C \otimes C$$

where the vertical arrow is the one which switches the factors;

$$
\begin{array}{ccc}
C & \xrightarrow{\ \ \delta\ \ } & C \otimes C \\
\downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle C \otimes \delta} \\
C \otimes C & \xrightarrow{\ \delta \otimes C\ } & C \otimes C \otimes C,
\end{array}
$$

where we have, as usual, eliminated from this diagram an isomorphism between $C \otimes (C \otimes C)$ and $(C \otimes C) \otimes C$;

$$
C
$$
$$
C \otimes K \xleftarrow{\ C \otimes \epsilon\ } C \otimes C \xrightarrow{\ \epsilon \otimes C\ } K \otimes C,
$$

where the diagonal arrows are the canonical isomorphisms.

600

A homomorphism between K-coalgebras is a K-module homomorphism which commutes with the structure maps ϵ and δ in an obvious way. With these definitions we get a category which we denote by K-Coalg. We denote by K-Mod the category of K-modules and homomorphisms. All modules will be considered simultaneously right and left modules with the same operations on both sides. This category of K-coalgebras has been studied extensively in [Sweedler], for the case in which K is a field.

It is a cartesian closed category, there are cofree coalgebras, many nice categories are enriched over it, etc. The proofs there rely heavily on the duality of finite dimensional vector spaces and to some extent on the partial duality in the infinite dimensional case. Here we establish many of the same results when K is an arbitrary commutative ring.

I do not give here any direct applications of these results, but I would like to mention, by way of example, one of the possibilities. We show that the category of associative K-algebras is enriched over K-Coalg. This means that if A and B are algebras there is a coalgebra $[A, B]$ representing the enriched Hom. This coalgebra includes all the ordinary homomorphisms as its group-like elements (those p for which $\delta(p) = p \otimes p$); for a given homomorphism p it includes all the p derivations as its p-primitive elements (those d for which $\delta(d) = p \otimes d + d \otimes p$) and similarly for higher derivations. The existence of cofree coalgebras allows a cosimplicial injective resolution of $[A, B]$. Cohomotopy and cohomology theories can now be constructed by applying various functors to this resolution.

From now on, we will usually omit explicit mention of the ring K, speaking of modules, coalgebras, linear maps, etc.

If $M' \subset M$ and $N' \subset N$ are pairs of modules and submodules, we let $M' \cdot N' \subset M \otimes N$ denote the image of the induced map $M' \otimes N' \to M \otimes N$.

1. PURE SUBMODULES

If $M' \subset M$ are a module and submodule, we say that M' is a *pure* submodule of M provided that for any module N, $M' \otimes N \to M \otimes N$ is mono.

THEOREM 1.1 (P. M. Cohn). *Let $M' \subset M$ be a module and submodule. Then M' is pure if and only if every finite system of linear equations*

$$\sum_{i=1}^{k_j} \lambda_{ij} x_i = m_j', \qquad j = 1,..., n, \tag{1}$$

with all $m_j' \in M'$, $\lambda_{ij} \in K$ has a solution in M' whenever it has one in M. (When $K = \mathbb{Z}$ or any other PID, the possibility of diagonalizing such a system allows reduction to the better known relative divisibility criterion.)

Proof. See [Cohn], Theorem 2.4.

PROPOSITION 1.2.   *Given $M' \subset M$, there is an $M'' \subset M$ such that $M' \subset M''$, such that every system of the type (1) which has a solution in M has one in M'', and such that $\#(M'') \leqslant \max(\#(M'), \#(K), \aleph_0)$. Here, of course, $\#(\cdot)$ denotes cardinality.*

Proof.   The set of equations of type (1) is

$$\bigcup K^{\Sigma k_j} \times M'^n,$$

where the union is over all finite sequences of k_j. The number of such systems is $\leqslant \sum \#(K)^n \times \#(M')^{n_2}$, where the sum may be taken over all pairs of integers n_1 and n_2. For each system of type (1) which does have a solution in M, choose one set of elements $m_1, ..., m_n$, which satisfy it. The number of such elements is $\leqslant \sum \#(K)^{n_1} \times \#(M')^{n_2} \times \#(M')^{n_3}$, again the sum taken over all 3-tuples of integers n_1, n_2, n_3. Let M'' be the submodule generated by all those elements together with the elements of M'. If both $\#(K)$ and $\#(M')$ are finite, then $\#(M'') \leqslant \aleph_0$, while if either is infinite, it is clear that $\#(M'') \leqslant \max(\#(M'), \#(K))$.

PROPOSITION 1.3.   *Given $M' \subset M$, there is an $M^* \subset M$ such that $M' \subset M^*$, such that M^* is a pure submodule of M, and such that*

$$\#(M^*) \leqslant \max(\#(M'), \#(K), \aleph_0).$$

Proof.   Apply the previous proposition to get a countable sequence

$$M' \subset M'' \subset M''' \subset \cdots \subset M^{(n)} \subset \cdots$$

such that each family of equations of type (1) with right-hand side in $M^{(n)}$ which has a solution in M has one in $M^{(n+1)}$ and such that $\#(M^{(n+1)})$ is small. Now let $M^* = \bigcup M^{(n)}$. One checks immediately that M^* satisfies Cohn's criterion and hence is pure. The cardinality limit is clear.

2. INVARIANT SUBMODULES

Let C be a coalgebra. A *submodule $M \subset C$* will be called *invariant under δ*, or simply *invariant*, provided $\delta(M) \subset M \cdot M$.

PROPOSITION 2.1.   *Let C be a coalgebra and M a submodule of C. Then there is a submodule $M' \subset C$ such that $M \subset M'$, such that $\delta(M) \subset M' \cdot M'$, and such that $\#(M') \leqslant \max(\#(M), \#(K), \aleph_0)$.*

Proof. For each $m \in M$, choose a representation $\delta(m) = \sum_{i=1}^{k_m} m_{i_{(1)}} \otimes m_{i_{(2)}}$. Let M' be the submodule generated by all these elements $m_{i_{(1)}}$, $m_{i_{(2)}}$ and the elements of M. Now all the claims of the conclusion are clear.

PROPOSITION 2.2. *Let C be a coalgebra and M a submodule of C. Then there is a submodule $M^1 \subset C$ such that $M \subset M^1$, such that M^1 is invariant, and such that $\#(M) \leqslant \max(\#(M), \#(K), \aleph_0)$.*

Proof. Iterate the above proposition to get a sequence

$$M \subset M' \subset M'' \subset \cdots \subset M^{(n)} \subset \cdots$$

such that $\delta(M^{(n)}) \subset M^{(n+1)} \cdot M^{(n+1)}$. If $M^1 = \bigcup M^{(n)}$, it clearly has all the claimed properties.

3. SUBCOALGEBRAS

THEOREM 3.1. *Let C be a coalgebra, M a submodule of C. Then there is a subcoalgebra $C' \subset C$ such that $M \subset C'$ and $\#(C') \leqslant \max(\#(M), \#(K), \aleph_0)$.*

Proof. Let $M' = M^*$, $M'' = (M')!,\ldots, M^{(n)} = (M^{(n-1)})^*$, when n is odd and $M^{(n)} = (M^{(n-1)})!$ when n is even. Then let $C' = \bigcup M^{(n)}$. It is clear that C' satisfies Cohn's criterion and hence is pure while it is also clearly invariant. Hence $\delta(C') \subset C' \cdot C'$ while $C' \otimes C' \to C \otimes C' \to C \otimes C$ are both mono, which implies that $C' \otimes C' = C \cdot C$, so $\delta(C') \subset C' \otimes C'$. A similar argument implies that $C' \otimes C' \otimes C' \to C \otimes C \otimes C$ is also mono, which means that the coassociativity law in C' can be deduced from that of C. Similar observations are valid for the cocommutativity. Needless to say, the ϵ of C' is just the composite $C' \subset C \to^\epsilon K$. The cardinality conclusion is clear.

COROLLARY 3.2. *The coalgebras whose cardinality is $\leqslant \max(\#(K), \aleph_0)$ generate the category K-coalg.*

Proof. Let $C_1 \rightrightarrows^f_g C_2$ be maps of coalgebras with $f \neq g$. Then there is some $c \in C_1$ with $f(c) \neq g(c)$. Let M be the submodule of C_1 generated by c. Then $\#(M) \leqslant \#(K)$ and the result follows easily from the previous theorem.

4. COFREE COALGEBRAS

THEOREM 4.1. *The obvious underlying functor $U: K\text{-Coalg} \to K\text{-Mod}$ has a right adjoint and is cotripleable.*

Proof. It is a straightforward exercise to show that U creates all colimits. It is based primarily on the fact that $\otimes$ preserves colimits. To show that U has a right adjoint, we invoke the special adjoint functor theorem. The only missing ingredient is that K-Coalg is co-well-powered. But that follows easily from the facts that U preserves epis (U preserves colimits), that K-Mod is co-well-powered, and that a given K-module can underlie only a small set of coalgebras. To see that it is cotripleable, we resort to the criterion of [Duskin], (3.2) and consider a pair of maps $C_1 \rightrightarrows_{d_1}^{d_0} C_2$ which constitute a coequivalence relation on C_1. All we really require is that d_0 and d_1 are coreflexive, i.e., have a common left inverse. We also suppose that

$$M \rightarrow C_1 \rightrightarrows C_2$$

is a split equalizer diagram in the category K. (We have suppressed the U to simplify matters). Since $M \rightarrow C_1$ is split, it is certainly pure. Now each row and each column of the diagram

$$
\begin{array}{ccccc}
M \otimes M & \longrightarrow & C_1 \otimes M & \rightrightarrows & C_2 \otimes M \\
\downarrow & & \downarrow & & \downarrow \\
M \otimes C_1 & \longrightarrow & C_1 \otimes C_1 & \rightrightarrows & C_2 \otimes M \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
M \otimes C_2 & \longrightarrow & C_1 \otimes C_2 & \rightrightarrows & C_2 \otimes C_2
\end{array}
$$

is an equalizer and always the equalizer of a coreflexive pair from which we see, by an easy diagram chase, that the diagonal is also an equalizer. Now by considering the diagram

$$
\begin{array}{ccccc}
M & \longrightarrow & C_1 & \rightrightarrows & C_2 \\
& & \downarrow & & \downarrow \\
M \otimes M & \longrightarrow & C_1 \otimes C_1 & \rightrightarrows & C_2 \otimes C_2
\end{array}
$$

we see that M is a subcoalgebra. Since U is faithful, it is trivial to see that

$$M \rightarrow C_1 \rightrightarrows C_2$$

is an equalizer diagram in K-coalg.

5. The Cartesian Closed Structure

If C_1 and C_2 are coalgebras, then $C_1 \otimes C_2$ can be given, in a natural way, the structure of a coalgebra by

$$C_1 \otimes C_2 \xrightarrow{\delta \otimes \delta} C_1 \otimes C_1 \otimes C_2 \otimes C_2 \to C_1 \otimes C_2 \otimes C_1 \otimes C_2 \,,$$

$$C_1 \otimes C_2 \xrightarrow{\epsilon \otimes \epsilon} K \otimes K \cong K.$$

Moreover there are natural maps

$$C_1 \otimes C_2 \xrightarrow{C_1 \otimes \epsilon} C_1 \otimes K \cong C_1 \quad \text{and} \quad C_1 \otimes C_2 \xrightarrow{\epsilon \otimes C_2} K \otimes C_2 \cong C_2 \,.$$

This leads to the following proposition, whose proof is left as an exercise.

PROPOSITION 5.1. *If C_1 and C_2 are coalgebras, then $C_1 \otimes C_2$ together with the maps as described above is the product of C_1 and C_2 in K-Coalg.*

PROPOSITION 5.2. *If C is a coalgebra, the functor $C \otimes -$: K-Coalg $\to$ K-Coalg commutes with all colimits.*

Proof. This is because colimits are created by the underlying module functor and $C \otimes -$ has that property in K-Mod.

THEOREM 5.3. *The category K-Coalg is a cartesian closed category.*

Proof. The special adjoint functor theorem provides each functor $C \otimes -$ with a right adjoint $(-)^C$.

6. Props and Categories Proppable over K-mod

For the definition of a prop we refer to [Mac Lane] Section 24. The idea is to generalize the notion of a finitary theory by taking a category whose objects are natural numbers and whose maps include the permutations (required to state commutative laws) but not necessarily other maps between the natural numbers. You also suppose that the prop, **P**, has a biproduct $\otimes$: **P** $\times$ **P** $\to$ **P** which at the object level satisfies $m \otimes n = m + n$. But this $\otimes$ is not assumed to be the product functor on **P** as it would be when **P** is a theory, so that in particular not everything is determined by the maps to 1. If **X** is some category with a commutative, associative, unitary product $\otimes$, then a **P** algebra in **X** is a functor **P** $\to$ **X** which preserves the $\otimes$. E.g., if **P** is the prop generated by (assuming that such a notion exists—it does) a map $1 \to 0$ and a map $1 \to 2$ and we add suitable identities, then a **P**-algebra in

K-Mod is precisely a K-Coalgebra. Or rather it would be if $\otimes$ in K-Mod were commutative, associative, and unitary. Thus the above description of prop represents a substantial simplification. You have to beef up the definition by choosing a standard associativity of the $\otimes$ both in the prop and in K-Mod. Then use [MacLane] 15.2 to see that nothing depends on the fixed association chosen. (I might add that this difficulty can be avoided when dealing with theories because the natural isomorphisms associated with product can all be described by universal mapping properties and are consequently unique. Or to put it another way, we can say what a product preserving functor is even if we have never heard the word "associative.") I thank C. Auderset for pointing out to me the complications that arise in the careful definition of a prop. At any rate, once you have a prop $\mathbf{P}$ and a tensored category $\mathbf{X}$, you can define the notion of a $\mathbf{P}$ algebra in $\mathbf{X}$ and, in the obvious way, of a morphism of $\mathbf{P}$-algebras which results in a category I will denote $\mathbf{X}^{\mathbf{P}}$. It comes equipped with a natural underlying functor to $\mathbf{X}$ which takes an algebra to its value at 1. If A is such an algebra, then we must have $A(n)$ be the nth tensor power of $A(1)$ and, in particular, $A(0)$ must be the unit for the tensor algebra. A category $\mathbf{Y}$ and a functor $\mathbf{Y} \to \mathbf{X}$ are called proppable over $\mathbf{X}$ (or, more carefully, over $(\mathbf{X}, \otimes, \cdot, \cdot, \cdot, \cdot)$ where the dots stand for the various isomorphisms assumed) if $\mathbf{Y}$ and the functor are equivalent to $\mathbf{X}^{\mathbf{P}}$ and its standard underlying functor for some prop $\mathbf{P}$.

As indicated above, the category of coalgebras is proppable over K-Mod. So is the category of algebras. (If $\mathbf{P}$ is a prop, then so is $\mathbf{P}^{op}$. Or, an opprop is a prop.) Another example is the category of inner product spaces. Just take the free prop generated by a map $2 \to 0$ subject to a symmetry law. That Lie algebras are not proppable (How could you state the Jacobi identity?) suggests that there is the notion of an additive prop and no doubt $\mathbf{V}$-props for any closed category $\mathbf{V}$. No doubt all the results we give have wide generalizations in such directions but we refrain from giving them.

Among the props we single out two special kinds.

DEFINITION 6.1. A prop $\mathbf{P}$ is said to be of algebra type if all the maps and all the commutative diagrams are generated by those with codomain 1. It is said to be of coalgebra type if $\mathbf{P}^{op}$ is of algebra type.

Clearly the prop whose algebras are ordinary K-algebras is of algebra type and dually for coalgebras.

DEFINITION 6.2. Let $\mathbf{P}$ be a prop, A, B be $\mathbf{P}$-algebras in K-Mod, C be a coalgebra. We make the usual abuse of notation and write A for $A(1)$, $A^n = A \otimes \cdots \otimes A$ for $A(n)$. If $\alpha: m \to n$ is a map in $\mathbf{P}$, then we let $A^{\alpha}: A^m \to A^n$ denote the corresponding structure map. Also, define for each $n \geqslant 0$ a map $\delta_n: C \to C^n$ as follows. $\delta_0 = \epsilon$, $\delta_1 = C$, $\delta_2 = \delta$ while for

$n > 2$, $\delta_n = \delta \otimes C^{n-2} \cdot \delta_{n-1}$. The properties of coalgebras that we suppose guarantee that up to the various given natural equivalences there is only one natural map $C \to C^n$ and this is what we call δ_n . Then a map $f \colon A \otimes C \to B$ is called a measuring if either of the following two equivalent diagrams commutes for all $\alpha \colon m \to n$ in $\mathbf{P}$:

$$
\begin{array}{ccccccc}
A^m \otimes C & \xrightarrow{A^m \otimes \delta_m} & A^m \otimes C^m & \xrightarrow{\;\cong\;} & (A \otimes C)^m & \xrightarrow{\;f^m\;} & B^m \\
{\scriptstyle A^\alpha \otimes C}\big\downarrow & & & & & & \big\downarrow{\scriptstyle B^\alpha} \\
A^n \otimes C & \xrightarrow{A^n \otimes \delta_n} & A^n \otimes C^n & \xrightarrow{\;\cong\;} & (A \otimes C)^n & \xrightarrow{\;f^n\;} & B^n
\end{array}
$$

$$
\begin{array}{ccccccc}
A^m & \xrightarrow{\;\bar{f}^m\;} & [C, B]^m & \longrightarrow & [C^m, B^m] & \xrightarrow{[\delta_m, B^m]} & [C, B^m] \\
{\scriptstyle A^\alpha}\big\downarrow & & & & & & \big\downarrow{\scriptstyle [C, B^m]^\alpha} \\
A^n & \xrightarrow{\;\bar{f}^n\;} & [C, B]^n & \longrightarrow & [C^m, B^m] & \xrightarrow{[\delta_n, B^n]} & [C, B^n].
\end{array}
$$

Here $[C, B]$ is just the K-linear form and $\hat{f} \colon A \to [C, B]$ is the map which corresponds to f under the adjunction.

The proof that these two definitions are equivalent is not immediate and the carrying out of it is an educational experience which we leave to the reader.

We let Meas $(A, C; B)$ denote the set of measurings of $A \otimes C \to B$.

THEOREM 6.3. *For any prop* $\mathbf{P}$ *and* $\mathbf{P}$-*algebras* A, B, *the functor*

$$\mathrm{Meas}(A, -; B) \colon K\text{-}\mathrm{Coalg}^{op} \to S$$

is representable.

Proof. Ignoring the measuring condition, the functor that associates to each C the linear maps $A \otimes C \to B$ preserves all colimits. In fact it is representable by the free $\mathbf{P}$-algebra generated by $[B, A]$. It is trivial to check that that imposition of the measuring condition gives a subfunctor which still preserves all colimits and hence, once more, the special adjoint functor theorem implies that it is representable.

7. TENSORS AND COTENSORS

Exactly as in the proof of (6.3), one can easily prove that the functor Meas$(-, -; -)$ preserves colimits as a functor of the first variable and limits as a functor of the third. Hence it is natural to enquire whether these functors are representable as well. Objects representing them are called tensors and cotensors respectively. When a category is enriched over $\mathbf{S}$, the corresponding constructions are copowers and powers, respectively.

In general we do not know, for a category proppable over K-Mod, whether tensors and cotensors (over K-Coalg) always exist; even when K is a field, I do not think the answer is known. However we shall prove that when the prop is either of algebra or coalgebra type, then both tensors and cotensors exist. We begin with coalgebra type props.

PROPOSITION 7.1.   *Let* **P** *be a coalgebra type prop and* A *be a* **P***-algebra in* K*-Mod. Let* C *be a* K*-coalgebra. Then the ordinary tensor product* $A \otimes C$ *can be endowed with a* **Z***-algebra structure such that* **Z***-algebra maps* $A \times C \to B$ *are the same as measures.*

Proof.   We need only consider maps in **P** of the form $\alpha: 1 \to n$. We use the map

$$A \otimes C \xrightarrow{\ A^{\alpha} \otimes \delta_n\ } A^n \otimes C^n \cong (A \otimes C)^n.$$

Since every diagram that can be buily with the δ_n commutes, this makes $A \otimes C$ into a **P**-algebra. The first definition of a measure makes the last statement of the proposition clear.

PROPOSITION 7.2.   *Let* **P** *be a prop of coalgebra type. Then for any* **P***-algebra* A *and coalgebra* C*, the functor* $\mathrm{Meas}(A, C; \ -): (K\text{-Mod})^{\mathbf{P}} \to \mathbf{S}$ *is representable.*

Proof.   The methods of Sections 1 and 2 can be repeated in any category of algebras over a coalgebras type (although the bounds involved would have to take into account the size of the homsets $\mathbf{P}(1, n)$ as well) to show that the category has a set of generators and is co-will-powered and cocomplete. Hence the special adjoint functor theorem implies that any limit preserving set-valued functor is representable.

COROLLARY 7.3 (to the proof).   *If* **P** *is a coalgebra type prop, then the underlying functor* $(K\text{-Mod})^{\mathbf{P}} \to K\text{-Mod}$ *has a right adjoint and is cotripleable.*

Proof.   Just mimic the proof of (4.1).
Next we turn to algebra type props. By analogy with (7.1) we have

PROPOSITION 7.4.   *Let* **P** *be an algebra type prop and* B *be a* **P***-algebra in* K*-Mod. Let* C *be a* K*-coalgebra. Then the ordinary vector space* $\mathrm{Hom}[C, B]$ *can be endowed with a* **P***-algebra structure such that* **P***-algebra maps* $A \to [C, B]$ *are the same as measures (or rather as adjoints of measures).*

Proof.   Just dualize the proof of 7.1 using, of course, the second definition of a measure.

PROPOSITION 7.5. *Let* **P** *be an algebra type prop. Then the underlying functor* $(K\text{-Mod})^{\mathbf{P}} \to K\text{-Mod}$ *has a left adjoint.*

Proof. First I claim that this functor preserves limits. E.g., for products. There is always a natural map $(\prod A_i)^n \to \prod A_i{}^n$, just using the universal mapping property of products. Then if $\alpha\colon n \to 1$ is a map in **P**, we use the composite

$$\left(\prod A_i\right)^n \to \prod A_i{}^n \xrightarrow{\ \Pi A_i{}^{\alpha}\ } \prod A_i$$

to put a **P**-structure on the product. A similar argument applies to equalizers. Hence we need only verify the solution set condition. If A is an algebra and $M \subset A$ is a submodule, then let

$$M' = \bigcup_{\alpha \in \mathbf{P}(n,1)} \mathrm{Im}(M^n \to A^n \to A).$$

Iterating this construction, we get a sequence

$$M \subset M' \subset M'' \subset \cdots \subset M^{(m)} \subset \cdots$$

such that for all $\alpha\colon n \to 1$ in **P** the image of $(M^{(m)})^n \to A^n \to A$ is contained in $M^{(m+1)}$. From this it easily follows that $A' = \bigcup M^{(m)}$ is a subalgebra of A containing M whose cardinality is not too large. Hence each K-module M can only generate a set of algebras so that the ordinary adjoint functor is satisfied.

PROPOSITION 7.6. *Let* **P** *be an algebra type prop and* $f\colon A \to B$ *a morphism of* **P**-*algebras. Then* $f(A)$ *is a subalgebra of* B.

Proof. Given an $\alpha\colon n \to 1$ in **P**, just use the diagonal fill-in in the diagram

$$
\begin{array}{ccc}
A^n & \longrightarrow & f(A)^n \\
{\scriptstyle A^{\alpha}}\downarrow & & \downarrow \\
A & & B^n \\
\downarrow & & \downarrow{\scriptstyle B^{\alpha}} \\
f(A) & \rightarrowtail & B
\end{array}
$$

PROPOSITION 7.7. *Let* **P** *be an algebra type prop and* $f\colon B \rightarrowtail B'$ *a* 1–1 *map of* **P** *algebras. Then for any* **P**-*algebra* A *and coalgebra* C, *a map* $A \otimes C \to B$ *is a measure if and only if the composite*

$$A \otimes C \to B \rightarrowtail B'$$

is.

Proof. This follows trivially from the first definition of a measure.

Let A be a **P**-algebra, C a coalgebra and $F(A \otimes C)$ the free **P**-algebra generated by the K-module $A \otimes C$. Then there is a K-linear map $A \otimes C \to F(A \otimes C)$ such that for any **P**-algebra B any K-linear map $A \otimes C \to B$ has unique extension to a **P**-algebra homomorphism $F(A \otimes C) \to B$. Now consider the set Γ of all **P**-algebras $G = F(A \otimes C)/\sim$ which are quotients of $F(A \otimes C)$ such that the composite $A \otimes C \to F(A \otimes C) \to G$ is a measure. I claim that the set of measures $A \otimes C \to G$ of this form is a solution set. In fact, let $A \otimes C \to B$ be a measure. It extends to a **P**-algebra map $F(A \otimes C) \to B$ which factors $F(A \otimes C) \twoheadrightarrow G \to B$ by (7.6), and by (7.7), $A \otimes C \to F(A \otimes C) \to G$ is a measure so that $A \otimes C \to G$ belongs to Γ. This completes the proof of the following:

PROPOSITION 7.8. *Let* **P** *be an algebra type prop. Then for all* **P**-*algebras A and coalgebras C,* $\mathrm{Meas}(A, C: -): (K\text{-Mod})^{\mathbf{P}} \to \mathbf{S}$ *is representable.*

Remark 7.9. The reader may suspect, and my student T. Fox has indeed shown, that the results of this paper remain valid when K-Mod is replaced by any $\otimes$-closed locally presentable category (see [Barr], Chapter II or [Gabriel, Ulmer]). Of course the main difficulty to overcome is to find a satisfactory replacement for Cohn's criterion.

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