COEQUALIZERS AND FREE TRIPLES, II

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ABSTRACT. This paper studies a category $\mathcal{X}$ with an endofunctor $T : \mathcal{X} \longrightarrow \mathcal{X}$. A T-algebra is given by a morphism $TX \longrightarrow X$ in $\mathcal{X}$. We examine the related questions of when T freely generates a triple (or monad) on $\mathcal{X}$; when an object $X \in \mathcal{X}$ freely generates a T-algebra; and when the category of T-algebras has coequalizers and other colimits. The paper defines a category of "T-horns" which effectively contains $\mathcal{X}$ as well as all T-algebras. It is assumed that $\mathcal{X}$ is cocomplete and has a factorization system $(\mathcal{E}, \mathcal{M})$ satisfying reasonable properties. An ordinal-indexed sequence of T-horns is then defined which provides successive approximations to a free T-algebra generated by an object $X \in \mathcal{X}$, as well as approximations to coequalizers and other colimits for the category of T-algebras. Using the notions of an $\mathcal{M}$-cone and a separated T-horn it is shown that if $\mathcal{X}$ is $\mathcal{M}$-well-powered, then the ordinal sequence stabilizes at the desired free algebra or coequalizer or other colimit whenever they exist. This paper is a successor to a paper written by the first author in 1970 that showed that T generates a free triple when every $X \in \mathcal{X}$ generates a free T-algebra. We also consider colimits in triple algebras and give some examples of functors T for which no $X \in \mathcal{X}$ generates a free T-algebra.

1. Introduction

Nearly fifty years ago, the first author published "Coequalizers and Free Triples" [Barr (1970)]. That paper examines endofunctors $T : \mathcal{X} \longrightarrow \mathcal{X}$ on a category $\mathcal{X}$, and defines the category of T-algebras, and the underlying functor $U : T\text{-Alg} \longrightarrow \mathcal{X}$, see 2.2 below. It shows that, given reasonable conditions on $\mathcal{X}$, if every object $X \in \mathcal{X}$ generates a free T-algebra, then T generates a free triple. Similar conditions were given for the category of T-algebras as well as for the category $\mathcal{X}^{\mathbf{T}}$ of algebras for a triple $\mathbf{T}$ to have coequalizers. This paper, as its title suggests, is a sequel to [Barr (1970)].

In 1979, the first author wrote, but did not publish, a follow-up paper, [Barr (1979)], We recently came across a copy of this paper (see below) and discovered serious difficulties in the exposition. The present paper is the result of cleaning it up. It turns out that the methods and results of the original paper, while clumsy, were fundamentally correct.

An important tool we use here is something we call a T-horn, 3.1. This enables us to confine the difficulties to one place and systematize the main construction. A transfinite process can be applied to any T-horn and, when it converges, it shows the existence of coequalizers and other colimits in the category T-Alg. Since $\mathcal{X}$ has a natural embedding into T-Horns, the transfinite process can be applied to each object of $\mathcal{X}$. When the process converges for each $X \in \mathcal{X}$, then each object of $\mathcal{X}$ generates a free T-Alg which implies that T generates a free triple. Some important tools in this paper are the notions of an

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$\mathcal{M}$-cone, 2.5, and of a separated T-horn, 3.5. This process was implicit in [Barr (1979)] but not formulated in terms of T-horns.

The basic construction is of the type of metaconstruction of [Koubek & Reiterman (1979)]. The particular form chosen, however, leads to an increasing ordinal chain that either provides the desired object or usually demonstrates its failure to exist by increasing indefinitely. See 4.15, but also 4.18. We note that our construction is definitely different from that of [Adámek & Trnková (2011)] since our Theorem 4.15 would otherwise be contradicted by their Example 3.1 (a).

Several interesting papers have appeared which get partial answers to our questions. See [Adámek (1977), Adámek & Koubek (1980), Adámek & Trnková (2011), Kelly (1980), Koubek & Reiterman (1979), Schubert (1972)]. We note that [Koubek & Reiterman (1979)] not only introduced the type of transfinite method we have used but also defined a "partial algebra" which in our notation is equivalent to a T-horn, (X, Y, v, w) for which the map v is either in $\mathcal{M}$, or is a mono—see section 3. This same paper also defines a "generalized partial algebra" which is the same as a T-horn, but the maps between generalized partial algebras are not the same as T-horn morphisms.

The paper [Kelly (1980)] was written at about the same time as the original paper, [Barr (1979)]. Kelly even has a reference to that manuscript. Kelly also writes about the important notion of an $\mathcal{M}$-cone, under the name of "cone in $\mathcal{M}$" and Kelly's Proposition 1.1(i) anticipates our Proposition 2.8. This proposition and related results are not only useful, they convey an intuitive sense of what an $\mathcal{M}$-cone is.

Section 5 of [Kelly (1980)] discusses "well-pointed endofunctors", which are more suitable for triples. Moreover, if $\mathcal{X}$ has finite coproducts, then every endofunctor generates a free well-pointed functor in a straightforward way, so the question of whether an endofunctor $T : \mathcal{X} \longrightarrow \mathcal{X}$ generates a free triple reduces to the case of a well-pointed functor. Section 14 of [Kelly (1980)] introduces a comma category which plays the same role in that paper as the category of T-Horns does in this paper.

The original paper might have gone undiscovered except that recently Camell Kachour, a student of Michael Batanin and Ross Street's, (see [Kachour (2013)]) asked if it was possible to get a copy of the original draft since none seemed available elsewhere. We eventually unearthed it, sent him a copy, and decided to write a substantially revised version.

We would like to thank the referees for helping to significantly improve the presentation of this paper.

2. Preliminaries

2.1. Notation. We will be using the following notational conventions throughout this paper. A double subscript such as u_{nm} will be written without commas in the subscript if the indices are single letters, while such as $u_{n+1,m}$, $u_{n,m+1}$, or $u_{n+1,m+1}$ will use commas as separators. Second, we will usually denote composition of arrows by simple juxtaposition without punctuation so long as a single character denotes each arrow. As soon as an arrow is denoted by two or more characters the composition will be denoted using periods. Thus $w_m v_{mn}$ but $u_{m+1,k}.w_{m+1}.Tu_{mn}$.

2.2. Definition. *Throughout this paper, we will be dealing with a category $\mathcal{X}$ equipped with an endofunctor T. By a T-***algebra** *(X, x), we mean an object X together with a map $x : TX \longrightarrow X$. A morphism $f : (X, x) \longrightarrow (Y, y)$ of T-algebras is a map $f : X \longrightarrow Y$ such that $fx = y.Tf$. We denote this category by T-Alg. The two questions we will be looking at are the existence of a free triple on $\mathcal{X}$ generated by T and the question of whether the category T-Alg of T-algebras has coequalizers. We define the functor $U : T\text{-Alg} \longrightarrow \mathcal{X}$ so that $U(Z, z) = Z$ and $U(f) = f$.*

2.3. Definition. *A pair $(\mathcal{E}, \mathcal{M})$ of subcategories of $\mathcal{X}$ is a* **factorization system** *if*

FS-1. $\mathcal{E} \cap \mathcal{M}$ is the class of isomorphisms of $\mathcal{X}$.

FS-2. Every $f \in \mathcal{X}$ can be factored as $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

FS-3. Every commutative square

$$\begin{tikzcd} A \arrow[r, two heads, "e"] \arrow[d] & B \arrow[d] \\ C \arrow[r, tail, "m"'] & D \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$ can be "filled in" by a map $B \longrightarrow C$ making both triangles commute.

Note that we will generally be using a double-headed arrow to denote a morphism in $\mathcal{E}$ and a tailed arrow to denote a morphism of $\mathcal{M}$. We will say that e **is left orthogonal to** m **and** m **is right orthogonal to** e provided FS-3 is satisfied for any square of that type. We may write $e \perp m$ if that holds.

2.4. Proposition. *Suppose $(\mathcal{E}, \mathcal{M})$ is a factorization system. Then*

1. *$f \in \mathcal{E}$ iff f is left orthogonal to every $m \in \mathcal{M}$; dually $f \in \mathcal{M}$ iff f is right orthogonal to every $e \in \mathcal{E}$.*

2. *$\mathcal{E}$ is closed under pushout; $\mathcal{M}$ is closed under pullback.*

3. *If every $e \in \mathcal{E}$ is epic and every $m \in \mathcal{M}$ is monic, then $gf \in \mathcal{E}$ implies $g \in \mathcal{E}$ and $gf \in \mathcal{M}$ implies $f \in \mathcal{M}$.*

4. *Under the same assumption, every regular epic is in $\mathcal{E}$ and every regular monic is in $\mathcal{M}$.*

The proofs are standard.

2.5. Definition. *By a* **cone**, *we mean a class of morphisms with the same source. Warning: it might be a proper class. The source will be called the* **vertex** *of the cone, the class of targets will be called the* **base** *of the cone, and the arrows will be called the* **elements** *of the cone.*

A cone $\{X \longrightarrow X_i\}$ *is called* **an** $\mathcal{M}$**-cone** *if it is orthogonal to* $\mathcal{E}$ *in the following sense: if* $e : Y \twoheadrightarrow Y' \in \mathcal{E}$, *if for each cone* $\{Y' \longrightarrow X_i\}$ *with the same base, and if for every index* i, *the diagram*

$$\begin{tikzcd} Y \arrow[r, two heads] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & X_i \end{tikzcd}$$

commutes, then there is a unique arrow $t : Y' \longrightarrow X$ *such that for each* i, *both triangles in the diagram*

$$\begin{tikzcd} Y \arrow[r, two heads] \arrow[d] & Y' \arrow[d] \arrow[dl] \\ X \arrow[r] & X_i \end{tikzcd}$$

commute.

2.6. Blanket assumptions. Henceforth we will suppose

(BA-1) $\mathcal{X}$ is cocomplete.

(BA-2) $(\mathcal{E}, \mathcal{M})$ is a factorization system.

(BA-3) Every map in $\mathcal{E}$ is epic.

(BA-4) Every map in $\mathcal{M}$ is monic.

(BA-5) Every cone $\{f_i : X \longrightarrow X_i\}$ factors $\{X \overset{e}{\twoheadrightarrow} X' \xrightarrow{m_i} X_i\}$ with $e \in \mathcal{E}$ and $\{X' \longrightarrow X_i\}$ an $\mathcal{M}$-cone.

2.7. Notation. We say that the cone $\{f_i : X \longrightarrow X_i\}$ **factors through** $e : X \longrightarrow Y$ **with** $e \in \mathcal{E}$ if for all i there exists g_i such that $f_i = g_i e$.

The following result can be found in Proposition 1.1(i) of [Kelly (1980)].

2.8. Proposition. *A cone is an* $\mathcal{M}$*-cone iff whenever it factors through* e *for* $e \in \mathcal{E}$, *then* e *is invertible.*

Proof. First assume that the cone $\{X \longrightarrow X_i\}$ satisfies the condition that whenever it factors through a map $e \in \mathcal{E}$ then e is invertible. Then let

$$\begin{tikzcd} Y \arrow[r, two heads] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & X_i \end{tikzcd}$$

be as in the definition of an $\mathcal{M}$-cone. Let

$$\begin{tikzcd} Y \arrow[r, two heads] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

be a pushout diagram. It follows that the map $e' : X \longrightarrow Z$ is in $\mathcal{E}$ and the cone factors through e' so e' is invertible. From this the existence of the map $t : Y' \longrightarrow X$ readily follows so the cone is an $\mathcal{M}$-cone.

Conversely, assume that $\{X \longrightarrow X_i\}$ is an $\mathcal{M}$-cone which factors through the map $e : X \twoheadrightarrow Y$ with $e \in \mathcal{E}$. Consider the diagram

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X_i \end{tikzcd}$$

where $X \longrightarrow X$ is the identity and $X \twoheadrightarrow Y$ is e. Since $\{X \longrightarrow X_i\}$ is an $\mathcal{M}$-cone, there exists $t : Y \longrightarrow X$ which is a left inverse to e which implies that e is invertible. ∎

2.9. Corollary. *Let $\{f_i : X \longrightarrow X_i\}$ be a cone. Factor each $f_i = g_i e_i$ with $g_i \in \mathcal{M}$ and $e_i \in \mathcal{E}$. Let $e_i : X \twoheadrightarrow Y_i$. Then the cone $\{f_i : X \longrightarrow X_i\}$ is an $\mathcal{M}$-cone iff the cone $\{e_i : X \twoheadrightarrow Y_i\}$ is an $\mathcal{M}$-cone.*

Proof. $\{f_i : X \longrightarrow X_i\}$ factors through $e \in \mathcal{E}$ iff $\{e_i : X \twoheadrightarrow Y_i\}$ factors through e. ∎

2.10. Corollary. *If $\mathcal{X}$ is $\mathcal{E}$-cowell-powered, then for every $\mathcal{M}$-cone $\{f_i : X \longrightarrow X_i \mid i \in I\}$ there exists a* **set** *$I_0 \subseteq I$ such that $\{f_i : X \longrightarrow X_i \mid i \in I_0\}$ is an $\mathcal{M}$-cone.*

Proof. Factor each $f_i = g_i e_i$ with $g_i \in \mathcal{M}$ and $e_i \in \mathcal{E}$. Then, by the above results, there is no non-invertible $e \in \mathcal{E}$ through which $\{e_i : X \twoheadrightarrow Y_i \mid i \in I\}$ factors. Since $\mathcal{X}$ is cowell-powered, we can find $I_0 \subseteq I$ such that there is no non-invertible $e \in \mathcal{E}$ through which $\{e_i : X \twoheadrightarrow Y_i \mid i \in I_0\}$ factors. And this implies that $\{f_i : X \longrightarrow X_i \mid i \in I_0\}$ is an $\mathcal{M}$-cone. ∎

2.11. Notation.

1. A **small diagram** is one which is indexed by a set, while a **large diagram** is one which is indexed by a proper class.

2. **Small colimits** are colimits of small diagrams and **large colimits** are colimits of large diagrams. The terms **small limits** and **large limits** are defined analogously.

3. A category is **Isbell cocomplete with respect to a class $\mathcal{E}$ of epis** if it is cocomplete (has all small colimits) and has colimits of any diagram, large or small, which consists of an object X together with a cone of maps $\{e_i : X \twoheadrightarrow Y_i\}$ with $e_i \in \mathcal{E}$ for all i.

2.12. Proposition. *Suppose that (BA-1)–(BA-4) hold and $\mathcal{X}$ is Isbell cocomplete with respect to $\mathcal{E}$. Then (BA-5) also holds.*

Proof. Let $\{X \longrightarrow X_i\}$ be a cone. We will say that $X \twoheadrightarrow Y \in \mathcal{E}$ is an $\mathcal{E}$-factorization of the cone if the cone factors through $X \twoheadrightarrow Y$. Since $\mathcal{X}$ is Isbell cocomplete, we can let X' be a colimit of all the $\mathcal{E}$-factorizations of the cone and $e : X \longrightarrow X'$ any of the composites $X \twoheadrightarrow Y \longrightarrow X'$, since they are all the same. We claim that $e \in \mathcal{E}$. In fact, if we have a square

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d] & X' \arrow[d] \\
Z \arrow[r, tail, "m"'] & Z'
\end{tikzcd}$$

with $m \in \mathcal{M}$, then for each $X \twoheadrightarrow Y \longrightarrow X_i$ as above, we have the indicated diagonal fill-in in the diagram

$$\begin{tikzcd}
X \arrow[r, two heads] \arrow[d] & Y \arrow[r] \arrow[dl, dashed] & X' \arrow[d] \\
Z \arrow[rr, tail] & & Z'
\end{tikzcd}$$

whence, since X' is a colimit of all such $X \twoheadrightarrow Y$, we get a diagonal fill-in $X' \longrightarrow Z$ and thus conclude that $e \in \mathcal{E}$. If there were a further $X' \overset{t}{\twoheadrightarrow} Y$ such that the composite $X \xrightarrow{e} X' \xrightarrow{t} Y$ is an $\mathcal{E}$-factorization of the cone, there is a map $v : Y \longrightarrow X'$ such that $vte = e$ and since e is epic we conclude that $vt = \mathrm{id}$ and since t is also epic, that t is an isomorphism. The result is that the cone $\{X' \longrightarrow X_i\}$ cannot be $\mathcal{E}$-factorized further, which shows it is an $\mathcal{M}$-cone in view of Proposition 2.8

2.13. Proposition. *Let $\{f_i : X \longrightarrow X_i\}$ with every $f_i \in \mathcal{E}$. Suppose that $\{g_i : X_i \longrightarrow X'\}$ is a colimit of the cone. Then every $g_i \in \mathcal{E}$.*

Proof. We observe that all composites $g_i f_i$ are the same and we denote it by f. It suffices to show that $f \in \mathcal{E}$. For each i, we can embed the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & X' \arrow[d, "k"] \\ Y \arrow[r, tail, "m"'] & Y' \end{tikzcd}$$

into the diagram

$$\begin{tikzcd} X \arrow[r, two heads, "f_i"] \arrow[d, "h"'] & X_i \arrow[r, "g_i"] \arrow[d, "kg_i"'] & X' \arrow[dl, "k"] \\ Y \arrow[r, tail, "m"'] & Y' & \end{tikzcd}$$

Then there is a map $\ell_i : X_i \longrightarrow Y$ such that $h = \ell_i f_i$ and $m\ell_i = kg_i$. Since all $\ell_i f_i = h$ are the same, it follows from the colimit property that there is a map $\ell : X' \longrightarrow Y$ such that $\ell g_i = \ell_i$ for all i. Then $\ell f = \ell g_i f_i = \ell_i f_i = h$. For $m\ell$, we compose with g_i to get $m\ell g_i = m\ell_i = kg_i$. Since $f = g_i f_i$ is epic, so is g_i and we conclude that $m\ell = k$. This shows that $f \perp m$, as required. ∎

3. The Basic Construction

We will define a transfinite process which can be viewed as a series of closer and closer approximations to the free T-algebra generated by an object of $\mathcal{X}$, or, in some cases, approximations to a desired coequalizer or other colimit. These approximations take place in the category of T-horns, which we will now define.

3.1. The category of T-horns.

A cone of the form

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "w"'] & TX \\ X & \end{tikzcd}$$

will be called a T**-horn**, which we will denote (X, Y, v, w). A map $(f, g) : (X, Y, v, w)$ to (X', Y', v', w') consists of a pair $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ such that

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[dd, "w"'] \arrow[dr, "g"] & TX \arrow[dr, "Tf"] & \\ & Y' \arrow[r, "v'"] \arrow[dd, "w'"] & TX' \\ X \arrow[dr, "f"'] & & \\ & X' & \end{tikzcd}$$

commutes. We often abbreviate T-horn to horn. We let $\mathcal{THorn}$ denote the category of T-horns.

3.2. The functors J and K. The categories $\mathcal{X}$ and T-Alg are both fully and faithfully embedded in $\mathcal{THorn}$ by the functors J and K respectively. If $X \in \mathcal{X}$ then $JX = (X, 0, v, w)$ where 0 denotes the initial object of $\mathcal{X}$ and v and w are the unique maps. If $(Z, z) \in T$-Alg then $K(Z, z) = (Z, TZ, \mathrm{id}, z)$.

3.3. T-algebra horns. We say that the horn H is a T**-algebra horn** if it is isomorphic to a horn of the form $K(Z, z)$. Clearly v is invertible if (X, Y, v, w) is a isomorphic to some $K(Z, z)$. The converse readily follows from:

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "w"'] \arrow[dr, "v"] & TX \arrow[dr, "\mathrm{id}"] & \\ X \arrow[dr, "\mathrm{id}"] & TX \arrow[r, "\mathrm{id}"] \arrow[d, "wv^{-1}"] & TX \\ & X & \end{tikzcd}$$

3.4. Admissible Maps. A horn morphism $(f, g) : (X, Y, v, w) \longrightarrow K(Z, z)$ is given by a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "w"'] \arrow[dr, "g"] & TX \arrow[dr, "Tf"] & \\ X \arrow[dr, "f"] & TZ \arrow[r, "\mathrm{id}"] \arrow[d, "z"'] & TZ \\ & Z & \end{tikzcd}$$

which implies that $g = Tf.v$. Accordingly, we will say that when (X, Y, v, w) is a horn, then $f : X \longrightarrow U(Z, z)$ is **admissible for** (X, Y, v, w) if $fw = z.Tf.v$. This characterizes maps of a horn to a T-algebra horn.

3.5. Separated Horns. We say that a horn (X, Y, v, w) is $\mathcal{M}$-**separated** if the cone $\{f_i : X \longrightarrow U(Z_i, z_i)\}$ of all admissible maps from X is an $\mathcal{M}$-cone. Since $\mathcal{M}$ will not change, we usually call such horns **separated**. We let $\mathcal{S}ep \subseteq \mathcal{T}\mathcal{H}orn$ denote the full subcategory of all separated horns.

3.6. Lemma.

1. *A horn which has an admissible map in $\mathcal{M}$ is separated. In particular, every T-algebra horn is separated.*

2. *If $\mathcal{X}$ has small products then so does the category T-*Alg

3. *If $\mathcal{X}$ is $\mathcal{E}$-cowell-powered and has products of set-indexed families of objects, then a horn is separated iff it has an admissible map in $\mathcal{M}$.*

Proof.

1. If (X, Y, v, w) is a horn and $f : X \longrightarrow U(Z, z)$ is admissible and $f \in \mathcal{M}$ then the cone consisting of the single map $\{f : X \longrightarrow Z\}$ is an $\mathcal{M}$-cone.

2. Let $\{(Z_i, z_i)\}$ be a set-indexed family of T-algebras. Let P together with projection maps $p_i : P \longrightarrow Z_i$ be a product of the objects Z_i. Let $z : TP \longrightarrow P$ be determined by $p_i z = z_i T(p_i)$ for all i. The remaining details are straightforward.

3. Let (X, Y, v, w) be a separated horn. Then there exists an $\mathcal{M}$-cone $\{f_i : X \longrightarrow Z_i\}$ of admissible maps. By Corollary 2.10, we may as well assume that I is a set. Let (P, z) be the product of the T-algebras (Z_i, z_I), as above. Let $f : X \longrightarrow P$ be such that $p_i f = f_i$ for all i. Factor $f = ge$ with $g \in \mathcal{M}$ and $e \in \mathcal{E}$. Since the $\mathcal{M}$-cone clearly factors through e, we see that e is invertible so $f \in \mathcal{M}$. ∎

3.7. Proposition. *Let $\{f_i : X \longrightarrow X_i \mid i \in I\}$ be a cone for which the product P of $\{X_i \mid i \in I\}$ exists. Let $\{p_i : P \longrightarrow X_i\}$ be the projections and let $f : X \longrightarrow P$ the map for which $p_i f = f_i$ for all $i \in I$. Then $\{f_i : X \longrightarrow X_i \mid i \in I\}$ is an $\mathcal{M}$-cone iff $f \in \mathcal{M}$.*

Proof. $\{f_i : X \longrightarrow X_i \mid i \in I\}$ is an $\mathcal{M}$-cone iff whenever it factors through $e \in \mathcal{E}$ then e is invertible iff whenever f factors through $e \in \mathcal{E}$ then e is invertible iff $f \in \mathcal{M}$. ∎

3.8. The Reflection L. In general, for any horn (X, Y, v, w), we can factor the cone $\{f_i : X \longrightarrow U(Z_i, z_i)\}$ of all admissible maps as $\{f_i : X \longrightarrow X' \longrightarrow U(Z_i, z_i)\}$ so that each $f_i = f_i'.e$ where $e : X \twoheadrightarrow X'$ is in $\mathcal{E}$ and $\{f_i' : X' \longrightarrow U(Z_i, z_i)\}$ is an $\mathcal{M}$-cone. We then define $L(X, Y, v, w) = (X', Y, Te.v, ew)$. It is readily shown that $L : \mathcal{T}\mathcal{H}orn \longrightarrow \mathcal{S}ep$ is a reflection functor of horns into separated horns and (e, id_Y) is the reflection map.

3.9. Definition. *We define the* **pre-successor** *of the horn* (X, Y, v, w), *as* (P, TX, Tr, s), *where* P *is determined so that the square in the diagram below is a pushout:*

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "w"'] & TX \arrow[r, "Tr"] \arrow[d, "s"] & TP \\ X \arrow[r, "r"'] & P & \end{tikzcd}$$

Note that (r, v) *is a map from* (X, Y, v, w) *to its pre-successor.*

We further define $\mathrm{Succ}(X, Y, v, w)$, *the* **successor** *of* (X, Y, v, w), *as* $L(P, TX, Tr, s)$, *the reflection of the pre-succesor into the subcategory of separated horns.*

Finally, we define the **successor map** *from a horn to its successor as* $\gamma = \eta(r, v)$, *where* η *is the reflection map from the pre-successor to its separated reflection.*

3.10. Lemma. *The successor map* $\gamma : H \longrightarrow \mathrm{Succ}(H)$ *is a isomorphism iff* H *is a* T*-algebra horn.*

Proof. If H is a T-algebra horn, then it is isomorphic to $K(Z, z)$ for some T-algebra (Z, z). It suffices to calculate the successor of $K(Z, z)$, using the fact that the pushout of an identity map is an identity

$$\begin{tikzcd} TZ \arrow[r, "\mathrm{id}"] \arrow[d, "z"'] & TZ \arrow[r, "\mathrm{id}"] \arrow[d, "z"] & TZ \\ Z \arrow[r, "\mathrm{id}"'] & Z & \end{tikzcd}$$

so $K(Z, z)$ is its own pre-successor. Since the above pre-successor is separated (by 3.6) it follows that $K(Z, z)$ is its own successor and $\gamma = (\mathrm{id}, \mathrm{id})$ is the identity horn map.

Conversely, suppose that $\gamma : (X, Y, v, w) \longrightarrow \mathrm{Succ}(X, Y, v, w)$ is an isomorphism. Let the pre-successor of (X, Y, v, w) be (P, TX, Tr, s) in the diagram below. Then the successor of (X, Y, v, w) is the separated reflection of (P, TX, Tr, s) which is $(P', TX, Te.Tr, es)$ in this diagram:

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "w"'] & TX \arrow[r, "Tr"] \arrow[d, "s"] & TP \arrow[r, "Te"] & TP' \\ X \arrow[r, "r"] \arrow[dr, "er"'] & P \arrow[d, "e"] & & \\ & P' & & \end{tikzcd}$$

If this sets up an isomorphism between (X, Y, v, w) and its successor, then, clearly, er must be an isomorphism. But then $Te.Tr$ is also an isomorphism which implies that $\mathrm{Succ}(X, Y, v, w) = (P', TX, Te.Tr, es)$ is a T-algebra horn, by 3.6. And since we are assuming that γ is an isomorphism, it follows that (X, Y, v, w) is a T-algebra horn. ∎

3.11. Definition. *A horn morphism $\theta : H_1 \longrightarrow H_2$ has the* **unique extension property with respect to T-algebra horns** *if every horn map $f : H_1 \longrightarrow K(Z, z)$ has a unique extension to a horn map $f' : H_2 \longrightarrow K(Z, z)$ for which $f'\theta = f$.*

3.12. Proposition. *Let H be a T-horn and let H' be its pre-successor. Then $(r, v) : H \longrightarrow H'$ is a map of horns and has the unique extension property with respect to T-algebra horns.*

Proof. Suppose $(f, Tf.v) : (X, Y, v, w) \longrightarrow U(Z, z)$ is a horn morphism. Then in the diagram

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "w"'] & TX \arrow[r, "Tr"] \arrow[d, "s"'] \arrow[dr, "Tf"] & TP \arrow[dr, dashed, "Tf'"] & \\
X \arrow[r, "r"] \arrow[drr, "f"'] & P \arrow[dr, dashed, "f'"] & TZ \arrow[r, "\mathrm{id}"] \arrow[d, "z"] & TZ \\
 & & Z &
\end{tikzcd}$$

the fact that $z.Tf.v = fw$ and the fact that the upper left square is a pushout implies that there is a unique $f' : P \longrightarrow Z$ such that $f's = z.Tf$ and $f'r = f$. The remaining commutativities readily follow so that $(f', Tf'.Tr)$ is a horn morphism.

If $(g, Tg.Tr)$ is another horn morphism extending $(f, Tf.v)$, then $gr = f$ which implies that $Tg.Tr = Tf$ and then $gs = z.Tf = f'$ which, together with $gr = f$ and the uniqueness of maps from a pushout, implies that $g = f'$. ∎

3.13. Corollary. *If H is separated, the successor map $\gamma : H \longrightarrow H' \longrightarrow \mathrm{Succ}(H)$ has the unique extension property with respect to T-algebra horns.* ∎

3.14. Proposition. *The categories of horns and of separated horns are cocomplete.*

Proof. Let $D : I \longrightarrow \mathcal{THorn}$ be a diagram. Suppose $D_i = (X_i, Y_i, v_i, w_i)$. Let $X = \operatorname{colim} X_i$ and $Y = \operatorname{colim} Y_i$ with transition maps $r_i : Y_i \longrightarrow Y$ and $s_i : X_i \longrightarrow X$. We define

$w : Y \longrightarrow X$ and $v : Y \longrightarrow TX$ as the unique maps for which the diagram

$$\begin{tikzcd}
Y_i \arrow[r, "v_i"] \arrow[d, "w_i"'] \arrow[dr, "r_i"] & TX_i \arrow[dr, "s_i"] & \\
X_i \arrow[dr, "s_i"] & Y \arrow[r, "v"] \arrow[d, "w"] & TX \\
& X &
\end{tikzcd}$$

commutes.

For a colimit in separated horns, apply the reflection L to the colimit in horns. ∎

3.15. The Basic Ordinal Sequence.

Recall that a partially ordered set, or class, can be regarded as a category with a single morphism from $x \longrightarrow y$ iff $x \leq y$ and no other morphisms.

If n is an ordinal number, then an **n-sequence** is a functor from n to a given category while an **ordinal sequence** is a functor from the class of all ordinals.

3.16. Definition. *Let H be a T-horn. We use transfinite induction to define a T-horn H_n for every ordinal n and maps $\theta_{mk} : H_k \longrightarrow H_m$ as follows:*

1. *We start by letting $H_0 = L(H)$, and letting $\eta : H \longrightarrow H_0$ be the reflection map.*

2. *If n is a non-limit ordinal, then we have $H_0, H_1, \ldots, H_{n-1}$ and can define $H_n = \mathrm{Succ}(H_{n-1})$, the successor of H_{n-1}. We use Proposition 3.12, to define $\theta_{n,n-1}$.*

3. *If $n > 0$ is a limit ordinal, then the n diagram $\{H_m \mid m < n\}$ has been defined and we let H_n be the colimit of this diagram in the category of separated T-horns. (Note this colimit is found by taking the colimit in $\mathcal{THorn}$ then reflecting that colimit into the subcategory of separated T-horns.) The maps in the colimit diagram give us the transition maps θ_{mn}.*

3.17. Proposition. *Given a T-horn H, the ordinal-indexed family $\{H_n\}$ together with the maps θ_{mk} defined above forms an ordinal sequence in the category of T-horns.*

Moreover, all of the maps $\theta_{m,k}$ as well as the map $\eta : H \longrightarrow H_0$ have the unique extension property with respect to T-algebra horns.

Proof. Using Proposition 3.12 and the properties of colimits, the details of the proof follow by a straightforward transfinite induction. ∎

3.18. Definition. *The* **basic ordinal sequence generated by a T-horn** *H is the ordinal sequence of Definition 3.16. The T-horn H is called the* **generator** *of the basic ordinal sequence.*

Sometimes we will let H_{-1} denote the generating horn H.

4. Applications

In this section we will show that if the basic ordinal sequence generated by JX stabilizes (4.9), then X generates a free T-algebra, and that if it does so for every $X \in \mathcal{X}$, then T generates a free triple. Moreover if the basic ordinal sequence for every T-horn stabilizes, then T-Alg is cocomplete.

Recall that $U : T\text{-Alg} \longrightarrow \mathcal{X}$ is the obvious underlying functor, that $K : T\text{-Alg} \longrightarrow \mathcal{THorn}$ is given by $K(Z, z) = (Z, TZ, \mathrm{id}, z)$, and that $J : \mathcal{X} \longrightarrow \mathcal{THorn}$ is given by $JX = (X, 0, v, w)$ where 0 is an initial object and v and w are forced.

4.1. Proposition. *For any $X \in \mathcal{X}$ and $(Z, z) \in T$-Alg, there is a one-one correspondence between admissible maps $X \longrightarrow Z$ in $\mathcal{X}$ and maps $JX \longrightarrow K(Z, z)$ in $\mathcal{THorn}$.*

Proof. This is obvious from the diagram

$$\begin{tikzcd}
0 \arrow[r, "v"] \arrow[d, "w"'] \arrow[dr] & TX \arrow[dr, "Tf"] & \\
X \arrow[dr, "f"'] & TZ \arrow[r, "\mathrm{id}"] \arrow[d, "z"] & TZ \\
& Z &
\end{tikzcd}$$

∎

Combining this with 3.17 we have:

4.2. Corollary. *Let $\{H_n\}$ be the basic ordinal sequence generated by JX. Then for any T-algebra (Z, z) and any ordinal n, there is a one-one correspondence between admissible maps $X \longrightarrow Z$ and maps $H_n \longrightarrow K(Z, z)$.* ∎

4.3. Definition. *A basic ordinal sequence $\{H_n, \theta_{mn}\}$ will be said to* **stabilize at** *n if for all $m > n$, the transition maps θ_{mn} are isomorphisms. We will often say that such a sequence* **converges**.

4.4. Definition. *The horn* (X, Y, v, w) **splits** *if there exists a map* $x : TX \longrightarrow X$ *such that both triangles commute in the pushout diagram*

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "w"'] & TX \arrow[dl, "x"'] \arrow[d, "s"] \\ X \arrow[r, "r"] & P \end{tikzcd}$$

4.5. Lemma. *The horn* (X, Y, v, w) *splits iff* $r : X \longrightarrow P$ *is an isomorphism in the diagram above.*

Proof. If r is an isomorphism, then it is trivial to see that $r^{-1}s$ makes both triangles commute. For the other direction, suppose such an x is given. In the diagram

$$\begin{tikzcd} Y \arrow[r, "v"] \arrow[d, "w"'] & TX \arrow[dl, "x"'] \arrow[d, "s"] \arrow[ddr, "x"] & \\ X \arrow[r, "r"] \arrow[drr, "\mathrm{id}"'] & P & \\ & & X \end{tikzcd}$$

in which the upper left square is a pushout, we conclude that there is a map $g : P \longrightarrow X$ such that $gr = \mathrm{id}$ and $gs = x$. Moreover $rg : P \longrightarrow P$ satisfies $rgr = r$ and $rgs = rx = s$. The uniqueness of maps from a colimit forces $rg = \mathrm{id}$ so that r is an isomorphism. ∎

4.6. Lemma. *The horn* (X, Y, v, w) *splits if* v *is epic and there is a map* $x : TX \longrightarrow X$ *such that* $xv = w$.

Proof. In the diagram of the previous lemma, r is split monic, while also epic, therefore an isomorphism. ∎

4.7. Corollary. *If* (Z, z) *is a* T*-algebra, then* $K(Z, z)$ *is split.*

Proof. Recall that $K(Z, z) = (Z, TZ, \mathrm{id}, z)$ so $v = \mathrm{id}$ is epic and we can take $x = z$. ∎

4.8. Proposition. *If the horn* (X, Y, v, w) *is split by* $x : TX \longrightarrow X$*, then* (X, x) *is a* T*-algebra and the successor of* (X, Y, v, w) *is* $K(X, x)$*.*

Proof. Since r is an isomorphism, we can construct the pushout so that $P = X$, $r = \mathrm{id}$ and $s = x$. Then the pre-successor is clearly (X, TX, id, x) because the square in the diagram below is a pushout:

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "w"'] & TX \arrow[r, "\mathrm{id}"] \arrow[d, "x"] & TX \\
X \arrow[r, "\mathrm{id}"'] & X &
\end{tikzcd}$$

Moreover, the pre-successor is clearly $K(X, x)$. Since the pre-successor is a T-algebra, it is clearly separated and therefore is the successor. ∎

4.9. Theorem. *Suppose that $\{H_n, \theta_{mn}\}$ is the ordinal sequence as in 3.15. If there is an ordinal n such that H_n splits, then the sequence stabilizes at H_{n+1}.*

Proof. Suppose H_n is split by $x : TX_n \longrightarrow X_n$. Then H_{n+1} is the successor of H_n which, by the above lemma is $K(X, x)$. We claim that H_m is $K(X, x)$ for all $m > n$. The proof is by transfinite induction, so we assume that $H_k = K(X, x)$ for all k with $n < k < m$. If m is a non-limit ordinal, the inductive step follows by considering the successor, as above. If m is a limit ordinal, we first compute $\mathrm{colim}_{k<m} H_k$. But this is clearly $K(X, x)$ as $H_k = K(X, x)$ for a cofinal subsequence. Then we take the separated reflection of $K(X, x)$ but this is clearly $K(X, x)$ as $K(X, x)$, being a T-algebra horn, is obviously separated. ∎

4.10. Corollary. *Suppose the sequence of $\{H_m\}$ stabilizes at n. Then the transition map $(u_{n+1,n}, v_{n+1,n})$ is a horn isomorphism. Suppose that $(f, g) : H_n \longrightarrow K(Z, z)$ is a map in $\mathcal{T}\mathcal{H}orn$, then $f : (X_n, u^{-1}_{n+1,n} w_{n+1}) \longrightarrow (Z, z)$ is a map in T-*Alg.

Proof. From the diagram

$$\begin{tikzcd}
Y_n \arrow[r, "v_{n+1,n}"] \arrow[d, "w_n"'] \arrow[drr, "g"] & TX_n \arrow[d, "w_{n+1}"'] \arrow[drr, "Tf"] & & \\
& & TZ \arrow[r, "\mathrm{id}"'] \arrow[dd, "z"] & TZ \\
X_n \arrow[r, "u_{n+1,n}"] \arrow[drr, "f"'] & X_{n+1} & & \\
& & Z &
\end{tikzcd}$$

we have $fu^{-1}_{n+1,n} w_{n+1} = f w_n v^{-1}_{n+1,n} = z g v^{-1}_{n+1,n} = z.Tf$. ∎

Combining this with 4.2, we have:

4.11. Theorem. *Given an object $X \in \mathcal{X}$ for which the basic ordinal sequence generated by $H_{-1} = JX$ stabilizes at n, then H_n is the free T-algebra horn generated by X.* ∎

4.12. Theorem. *Suppose n is a limit ordinal such that T preserves the colimit of n-sequences, so that the canonical map*

$$\operatorname{colim}_{m<n} TX_m \longrightarrow T(\operatorname{colim}_{m<n} X_m)$$

is an isomorphism. Then every basic ordinal sequence stabilizes at $n+1$.

Proof. We compute $H_{n+1} = (X_{n+1}, Y_{n+1}, v_{n+1}, w_{n+1})$ by first finding the $\operatorname{colim}_{m<n} H_n$ then taking its separated reflection. But the colimit is (X, Y, v, w) where $X = \operatorname{colim} X_m$ and $Y = \operatorname{colim} Y_m = \operatorname{colim} TX_m$ and v is the obvious map from $Y \longrightarrow TX$. But, by hypothesis, v is an isomorphism. Therefore the colimit is a T-algebra horn, see 4.8. It follows that this T-algbra horn is separated, so it coincides with H_{n+1} and, by the above theorem, the sequence stabilizes at H_{n+1} . ∎

4.13. Corollary. *If there exists a limit ordinal n such that T commutes with colimits over n, then all basic ordinal sequences stabilize and every $X \in \mathcal{X}$ generates a free T-algebra. Thus $U : T\text{-Alg} \longrightarrow \mathcal{X}$ has a left adjoint and T generates a free triple.*

Proof. The last claim follows from [Barr (1970), Theorem 5.4]. ∎

Putting this all together we get

4.14. Theorem. *If for each object $X \in \mathcal{X}$, the basic ordinal sequence generated by JX stabilizes, then T generates a free triple. Moreover, if the basic ordinal sequence generated by H_{-1} converges, then the T-algebra horn to which it converges is the reflection of H_{-1} in the subcategory of T-algebra horns.* ∎

In the case when $\mathcal{X}$ is $\mathcal{M}$-well-powered, the converse is also true.

4.15. Theorem. *Suppose that $\mathcal{X}$ is $\mathcal{M}$-well-powered and T is an endofunctor for which a free triple exists. Then the basic ordinal sequences of T-horns generated by horns JX converge for all $X \in \mathcal{X}$. Moreover, if a horn H has a reflection in the subcategory of T-algebra horns, then the basic ordinal sequence generated by H converges to that reflection.*

Proof. Suppose T generates the free triple $\mathbf{F} = (F, \eta, \mu)$. For any object X, we form the basic ordinal sequence of horns $\{H_n\}$ as described above. The map $\eta : X \longrightarrow FX$ generates a horn morphism $H \longrightarrow K(FX, \mu X)$ which then generates unique maps $\theta_n : H_n \longrightarrow K(FX, \mu X)$ for each n. Let $\{H_n \longrightarrow K(Z_i, z_i)\}$ be a cone of $\mathcal{THorn}$-morphisms for which the associated cone $\{X_n \longrightarrow Z_i\}$ is an $\mathcal{M}$-cone. Since $(FX, \mu X)$ is free, each of the maps $H_n \longrightarrow K(Z_i, z_i)$ factors through $K(FX, \mu X)$. But then the map $X_n \longrightarrow FX$ must be in $\mathcal{M}$, for any first factor in $\mathcal{E}$ would also be a first factor in every $X \longrightarrow Z_i$ which implies that such a factor would be an isomorphism. But since FX has only a set of $\mathcal{M}$-subobjects, the class of all these $X_n \longrightarrow FX$ can only be a set, which implies that the sequence of X_n stabilizes. ∎

4.16. Remark. Recall that if H is a horn, then a horn morphism $\theta : H \longrightarrow K(Z_H, z_H)$ reflects H into the T-algebra horns if every horn map $H \longrightarrow K(Z, z)$ factors uniquely through θ.

4.17. Theorem. *If every basic ordinal sequence of horns stabilizes, then* T-Alg *is cocomplete.*

Note that this hypothesis differs from that of 4.14 in that it requires **every** basic ordinal sequence of horns to converge. See [Adámek (1977), Section III].

Proof. Since $K : T\text{-Alg} \longrightarrow \mathcal{THorn}$ is full and faithful and $\mathcal{THorn}$ is cocomplete, the conclusion follows as soon as K has a left adjoint. ∎

4.18. Example. Let $\mathcal{X}$ be the ordered category whose objects are the ordinals, together with one more object ∞ such that $n < \infty$ for all ordinals n. We let $\mathcal{M}$ consist of all the morphisms, while $\mathcal{E}$ consists of the isomorphisms (which are all identities). Define a functor $T : \mathcal{X} \longrightarrow \mathcal{X}$ by $Tn = n + 1$ and $T\infty = \infty$. The only T-algebra is ∞ since there can be no maps $n + 1 \longrightarrow n$. Clearly, the free triple generated by T exists and is the constant functor at ∞, while the ordinal chain never stabilizes. Note that $\mathcal{X}$ is not $\mathcal{M}$-well-powered.

4.19. Colimits in tripleable categories. Although not precisely an application of the above, we can apply the same type of construction to colimits in a category of algebras for a triple $\mathbf{T} = (T, \eta, \mu)$. The main change is to replace the category of T-algebras for the triple by the category of $\mathbf{T}$-algebras which we will denote $\mathcal{X}^{\mathbf{T}}$. We use the same definition of horn, but say that a horn is **T-separated** if it has an $\mathcal{M}$-cone whose base consists of $\mathbf{T}$-algebras. We denote by $V : \mathcal{X}^{\mathbf{T}} \longrightarrow \mathcal{X}$ the underlying functor.

Let $D : \mathcal{J} \longrightarrow \mathcal{X}^{\mathbf{T}}$ be a diagram in $\mathcal{X}^{\mathbf{T}}$. Let $X = \operatorname{colim} VD$. Let $\{X \longrightarrow V(Z_i, z_i)\}$ be the cone of all maps from X to the object underlying a $\mathbf{T}$-algebra such that, for all $j \in \mathcal{J}$, the composite $VD_j \longrightarrow X \longrightarrow Z_i$ underlies a map in $\mathcal{X}^{\mathbf{T}}$. $H = JX$ and let H_0 be its $\mathbf{T}$-separated reflection. Continue to build an ordinal sequence $\{H_n\}$, using at all stages the $\mathbf{T}$-separated reflection. If this stabilizes at n, then $(R, r) = (X_n, u_{n+1,n}^{-1} w_{n+1})$ is a T-algebra as above. We wish to show that it is a $\mathbf{T}$-algebra in which case it is a colimit in $\mathcal{X}^{\mathbf{T}}$.

The first condition that has to be satisfied is that $r.\eta R = \mathrm{id}$. Let $R \underset{r.\eta R}{\overset{\mathrm{id}}{\rightrightarrows}} R \xrightarrow{e} S$ be a coequalizer. For any $f_i : R \longrightarrow U(Z_i, z_i)$ where (Z_i, z_i) is a $\mathbf{T}$-algebra, we have a commutative diagram

$$\begin{tikzcd}
R \arrow[r, "\eta R"] \arrow[d, "f_i"'] & TR \arrow[r, "r"] \arrow[d, "Tf_i"'] & R \arrow[r, "e"] \arrow[d, "f_i"'] & S \\
Z_i \arrow[r, "\eta Z_i"] & TZ_i \arrow[r, "z_i"] & Z_i &
\end{tikzcd}$$

The left hand square commutes because of the naturality of η and the right hand one does because f_i is a morphism of T-algebras. It then follows that $f_i.r.\eta R = f_i.\mathrm{id}$ so that there is a map $g_i : S \longrightarrow Z$ such that $g_i e = f_i$. Then for each i in an $\mathcal{M}$-cone of $\mathbf{T}$-algebras we have a commutative diagram

$$\begin{tikzcd} R \arrow[r, two heads, "e"] \arrow[d, "\mathrm{id}"'] & S \arrow[d, "g_i"] \\ R \arrow[r, "f_i"'] & Z_i \end{tikzcd}$$

and the diagonal fill-in forces e to be an isomorphism, so that $r.\eta R = \mathrm{id}$. The argument for the identity $r.Tr = r.\mu R$ is similar. Let $T^2R \underset{r.\mu R}{\overset{r.Tr}{\rightrightarrows}} R \longrightarrow S$ be a coequalizer and argue as above.

5. Endofunctors on $\mathcal{S}et$

5.1. Ordinal Rank. Suppose $\{H_n = (X_n, Y_n, v_n, w_n)\}$ is the n-th horn of a basic ordinal sequence. An element of X_n or Y_n may have an "ancestry", in some cases going back to elements in X_0 and Y_0. Other elements may be more recent, only going back as far as elements in X_m or Y_m where m is close to n. And some elements of X_n and Y_n may be new—not being the image of any elements from H_m for any m with $m < n$. The ordinal rank of an element in X_n or Y_n is a way of measuring the "ancestry" of an element. We can sometimes show that an ordinal sequence $H_0, H_1, \ldots, H_n, \ldots$ converges or show that it fails to converge by keeping track of "ordinal rank".

5.2. Notation.

1. In what follows, T is an endofunctor on the category of sets.
2. We recall that each ordinal n is the set of all strictly smaller ordinals. It follows that $m \leq n$ iff $m \subseteq n$.
3. If $m \leq n$, we let $i_{nm} : m \longrightarrow n$ be the inclusion of the subset $m \subseteq n$.
4. We let 0 denote the empty set.

5.3. Definition. *Let an ordinal n and an element $b \in Tn$ be given. Then the* **ordinal rank**, *or simply the* **rank**, *of b, denoted by* $\operatorname{rank} b$, *is the smallest ordinal $m \leq n$ for which b is in the image of Tm under the map Ti_{nm}.*

5.4. Lemma. *If $0 < m \leq n$ then the map Ti_{nm} is an injection.*

Proof. The function i_{nm} has a left inverse in Sets, so Ti_{nm} also has a left inverse. ∎

5.5. Lemma. *The functions $Ti_{nm} : Tm \longrightarrow Tn$ are always rank-preserving.*

Proof. Suppose that $b \in Tm$ has rank r. Then b is in the image of Tr under $Ti_{mr} : Tr \longrightarrow Tm$. Let $c = Ti_{nm}b$. Then c is clearly in the image of Tr under the map Ti_{nr} so $\operatorname{rank} c \leq \operatorname{rank} b$. So if the $\operatorname{rank} b = 0$, then clearly $\operatorname{rank} c = 0$.

Now suppose that $\operatorname{rank} b = r > 0$ and assume that $\operatorname{rank} c = k < r$. Then there exists $d \in Tk$ such that $Ti_{nk}d = c$. Let $b' = Ti_{mk}(d)$. Then $Ti_{nm}(b') = c = T'i_{nm}(b)$. By Lemma 5.4, this implies that $b' = b$ which implies that $\operatorname{rank} b \leq k$ which is a contradiction. ∎

Next, we will assume that $\{H_n = (X_n, Y_n, v, w)\}$ is the basic ordinal sequence starting with $H_0 = J0 = (0, 0, v, w)$. We will, in effect, inductively define the rank of each element of Y_n and each element of X_n. To get a conceptual sense of how the induction will proceed, suppose we had a notion of rank for elements of X_n and Y_n and suppose that $p_n x$ denotes the rank of x for every $x \in X_n$ and that $q_n y$ denotes the rank of y for every $y \in Y_n$. Suppose $p_m : X_m \longrightarrow m$ and $q_m : Y_m \longrightarrow m$ have already been defined for all ordinals $m < n$.

Then, if n is a non-limit ordinal, $Y_n = TX_{n-1}$. It follows that $Tp_{n-1} : Y_n = TX_{n-1} \longrightarrow T(n-1)$. Let $r_{n-1} : T(n-1) \longrightarrow n-1$ be the ranking function defined above for members of $T(n-1)$. Then $q_n = r_{n-1}.Tp_{n-1} : Y_n \longrightarrow n-1$ is the ranking function for Y_n. The ranking function $p_n : X_n \longrightarrow n$ can now be defined as follows: If $x \in X_n$ and $x = w_n y$ then $p_n x = q_n y$. Of course it remains to show that this definition does not depend on which $y \in Y_n$ was chosen in case more than one element of Y_n maps to x. And if there is no $y \in Y_n$ with $wy = x$, then x lacks any "ancestry" going back to ordinals less than n, so, in this case, x is a recent addition to X_n and $p_n x = n$.

We would also have to consider the case when n is a limit ordinal. It turns out that the maps (p_n, q_n) give us a horn morphism from the H_n to a special type of horn we will call an "ordinal horn", see below. **However, it is easier to directly define the horn morphisms from H_n to the ordinal horn.**

5.6. Definition. *An element $b \in Tn$ is said to be* **of maximal rank** *if* $\operatorname{rank} b = n$.

5.7. Definition. *We say that T is* **maximal** *if, for every ordinal n, the set Tn has elements of maximal rank.*

5.8. Notation. For every ordinal n, we let $T_{\max}n$ be the set of all maximal elements of Tn, and we let $T_0 n$ be the set of all non-maximal elements of Tn. So Tn is the disjoint union $Tn = T_{\max}n \cup T_0 n$.

The Ordinal horns. For each ordinal n, we let $\mathcal{O}\hbar_n$, the **ordinal horn of degree** n, be $(n, T_0 n, i, r)$ where $i : T_0 n \longrightarrow Tn$ is the inclusion and $r : T_0 n \longrightarrow n$ be defined by $rb = \operatorname{rank} b$.

5.9. Lemma. *For every ordinal n, the horn $\mathcal{O}\hbar_n$ is separated.*

Proof. If $n > 0$, we can easily embed $\mathcal{O}\hbar_n$ into a T-algebra horn by extending the map $r : T_0 n \longrightarrow n$ to any function $Tn \longrightarrow n$.

The case $n = 0$ then n is the empty set and $T_0 n$ is empty too. But we can start by embedding $\mathcal{O}\hbar_0$ into $\mathcal{O}\hbar_1$ then proceeding as above. ∎

5.10. Lemma. *If T is maximal, then for every ordinal $m \le n$, there is an element $b \in Tn$ with* $\operatorname{rank} b = m$.

Proof. By definition, there exists $a \in Tm$ with $\operatorname{rank} a = m$. Since $m \subseteq n$ we see that $i_{nm} a$ is an element of Tn of rank m. ∎

The Ordinal sequence of Ordinal Horns. It is not hard to find a horn map $\lambda_{n+1,n} : \mathcal{O}\hbar_n \longrightarrow \mathcal{O}\hbar_{n+1}$. We note that $\lambda_{n+1,n}$ factors through the pre-successor of $\mathcal{O}\hbar_n$, whose construction is indicated by the following diagram, where the square is a pushout.

$$\begin{tikzcd}
T_0 n \arrow[r, "i"] \arrow[d, "r"'] & Tn \arrow[r, "Tu"] \arrow[d, "s"] & TP \\
n \arrow[r, "u"'] & P &
\end{tikzcd}$$

By Lemma 5.10 we see that Tn has elements of rank m for all $m \le n$. So $T_0 n$, which excludes all elements of Tn of rank n, has elements of rank m for all $m < n$. But recall that n is precisely the set of all such ordinals m, so $r : T_0 n \twoheadrightarrow n$ is onto.

For $m < n$, note that $r^{-1}m$, the set of all elements of rank m, is a subset of both $T_0 n$ and Tn. Since the entire subset gets mapped to the element $m \in n$, it is readily shown that $s : Tn \longrightarrow P$ must map the entire subset $r^{-1}m$ to a single element of P. We will let m denote that element. It is readily shown that P is the union of n, the set of all m for which $m < n$ with the set $T_{\max} n$. Recall that the successor horn to $\mathcal{O}\hbar_n$ is $L(P, Tn, Tu, s)$. But notice that there is a map $r' : P \longrightarrow n+1$ for which $r'm = m$ and $r'b = n$ for all maximal elements $b \in Tn$. It is easily verified that this gives us an onto horn map from (P, Tn, Tu, s) to $\mathcal{O}\hbar_{n+1}$. We then define $\lambda_{n+1,n}$ as the canonical horn map from $\mathcal{O}\hbar_n$ to its pre-successor followed by the onto map mentioned above.

Similarly, for n is a limit ordinal, there is an obvious onto map from $\operatorname{colim}_{m<n} \mathcal{O}\hbar_m \longrightarrow \mathcal{O}\hbar_n$. It is readily shown that these maps can be used to define the maps $\lambda_{m,k} : \mathcal{O}\hbar_k \longrightarrow \mathcal{O}\hbar_m$ for $k \le m$, such that we have a functor from the ordered category of all ordinals to the ordinal horns.

5.11. Proposition. *If T is a maximal functor, and if $H_0, H_1, \ldots, H_n, \ldots$ is the basic ordinal sequence for the empty set, then, for all $n > 0$, there is a surjection from $H_n \longrightarrow \mathcal{O}\hbar_n$.*

Proof. We use transfinite induction to define a natural transformation ϕ from the basic ordinal sequence $\{H_n\}$ to the ordinal sequence $\{\mathcal{O}\hbar_n\}$.

For $n = 0$ observe that $H_0 = (0, 0, v, w) = \mathcal{O}\hbar_0$ and let ϕ_0 be the identity map.

Assume $\phi_k : H_k \longrightarrow \mathcal{O}\!\mathcal{h}_k$ has been defined for $0 \le k < n$. We then will define $\phi_n : H_n \longrightarrow \mathcal{O}\!\mathcal{h}_n$. First assume that n is a non-limit ordinal,. Then $\phi_{n-1} : H_{n-1} \longrightarrow \mathcal{O}\!\mathcal{h}_{n-1}$ has been defined. it is readily shown that ϕ_{n-1} induces a horn map from the pre-successor of H_{n-1} to the pre-successor of $\mathcal{O}\!\mathcal{h}_{n-1}$. In turn, the pre-successor of $\mathcal{O}\!\mathcal{h}_{n-1}$ maps onto $\mathcal{O}\!\mathcal{h}_n$, giving us a map ρ from the pre-successor of H_{n-1} to $\mathcal{O}\!\mathcal{h}_n$. Since $\mathcal{O}\!\mathcal{h}_n$ is separated, the map ρ extends to a map from H_n, the separated reflection of the pre-successor of H_{n-1} to $\mathcal{O}\!\mathcal{h}_n$. Let $\phi_n : H_n \longrightarrow \mathcal{O}\!\mathcal{h}_n$ be this extension of ρ.

Finally, if n is a limit ordinal, then H_n is the colimit of $\{H_k \mid k < n\}$ and the colimit property enables us to define ϕ_n using the maps $\{\lambda_{n,k}\phi_k : H_k \longrightarrow \mathcal{O}\!\mathcal{h}_n \mid k < n\}$. Letting $\phi_n = (p_n, q_n)$, it is straightforward, in view of the above discussion, that $p_n : X_n \longrightarrow n$ is onto for all n. ∎

5.12. Corollary. *If T is maximal, the basic ordinal sequence for the empty set does not stabilize.*

Proof. In view of the above, the horns in the basic ordinal sequence for the empty set must have arbitrarily large cardinality since for every n we see that H_n must map onto $\mathcal{O}\!\mathcal{h}_n$. This implies that it cannot stabilize. ∎

5.13. Remark. Some functors T may have the property Tn is maximal once n is sufficiently large. A similar argument then shows that the ordinal sequence for JS cannot converge if S is sufficiently large. See example 2 below.

5.14. Examples.

1. Let T be the covariant power set functor. If $f : R \longrightarrow S$, then Tf is defined as the direct image, so that $Tf(A) = \{f(a) \mid a \in A\}$. Then T is easily seen to be maximal so the basic ordinal sequence for $J(0)$ does not converge.

2. On the other hand, if we take T to be the **finite** power set functor, with direct image, then it is clear that T commutes with colimits over ω so that the existence of a free triple follows from 4.13. Any restriction of the sizes of the subsets allowed will have the same effect.

3. Let T be the ultrafilter functor for which $T(R)$ is the set of all ultrafilters on R. If $f : R \longrightarrow S$ and if $\mathcal{U}$ is an ultrafilter on R, define $Tf(\mathcal{U}) = \mathcal{V}$ where $V \subseteq S$ is in $\mathcal{V}$ iff $f^{-1}(V) \in \mathcal{U}$. Then T is not maximal as $T(0) = 0$. The basic ordinal sequence for $J(0)$ converges immediately. But if R is an infinite set, the basic ordinal sequence for $J(R)$ does not converge as the argument given above applies. Even if R has only one element, it can be shown that the sequence for $J(R)$ fails to converge. (Suppose $\{H_0, H_2, \ldots, H_n = (X_n, Y_n, v_n, w_n), \ldots\}$ is the basic ordinal sequence for a one-point set. Then, for finite n the set X_n has $n+1$ elements and has infinitely many elements whenever n is infinite and the above argument applies.)

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