# SEARCHING FOR MORE ABSOLUTE $\mathcal{CR}$-EPIC SPACES

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Abstract. Our examination continues of spaces that have the property that any dense embedding induces an epimorphism in the category of commutative rings of the rings of real-valued functions.

## 1. Introduction

In several earlier papers, [Barr, *et al.* (2003), Barr, *et al.* (2005), Barr, *et al.* (2006)], we have explored the concept of embeddings of (completely regular Hausdorff) spaces that induce an epimorphism in the category of commutative rings on their rings of continuous real-valued functions. Such embeddings are called $\mathcal{CR}$-epic. A space is absolute $\mathcal{CR}$-epic if every embedding is $\mathcal{CR}$-epic. We are particularly interested in absolute $\mathcal{CR}$-epic spaces.

When $X$ is Lindelöf, we know that $X \longrightarrow Y$ is $\mathcal{CR}$-epic if and only if every bounded real-valued function on $X$ extends to an open subset of $Y$ that contains $X$ ([Barr, *et al.* (2005), Corollary 2.14]). When $X$ is not Lindelöf it is still necessary, but not sufficient, that every such function extend to an open set. Clearly, the embedding $X \longrightarrow \beta X$ is always $\mathcal{CR}$-epic, but a space cannot be absolute $\mathcal{CR}$-epic unless it is almost Lindelöf. meaning that of any two disjoint closed subsets, at least one is Lindelöf. One of the things we do in this paper is find more examples of almost Lindelöf spaces that are absolute $\mathcal{CR}$-epic but not Lindelöf.

It is easy to see that $X$ is absolute $\mathcal{CR}$-epic if and only if the embedding into every compactification of $X$ is absolute $\mathcal{CR}$-epic. For Lindelöf spaces, this means that every bounded function on $X$ extends to an open set of every compactification. We have seen in [Barr, *et al.* (2006), ] that every bounded function extends to a $G_\delta$ set containing $X$. Thus one way to guarantee that $X$ is absolute $\mathcal{CR}$-epic is to show that in any compactification, every $G_\delta$ containing $X$ contains an open set containing $X$. In fact, it suffices that this happen in $\beta X$. We call this the **countable neighbourhood property (CNP)**. See Section 3 of [Barr, *et al.* (2006)], especially Proposition 3.2, for fuller explanation. This seemed so fundamental that we wondered if this sufficient condition (in the Lindelöf case) was also necessary. In this paper we will show by example that it is not by exhibiting a Lindelöf absolute $\mathcal{CR}$-epic space that does not satisfy the CNP. Nonetheless, the CNP condition is necessary in order to have a good product theorem (8.1, below and [Barr,

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*et al.* (2006), Theorem 4.7]). This is especially useful for a space that satisfies Alster's condition (see [Alster (1988)]).

Other topics in this paper include alternate characterizations of CNP and study of the relation between $X$ and $\beta X - X$ (the "outgrowth" of $X$).

We recall some definitions from [Barr, *et al.* (2006)] that will be mentioned in this paper. As usual, all spaces will be completely regular Hausdorff. We said that a space was *amply Lindelöf* if every $G_\delta$ cover that covers each compact set finitely, contains a countable subcover. We have subsequently discovered that an equivalent condition was first discovered and exploited by K. Alster, [Alster (1988), Condition $(*)$] in connection with the question of which spaces have a Lindelöf product with every Lindelöf space. Accordingly, we will rename this as Alster's condition. (It must be admitted that we were never happy with "amply Lindelöf" in the first place.) A space that satisfies Alster's condition will be called an **Alster space**. We will continue to call a cover of a space ample if every compact set is finitely covered. Since $G_\delta$ sets are closed under finite unions, we will often suppose, without explicit mentions that our $G_\delta$ covers are closed under finite unions. For an ample $G_\delta$ cover, this means that every compact set is contained in some element of the cover.

We will be studying Alster spaces in a subsequent paper that concentrates on Alster's original context, namely that the product of an Alster space and a Lindelöf space is Lindelöf. See [Barr, *et al.*, (to appear)].

## 2. Some (punctured) planks are absolute $\mathcal{CR}$-epic

A topological space that is a product of two total orders is sometimes called a **plank**. Sometimes the word is used for what we call a **punctured plank**, a plank with one point removed. For example, the **Dieudonné plank** is the space $(\omega+1)\times(\omega_1+1)-\{(\omega,\omega_1)\}$. In [Barr, *et al.* (2006), 7.14, 7.15] we showed that the Dieudonné plank and certain other punctured planks are absolute $\mathcal{CR}$-epic. Here we extend that result.

We will be dealing with complete lattices. Each such lattice $X$ has a compact topology in which the subbasic closed sets are the closed intervals $[x,y]=\{u \mid x\le u\le y\}$. The topology is not necessarily Hausdorff, but it is when the lattice can be embedded into a product of chains. These lattices, being complete, will have top and bottom elements that we will denote by $\top$ and $\bot$, respectively.

2.1. Theorem. *Suppose $X$ is a complete lattice. Then the closed interval topology described above is compact.*

Proof. We use [Kelley (1955), 5.6] which states that a space is compact if any collection of subbasic closed sets with the finite intersection property has non-vacuous intersection. But if for some index set $I$ we have a collection of closed intervals $[x_i,y_i]$ with the finite intersection property, then we must have for all $i,j\in I$ that $x_i\le y_j$. If not, we would have $[x_i,y_i]\cap[x_j,y_j]=[x_i\vee x_j, y_i\wedge y_j]=\emptyset$. But then $\bigcap[x_i,y_i]=[\bigvee x_i,\bigwedge y_i]\ne\emptyset$. ∎

For the purposes of the next theorem, let us say that a complete chain $X$ **has (resp. lacks) a proper countable cofinal subset** when $X - \{\top\}$ does.

2.2. Theorem. *Let $X$ be a finite product of complete chains, none of which has a proper countable cofinal subset. Then $X - \{\top\}$ is almost compact and $\top$ is a P-point of $X$.*

Proof. Suppose $X$ is a chain that lacks a proper countable cofinal set. We need to prove that $\beta(X - \{\top\}) = X$, or that $X - \{\top\}$ is C-embedded in $X$. Suppose $f : X - \{\top\} \longrightarrow [0, 1]$. Let $t_0 = \liminf_{x \longrightarrow \top} f(x)$ and $t_1 = \limsup_{x \longrightarrow \top} f(x)$. We claim that $t_0 = t_1$. Choose $x_1 \in X$ arbitrarily. Let $x_2 > x_1$ such that $|f(x_2) - t_0| < 1/2$. Continuing in this way choose $x_n > x_{n-1}$ such that $|f(x_n) - t_{n \bmod 2}| < 1/n$. Let $x = \sup x_n$ and it is immediate that $t_0 = f(x) = t_1$. This allows us to extend $f$ to all of $X$ and it will obviously be continuous. The case of a finite product follows from

2.3. Proposition. *Suppose $X$ and $Y$ are compact spaces and $x_0 \in X$ and $y_0 \in Y$ are such that $X - \{x_0\}$ and $Y - \{y_0\}$ are almost compact. Then $X \times Y - \{(x_0, y_0)\}$ is almost compact.*

Proof. According to [Gillman & Jerison (1960), 9.14] the product of a compact space and a pseudo-compact space is pseudo-compact. In particular, $X \times (Y - \{y_0\})$ is pseudo-compact. According to [Glicksberg, 1959, Theorem 1], this implies that

$$\beta(X \times (Y - \{y_0\})) = X \times Y$$

. Since the $\beta$-compactification of any space between $X$ and $\beta X$ is $\beta X$, we conclude in this case that

$$\beta(X \times Y - \{(x_0, y_0))\} = X \times Y$$

Finally, we want to prove that $\top$ is a P-point of $X$. If $X = \prod_{i=1}^{n} X_i$ of complete chains, none having a proper countable cofinal set, then a countable sup of elements less than $\top$ is less than $\top$. A basic neighbourhood of $\top$ in $X_i$ has the form $(x_i, \top]$ and then a basic neighbourhood of $\top$ in $X$ is $\prod_{i=1}^{n}(x_i, \top]$. A countable set of such neighbourhoods has the form $\prod_{i=1}^{n}(x_{i,j}, \top]$ for $j = 1, 2, 3, \ldots$. The intersection of these neighbourhoods contains $\prod_{i=1}^{n}(\sup_j x_{ij}, \top]$, which is a neighbourhood of $\top$. ∎

2.4. Theorem. *Suppose that $X = \prod X_i$ is a product of finitely many complete chains, each of which has the property that $X_i - \{\top\}$ has a proper countable cofinal set. Then $X - \{\top\}$ is locally compact and $\sigma$-compact.*

Proof. Let $S_i = \{x_{i1}, x_{i2}, \ldots\}$ be a countable cofinal set in $X_i - \{\top\}$. Then it is evident that $S = \prod(S_i \cup \{\top\}) - \{(\top, \top, \ldots, \top)\}$ is a countable cofinal set in $X - \{\top\}$. For every $s \in S$, it is evident that $[\bot, s]$ is compact so that $X - \{\top\}$ is $\sigma$-compact. For any element $x = (x_1, \ldots, x_n) \in X - \{\top\}$, there is a compact neighbourhood of $x_i$ in $X_i$. At least one $x_i \neq \top$ and for that one, the compact neighbourhood can be chosen to exclude $\top$. The product of these compact neighbourhoods is a compact neighbourhood of $x$ in $X - \{\top\}$. ∎

Recall that a space is weakly Lindelöf if from every open cover a countable subset can be found whose union is dense. Obviously a Lindelöf space is weakly Lindelöf and so is any space with a dense Lindelöf subspace.

2.5. THEOREM. *Suppose that $Y$ is a space and $y_0$ is a non-isolated P-point. Suppose $Z$ is a space with a point $z_0$ such that $Z - \{z_0\}$ is weakly Lindelöf. Then $Y \times Z - \{(y_0, z_0)\}$ is $C$-embedded in $Y \times Z$.*

PROOF. Let $f \in C(Y \times Z - \{(y_0, z_0)\})$. For each $z \in Z - \{z_0\}$ and each $n \in \mathbf{N}$, there is a neighbourhood $V(z, n)$ of $y_0$ in $Y$ and a neighbourhood $W(z, n)$ of $z$ in $Z - \{z_0\}$ such that the oscillation of $f$ in $V(z, n) \times W(z, n)$ is less than $1/n$. Suppose $z(1, n), z(2, n), \ldots$ is a countable set of points of $Z - \{z_0\}$ such that $W(n) = \bigcup_{m \in \mathbf{N}} W(z(m, n), n)$ is dense in $Z - \{z_0\}$. Let $V(n) = \bigcap_{m \in \mathbf{N}} V(z(m, n), n)$. It follows that the oscillation of $f$ in $V(n) \times W(n)$ is at most $1/n$. Since $y_0$ is a P-point, $V(n)$ is a neighbourhood of $y_0$. For any $y \in Y - \{y_0\}$ both functions $f(y, -)$ and $f(y_0, -)$ are continuous on $Z - \{z_0\}$ and hence, for any $z \in Z - \{z_0\}$ and any $m \in \mathbf{N}$, there is a neighbourhood $T(m)$ of $z$ such that the oscillation in $T_m$ of both $f(y, -)$ and $f(y_0, -)$ is at most $1/m$. There is some $p \in W(n) \cap T(m)$ and we have

$$\begin{aligned} |f(y, z) - f(y_0, z)| &\leq |f(y, z) - f(y, p)| + |f(y, p) - f(y_0, p)| + |f(y_0, p) - f(y_0, z)| \\ &< 1/m + 1/n + 1/m = 2/m + 1/n \end{aligned}$$

Since the left hand side does not depend on $m$ this implies that $|f(y, z) - f(y_0, z)| \leq 1/n$. Finally, let $V = \bigcap_{n \in \mathbf{N}} V(n)$. Then for $y \in V$, we have $f(y, z) = f(y, z_0)$ and we can extend $f$ by $f(y_0, z_0) = f(y, z_0)$ for any $y \in V$. ∎

This works, in particular, if $Z - \{z_0\}$ is Lindelöf or if it contains a dense Lindelöf subspace.

From this, we can show the following generalization of [Barr, *et al.* (2006), Theorem 7.14].

2.6. THEOREM. *Suppose the Lindelöf space $Y$ is the union of a locally compact subspace and a non-isolated P-point $y_0$. Suppose $Z$ is a compact space that has a proper dense Lindelöf subspace and $z_0$ is a point not in that subspace. Then $Y \times Z - \{(y_0, z_0)\}$ is absolute $\mathcal{CR}$-epic.*

PROOF. Since $D = Y \times Z - \{(y_0, z_0)\}$ is $C$-embedded in $Y \times Z$, it follows that the realcompactification $\upsilon(D) = Y \times Z$. Since $Y \times Z$ is the union of a locally compact space and a compact space, the result follows from [Barr, *et al.* (2006), Theorem 7.11]. ∎

2.7. THEOREM. *Suppose that $X = \prod_{i=1}^{n} X_i$ is a finite product of complete chains. Assume that $\top$ is not an isolated point of any of the chains. Then $X - \{\top\}$ is absolute $\mathcal{CR}$-epic.*

Proof. Divide the spaces into two classes, $Y_1, Y_2, \ldots, Y_k$ that lack proper countable cofinal sets and $Z_1, Z_2, \ldots, Z_l$ that have them. Let $Y = \prod Y_i$ and $Z = \prod Z_j$. We know that $Y - \{\top\}$ is almost compact and $\top$ is an non-isolated P-point of $Y$. We know that $Z - \{\top\}$ is locally compact and $\sigma$-compact and $\top$ is not isolated in $Z$. It follows that $Y \times Z - \{(\top, \top)\}$ is absolute $\mathcal{CR}$-epic. ∎

## 3. Alster's condition

Most of the following theorem was proved as [Barr, *et al.* (2006), Theorem 4.7] under the additional hypothesis that the spaces satsfied the CNP. That condition was not used in the proofs; it was simply that we never studied Alster's condition separately.

3.1. Theorem.

1. *The product of two Alster spaces is Alster space.*
2. *A closed subspace of an Alster space is Alster space.*
3. *A Lindelöf space is an Alster space if every point has a neighbourhood that satisfies Alster's condition.*
4. *A union of countably many Alster spaces is an Alster space.*
5. *A cozero-subspace of an Alster space is an Alster space.*
6. *If $\theta : Y \longrightarrow X$ is a perfect surjection, then $X$ satisfies Alster's condition if and only if $Y$ does.*

Proof. We will mention briefly what, if any changes are needed from the proofs in [Barr, *et al.* (2006), Theorem 4.7].

1. This is just [Barr, *et al.* (2006), Theorem 4.5].
2. See [Barr, *et al.* (2006), proof of 4.7.2]
3. See [Barr, *et al.* (2006), proof of 4.7.3]
4. This is a stronger claim (union, not sum) than the corresponding part of [Barr, *et al.* (2006)] and requires its own proof. Suppose $X = \bigcup X_n$ is such a union. If $\mathcal{U}$ is an ample $G_\delta$ cover of $X$ then $\mathcal{U}_n = \{U \cap X_n \mid U \in \mathcal{U}\}$ is a $G_\delta$ cover of $X_n$ for all $n$. If $K$ is a compact subset of $X_n$, it is a compact subset of $X$ and hence covered by a finite subset of $\mathcal{U}$ and therefore by a finite subset of $\mathcal{U}_n$. Thus $\mathcal{U}_n$ is an ample cover of $X_n$ and therefore has a countable refinement. Thus there is a countable subset of $\mathcal{U}$ whose union contains $X_n$. Since there are countably many $X_n$ we conclude that $\mathcal{U}$ has a countable refinement.
5. See [Barr, *et al.* (2006), proof of 4.7.5]

6. The proof of [Barr, *et al.* (2006), 4.7.5] does not use the CNP condition. However the first part of the proof contains an unneeded gap, which we fill here. Suppose that $\theta : Y \longrightarrow X$ is perfect and $X$ satisfies Alster's condition. Suppose that $\mathcal{V}$ is an ample cover of $Y$. Since, for each point $p \in Y$ the set $\theta^{-1}(\theta(y))$ is compact, we can assume that some member $V \in \mathcal{V}$ contains that set. But then $\theta(p) \in \theta_{\#}(V)$. Since $\theta^{-1}(\theta_{\#}(V)) \subseteq V$ the set

$$\mathcal{W} = \{\theta^{-1}(\theta_{\#}(V)) \mid V \in \mathcal{V}\}$$

is a cover of $Y$ that refines $\mathcal{V}$. Moreover since both $\theta_{\#}$ and $\theta^{-1}$ preserve both open sets and arbitrary intersections, they preserve $G_\delta$ sets so that $\mathcal{W}$ is a $G_\delta$ cover. It is also ample since for any compact set $A$ the sets $\theta(A)$ and $\theta^{-1}(\theta(A))$ are compact and if $V \in \mathcal{V}$ contains $\theta^{-1}(\theta(A))$, then

$$\theta^{-1}(\theta_{\#}(V)) \supseteq \theta^{-1}(\theta_{\#}(\theta^{-1}(\theta(A)))) = (\theta^{-1}(\theta(A)) \supseteq A$$

The rest of the proof goes through with $\mathcal{W}$ in place of $\mathcal{V}$. ∎

3.2. Theorem. *Suppose that the Lindelöf space $X$ has the property that when $U$ is a $G_\delta$ set of $\beta X$ that contains a point $p \in X$, then $U \cup X$ is a $\beta X$-neighbourhood of $p$. Then $X$ is a CNP space that satisfies Alster's condition.*

Proof. CNP is immediate. Let $\mathcal{U}$ be an ample cover of $X$ by $G_\delta$ sets of $\beta X$. We can suppose, without loss of generality, that $\mathcal{U}$ is closed under finite union. We will show that there is an $X$-neighbourhood of $p$ that is covered by a countable subset of $\mathcal{U}$. Let $p \in U_1 \in \mathcal{U}$. We claim there is a continuous $f : \beta X \longrightarrow [0, 1]$ such that $A \subseteq \mathcal{Z}[f] \subseteq U_1$. In fact, this is immediate when $U_1$ is open and it follows for $G_\delta$ sets since a countable intersection of zero-sets is a zero-set. Then $X \cup \mathcal{Z}[f]$ is a $\beta X$-neighbourhood of $p$ so that there is a compact $\beta X$-neighbourhood $W$ of $p$ with $W \subseteq X \cup \mathcal{Z}[f]$. It follows that $W \cap \mathrm{coz}(f) \subseteq X$. But $W \cap \mathrm{coz}(f)$ is $\sigma$-compact and hence there are $U_2, U_3, \ldots \in \mathcal{U}$ such that $W \cap \mathrm{coz}(f) \subseteq \bigcup_{n \geq 2} U_n$ and then $W \subseteq \bigcup_{n \in \mathbf{N}} U_n$. Having done this for each point $p \in X$, the Lindelöf property implies that there are countably many points for which the corresponding $W$ covers $X$ and then so do the corresponding sequences of $U_n$. ∎

For example, this theorem implies that when $p \in \beta\mathbf{N} - \mathbf{N}$ is a P-point, then $\mathbf{N} \cup \{p\}$ is good. See also [Barr, *et al.* (2006), Theorem 5.4].

## 4. Countable unions of absolute $\mathcal{CR}$-epic spaces

If $X \subseteq Y$ is dense, any function in $f \in C(X)$ has a largest extension to a subset of $X$ which we will call the maximal extension of $f$. Although this can be worked out from [Fine, Gillman, & Lambek (1965), Section 3.7], it is very easy to check that the following formula works. Assuming $X$ is a dense subspace of $Y$, $f \in C(X)$ can be extended to a $y \in Y$ if and only if $\bigcap \mathrm{cl}(f(U \cap X))$. taken over all neighbourhoods $U$ of $y$, is a singleton.

4.1. THEOREM. *The classes of Lindelöf absolute $\mathcal{CR}$-epic spaces, Lindelöf CNP spaces, and Alster spaces are all closed under countable open unions.*

PROOF. Let $X = \bigcup X_n$ with each $X_n$ open in $X$ and suppose that $K$ is a compactification of $X$. Let $K_n = \mathrm{cl}_K(X_n)$. It is standard that $K_n$ is a $K$-neighbourhood of $X_n$ since $X_n$ is open in $X$. Let $\hat{f}$ be the maximum extension of $f$ into $K$ and $\hat{f}_n$ be maximum extension of $f|K_n$ into $K_n$. We claim that if $p \in \mathrm{dom}(\hat{f}_n)$, then $p \in \mathrm{dom}(\hat{f})$. In fact, in order that $p \in \mathrm{dom}(\hat{f}_n)$ it is required that for any $\epsilon > 0$ there exist an open neighbourhood $U \subseteq K_n$ of $p$ such that the oscillation of $f$ in $U \cap X_n$ be less than $\epsilon$. But $U \cap X_n$ is open in $X_n$, which is open in $X$ so there is an open neighbourhood $V \subseteq K$ such that $V \cap X = U \cap X_n$. Thus for any $\epsilon > 0$, there is an open neighbourhood $V \subseteq K$ such that the oscillation of $f$ in $V \cap X$ is less than $\epsilon$ and hence $p \in \mathrm{dom}(\hat{f})$. But $\mathrm{dom}(\hat{f}_n)$ is a neighbourhood of $X_n$ in $K_n$ and hence $\mathrm{dom}(\hat{f}) \supseteq \bigcup \mathrm{dom}(\hat{f}_n)$ is a neighbourhood of $X$ in $K$, using once more the fact that each $X_n$ is open in $X$. Since $X$ is Lindelöf, it is absolute $\mathcal{CR}$-epic, [Barr, *et al.* (2005), Corollary 2.14].

Now let us consider the case that the $X_n$ are all Lindelöf CNP. Let $U \subseteq K$ be a $G_\delta$ set containing $X$. Then $U \cap K_n$ is a $G_\delta$ set in $K_n$ and hence contains a $K_n$-neighbourhood of $X_n$ which contains a $K$-neighbourhood of $X_n$. Then $\bigcup V_n$ is a $K$-neighbourhood of $X$ contained in $U$.

The case of Alster spaces is Theorem 3.1.4. ■

4.2. THEOREM. *A countable, locally finite union of closed Lindelöf (resp. Alster) CNP spaces is Lindelöf CNP (resp. Alster).*

PROOF. By [Barr, *et al.* (2006), Theorem 3.6.3], a countable sum of Lindelöf CNP spaces is Lindelöf CNP and we show in [Barr, *et al.*, (to appear)] that CNP spaces are closed under perfect image (and it is well-known that being a Lindelöf space is as well.) Thus it is sufficient to show that the map from the sum to the union is perfect. The inverse image of each point is finite, hence compact. Let $A = \sum A_n$ be a closed subset of the sum with $A_n$ closed in $X_n$. If $p \notin \bigcup A_n$ there is a neighbourhood $U$ of $p$ that meets only finitely many of the $X_n$, say $X_1, X_2, \ldots, X_m$. For each $m \le n$, the set $A_n$ is closed in $X_n$, which is closed in $X$ and hence $\bigcup_{n=1}^m A_n$ is closed in $X$. Since $p \notin \bigcup_{n=1}^m A_n$, there is a neighbourhood $V$ of $p$ which misses that union. Since $U$ does not meet any $X_n$ for $n > m$, neither does $V$ so there is a neighbourhood of $p$ that does not meet $A$. Since $p$ was an arbitrary point not in $A$, we conclude that $A$ is closed. ■

By contrast, we will see in Section 7 that even a finite union of Lindelöf absolute $\mathcal{CR}$-epic, but not CNP spaces, need not be absolute $\mathcal{CR}$-epic.

## 5. Lindelöf Absolute $\mathcal{CR}$-epic spaces that are not CNP

5.1. THEOREM. *Let $\{X_n\}$ be any countable family of non-compact absolute $\mathcal{CR}$-epic Lindelöf spaces. The space*

$$X = \beta\left(\sum X_n\right) - \sum(\beta X_n - X_n)$$

*is absolute $\mathcal{CR}$-epic, but does not satisfy the CNP.*

Proof. We see that $\beta X = \beta\left(\sum X_n\right)$ and $\beta X - X = \sum(\beta X_n - X_n)$. An element of $p \in \beta X$ is said to be at **level** $n$ if $p \in \beta X_n - X_n$ and we write $n = \ell(p)$. We will also say that $\ell(p) = \infty$ if $p \in \beta(\sum X_n) - \sum \beta X_n$. Let $E$ be an admissible equivalence relation on $\beta X$. This means that $\beta X/E$ is Hausdorff and that $X$ is embedded in it, see [Barr, *et al.* (2006), Definition 2.4] for a fuller explanation. Let $\theta : \beta X \longrightarrow \beta X/E$ be the projection. We will say that an element $p$ is **fused** with an element $q$ whenever $(p, q) \in E$, which is the same as $\theta(p) = \theta(q)$.

We interrupt this with:

5.2. Lemma. *There is an $N \in \mathbf{N}$ such that whenever $p$ is fused with $q$, then either $\ell(p) = \ell(q)$ or $\ell(p)$, $\ell(q) < N$.*

Proof. Suppose we can find elements of arbitrarily high levels that are fused with elements of levels other than their own. We will consider two cases. First suppose that for some $N \in \mathbf{N}$ there are elements $(p, q) \in E$ such that $p$ has arbitrarily high order, while $q$ is limited to being below some level, say $N$.

In that case, we can choose elements $p_n$, $q_n$ for all $n \in \mathbf{N}$ such that $p_n$ is fused with $q_n$, $N < \ell(p_1) < \ell(p_2) < \cdots$ while $\ell(q_n) \leq N$ for all $n$. The set $\{(p_n, q_n)\} \subseteq E$ thus constructed is discrete since $\beta X_{\ell(p_n)} \times \beta X_{\ell(q_n)}$ are a family of disjoint open sets each containing one element of the set. But then it has a limit point $(p, q)$ and it is clear that $\ell(p) = \infty$, which implies that $p \in X$, while $\ell(q) \leq N$ and the result is that an element of $X$ is fused, which contradicts the fact that $E$ is admissible.

In the other case, there are pairs $(p, q) \in E$ in which $\ell(p) \neq \ell(q)$ and both levels are arbitrarily high. In that case, proceed as above, but assume that $\ell(p_n), \ell(q_n) > \ell(p_{n-1}), \ell(q_{n-1})$. Again this is a discrete set and hence has a limit point $(p, q)$ but now both elements belong to $X$ and we must show that $p \neq q$. We do this as follows. Since $p_1 \neq q_1$, there is a function $f_1$, defined on all the elements $\sum \beta X_n$ of level $\ell(p_1) \vee \ell(q_1)$ such that $f_1(p_1) = 0$ and $f_1(q_1) = 1$. Since the levels of both $p_2$ and $q_2$ are above the levels of both $p_1$ and $q_1$, this function can be extended to a function $f_2$ defined on those summands of levels up to $\ell(p_2) \vee \ell(q_2)$ and in such a way that $f_2(p_2) = 0$ and $f_2(q_2) = 1$. Continuing in this way, we define a function $f$ on the sum of the finite levels such that $f(p_n) = 0$ and $f(q_n) = 1$. This function has a unique extension to the elements at infinite level and it is clear that $f(p) = 0$, while $f(q) = 1$. Again, this contradicts the admissibility of $E$. ∎

Proof of 5.1, concluded. Since fused elements are confined to $\sum_{n=1}^{N} \beta X_n$ and that space is absolute $\mathcal{CR}$-epic, there is an open set around $\sum X_n$ that excludes fused elements. Since every function in $C(X)$ will extend to that open set and since $X$ is Lindelöf, follows from [Barr, *et al.* (2005), Corollary 2.14] that $X$ is absolute $\mathcal{CR}$-epic. Finally, we observe that the set $U_n = \sum_{i=1}^{n} \beta X_n \cup \beta\left(\sum_{i>n} X_n\right)$ is open, but $\bigcap_{n=1}^{\infty} U_n$ consists of the elements at level infinity and contains no neighbourhood of $X$ and thus $X$ does not satisfy the CNP. ∎

## 6. The Bohr topology of $\mathbf{Z}$ is not absolute $\mathcal{CR}$-epic

In this section, we will establish the claim of its header. The Bohr compactification $b(\mathbf{Z})$ is the reflection of $\mathbf{Z}$ into the category of compact groups. The map of $\mathbf{Z}$ into $b(\mathbf{Z})$ is monic (as soon as there is a monomorphism of $\mathbf{Z}$ into any compact group, which there is). It is not, however, a topological embedding since no compact group can contain an infinite discrete subgroup. For if $G$ is compact and $D$ is an infinite discrete subgroup, there is a neighbourhood $U$ of 0 in $G$ that contains no non-zero element of $D$. If we choose $V$ so that $VV \subseteq U$ then for any $x$, $y \in D$, we see that $xV \cap yV = \emptyset$. Then the cover of $G$ by all its translates has no finite refinement since each translate contains at most one element of $D$. It follows that the map $\theta : \beta(\mathbf{Z}) \longrightarrow b(\mathbf{Z})$ cannot be injective since an injective continuous map between compact sets is closed. But since $\mathbf{Z}$ is embedded in $\beta(Z)$ and not in $b(Z)$, this is impossible. Since a group is homogeneous, $\theta^{-1}$ of every point is not a singleton.

Now let $\mathbf{Z}_B$ be the group of integers with the topology induced by $b(\mathbf{Z})$. Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ and $\theta : \mathbf{Z} \longrightarrow \mathbf{T}$ be an injective homomorphism gotten by letting $\phi(1)$ be an irrational angle. This extends to $\phi : b(\mathbf{Z}) \longrightarrow \mathbf{T}$. Clearly $\hat{\phi}$ is surjective since $\mathbf{Z}$ is dense in each. Suppose $t_1$, $t_2$, ... is a sequence of of elements of $\mathbf{T}$ not in the range of $\theta$ that converges to 0. Choose elements $p_n \in b(\mathbf{Z})$ so that $\hat{\phi}(p_n) = t_n$. Let $p$ be a limit point of $\{p_n\}$. Then $\hat{\phi}(p) = 0$. By [Barr, *et al.* (2005), Theorem 4.1], it will be sufficient to show that $b(\mathbf{Z}) - \{p_n\}$ is not almost compact for any $n$. But this follows from the fact that each $\phi^{-1}(p_n)$ has more than one element, [Gillman & Jerison (1960), Problem 6J].

## 7. A perfect quotient of a Lindelöf absolute $\mathcal{CR}$-epic space that is not absolute $\mathcal{CR}$-epic

It is shown in [Barr, *et al.* (2006), Theorem 3.5.5] that a perfect quotient of a CNP space is CNP. Thus it is of some interest to show that the corresponding result for absolute $\mathcal{CR}$-epic spaces is false. In fact, the space is one of the spaces of Section 5. If we take every $X_i = \mathbf{N}$, we just get

$$X = \beta(\mathbf{N} \times \mathbf{N}) - \sum_{n \in \mathbf{N}} (\beta(\{n\} \times \mathbf{N}) - (\{n\} \times \mathbf{N}))$$

It might help to visualize $\beta(\mathbf{N} \times \mathbf{N})$ as being made of four rectangles

1. $\mathbf{N} \times \mathbf{N}$
2. $\mathbf{N} \times (\beta\mathbf{N} - \mathbf{N})$ (the vertical outgrowth)
3. $(\beta\mathbf{N} - \mathbf{N}) \times \mathbf{N}$ (the horizontal outgrowth)
4. $\beta(\mathbf{N} \times \mathbf{N}) - (\mathbf{N} \times (\beta\mathbf{N} - \mathbf{N})) - ((\beta\mathbf{N} - \mathbf{N}) \times \mathbf{N})$

Here is a sketch of $\beta(\mathbf{N}\times\mathbf{N})$:

| | |
|---|---|
| $\mathbf{N}\times(\beta\mathbf{N}-\mathbf{N})$ | $\beta(\mathbf{N}\times\mathbf{N})-(\mathbf{N}\times(\beta\mathbf{N}-\mathbf{N}))-((\beta\mathbf{N}-\mathbf{N})\times\mathbf{N})$ |
| $\mathbf{N}\times\mathbf{N}$ | $(\beta\mathbf{N}-\mathbf{N})\times\mathbf{N}$ |

Our space $X$ consists of all but the lower right rectangle. Since $\mathbf{N}$ is open in $\beta\mathbf{N}$, it follows readily that $\mathbf{N}\times\mathbf{N}$ is open in $X$ so that the union of the second and fourth rectangles is a compact subspace of $L\subseteq X$. Let $Y$ be the quotient of $X$ gotten by identifying the subset $L$ to a single point. The quotient map is perfect. But $Y$ is not absolute $\mathcal{CR}$-epic since we can find a sequence of elements of $\beta Y-Y$ that converges to a point of $Y$ (see [Barr, *et al.* (2005), Theorem 2.22]). Let $p\in\beta\mathbf{N}-\mathbf{N}$ and look at the sequence $(p,1),(p,2),(p,3),\ldots$ which converges to $\{L\}$.

Here is another example of the same thing, which also illustrates the fact that even a space that is the union of two closed Lindelöf absolute $\mathcal{CR}$-epic subspaces need not be absolute $\mathcal{CR}$-epic. Such a union is a perfect quotient of the sum and the latter is certainly absolute $\mathcal{CR}$-epic.

Let

$$X=Y=\beta(\mathbf{N}\times\mathbf{N})-\sum_{n\in\mathbf{N}}(\beta(\{n\}\times\mathbf{N})-(\{n\}\times\mathbf{N}))$$

the space that we began this section with. Since $\mathbf{N}\times\mathbf{N}$ is locally compact, it is open in $\beta(\mathbf{N}\times\mathbf{N})$ and hence also open in $X$ and $Y$. Thus the complements of $\mathbf{N}\times\mathbf{N}$ in $X$ and $Y$ are closed. The space $Z$ is gotten by amalgamating $X$ with $Y$ along that complement, that is identifying each point of $X-\mathbf{N}\times\mathbf{N}$ with its mate in $Y-\mathbf{N}\times\mathbf{N}$. This amalgamated space is a pushout

$$\begin{tikzcd}
\beta\mathbf{N}\times(\beta\mathbf{N}-\mathbf{N}) \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

Since $\mathbf{N}\times\mathbf{N}$ is open in $X$ and $Y$, $\beta\mathbf{N}\times(\beta\mathbf{N}-\mathbf{N})$ is closed and the pushout of two normal spaces by amalgating a common closed subspace is normal. Moreover since $\beta$ is a left adjoint it preserves pushouts, so that $\beta Z$ is the pushout

$$\begin{tikzcd}
\beta\mathbf{N}\times(\beta\mathbf{N}-\mathbf{N}) \arrow[r] \arrow[d] & \beta X \arrow[d] \\
\beta Y \arrow[r] & \beta Z
\end{tikzcd}$$

and $\beta X = \beta Y = \beta(\mathbf{N} \times \mathbf{N})$. Now let $K$ be the pushout

$$\begin{CD}
\beta\mathbf{N} \times (\beta\mathbf{N} - \mathbf{N}) \cup (\beta\mathbf{N} - \mathbf{N}) \times \beta\mathbf{N} @>>> \beta X \\
@VVV @VVV \\
\beta Y @>>> K
\end{CD}$$

which is obviously a compactification of $Z$. Now let $f \in C(Z)$ be defined by $f(n,m) = 1/n$ on $(n,m) \in X$ while $f(n,m) = 0$ on $(n,m) \in Y$, while $f = 0$ on the single copy of $\beta\mathbf{N} \times (\beta\mathbf{N} - \mathbf{N})$. Then it is obvious that $f$ cannot be extended to any point of $K - Z$. Since $Z$ is not open in $K$, this shows that $Z$ is not absolute $\mathcal{CR}$-epic.

## 8. Charactizations of Lindelöf CNP spaces

In the following theorem, $L$ denotes the convergent sequence $1, 1/2, 1/3, \ldots, 0$.

8.1. Theorem. *A Lindelöf space satisfies the CNP if and only if its product with $L$ is absolute $\mathcal{CR}$-epic.*

Proof. The product of Lindelöf CNP spaces is Lindelöf CNP and therefore absolute $\mathcal{CR}$-epic, so that direction is trivial. Conversely, assume that $X \times L$ is absolute $\mathcal{CR}$-epic. Suppose $\{U_n\}$ is a countable family of $\beta X$-open neighbourhoods of $X$. We may assume, without loss of generality, that the sequence is nested. We must show that $U = \bigcap U_n$ is a $\beta X$-neighbourhood of $X$. Define an equivalence relation $E_n$ on $\beta X \times L$. Define $A_n = \beta X - U_n$ and $E_n$ is the equivalence generated by $((p, 1/n), (p, 0)) \in E_n$ whenever $p \in A_n$. Let $E = \bigcup E_n$.

8.2. Lemma. *The set $E$ is closed in $(\beta X \times L) \times (\beta X \times L)$.*

Proof. The map $\beta X \times L \longrightarrow (\beta X \times L) \times (\beta X \times L)$ that sends $(p, s, t)$ to $((p, s), (p, t))$ is clearly a closed embedding. Let $D$ denote the subset consisting of the elements $(p, t, t)$. It is them readily verified that $E$ is the direct image under this map of the set $B$ that is the union of the following four sets:

1. $\{(p, 1/n, 1/m) \mid p \in A_{n \wedge m}\}$;
2. $\{(p, 1/n, 0) \mid p \in A_n\}$;
3. $\{(p, 0, 1/n) \mid p \in A_n\}$;
4. $D$

It suffices to show that $B$ is a closed subset of $X \times L \times L$. Let $(q, 1/n, 1/m) \notin B$ be given. Let $L_n = \{1, 1/2, \ldots, 1/n\}$. Given $(q, 1/n, 1/m) \notin B$, we must have $q \in U_{n \wedge m}$. Clearly $(q, 1/n, 1/m) \in (U_{n \wedge m} \times L_n \times L_m) - D$ and the latter is an open set since $D$ is obviously closed. Suppose that $(p, 1/k, 1/l) \in U_{n\wedge m \times L_n \times L_m} - D$. Then we must have $k \le n$ and $l \le m$, which implies that $k \wedge l \le m \wedge n$, so that $U_{k\wedge l} \supseteq U_{n\wedge m}$ and $p \in U_{k\wedge l}$. It follows that $p \notin B$. No element of the second or third type lies in $(U_{n\wedge m} \times L_n \times L_m)$ since it has 0 in the second or third place and the elements with identical second and third place are explicitly eliminated. The case of an element of the form $(q, 1/n, 0)$ or $(q, 0, 1/n)$ is similar and we will not repeat it. ∎

Proof of 8.1, concluded. Assume that $X \times L$ is absolute $\mathcal{CR}$-epic, We see that $(\beta X \times L)/E$ is a compactification of $X \times L$. Let $f : X \times L \longrightarrow \mathbf{R}$ be the second projection (recall that $L \subseteq [0, 1]$). Since $X \times L$ is absolute $\mathcal{CR}$-epic, $f$ extends to an open set $W \subseteq (\beta X \times L)/E$. Consider the map $\beta X \longrightarrow \beta X \times L$ which sends $p$ to $(p, 0)$. Let $V$ be the inverse image of $W$ under this map. Clearly $V$ is an open subset of $\beta X$ which contains $X$. It suffices to show that $V$ is contained in $\bigcap U_n$ . But if $(p, 0) \in V$ then $p$ must be in $\bigwedge U_n$ otherwise $p \in A_n$ for some $n$ and so $((p, 0), (p, 1/n)) \in E$ which shows that $f$ cannot extend to $(p, 0)$ as $f(p, 0) = 0$ but $f(p, 1/n) = 1/n$. ∎

8.3. Remark. Since a closed subspace of a Lindelöf absolute $\mathcal{CR}$-epic space is absolute $\mathcal{CR}$-epic (use [Barr, *et al.* (2006), Theorem 6.1] in conjunction with the fact that a closed subspace of a Lindelöf space is Lindelöf and therefore $C^*$-embedded), one readily sees that if $L$ is any space that contains a proper convergent sequence and $X \times L$ is Lindelöf absolute $\mathcal{CR}$-epic, then $X$ is Lindelöf CNP.

A space $X$ is said to have the **sequential bounded property** or SBP at the point $p$ if for any sequence $\{f_n\}$ of functions in $C(X)$ there is a neighbourhood of $p$ on which each of the functions is bounded. A space has the SBP if it does so at every point. For example, every locally compact space has this property.

So does every P-space. The easiest way to see this is to let $p \in X$ and let $U_n = \{x \in X \mid |f_n(p) - f_n(x)| < 1\}$. Then $\bigcap U_n$ is a $G_\delta$ containing $p$ and in a P-space, every $G_\delta$ is open.

Since both of these classes of spaces have the CNP, the following characterization comes as no surprise.

8.4. Theorem. *A Lindelöf space is CNP iff it has the SBP at every point.*

Proof. Suppose $X$ is Lindelöf with the CNP and $K$ is a compactification of $X$. Let $f_1, f_2, \ldots$ be a sequence of functions in $C(X)$. We can replace each $f_n$ by $1 + |f_n|$ and assume that they are all positive and bounded away from 0. Let $g_n = 1/f_n$ and $U_n = \operatorname{coz}(g_n)$. The CNP implies that $\bigcap U_n$ is a neighbourhood of $X$. Now let $p \in X$. There is a closed, hence compact, neighbourhood $V$ of $p$ inside $\bigcap U_n$. Since every $g_n$ is non-zero on $V$, it follows that every $f_n$ is bounded there. In particular, every $f_n$ is bounded in $V \cap X$, which is an $X$-neighbourhood of $p$.

Conversely, suppose $X$ satisfies the SBP. If $\{U_n\}$ is a sequence of $K$-neighbourhoods of $X$, the Lindelöf property allows us to choose, for each $n$, a function $f_n$ such that $X \subseteq \operatorname{fin}(f_n) \subseteq U_n$. For each $p \in X$, there is an $X$-open set $V_p$ on which each $f_n$ is bounded, from which it is clear that each $f_n$ is bounded on $W_p = \operatorname{cl}_K(V_p)$, which is a

$K$-neighbourhood of $p$. Since $f_n$ is bounded on $W_p$, it follows that $W_p \subseteq \operatorname{fin}(f_n) \subseteq U_n$ for each $p$ and each $n$, so that $W = \bigcup_{p\in X} V_p$ is a $K$-neighbourhood of $X$ contained in $\bigcap U_n$. ∎

8.5. Definition. *If $f$, $g \in C(X)$ let us say that $g$* **surpasses** *$f$ and write $f \prec g$ if there is a real number $b > 0$ such that $f < bg$.*

8.6. Theorem. *A Lindelöf space has CNP if and only if whenever $f_1$, $f_2$, $\cdots$ is a sequence of functions in $C(X)$, there is a $g \in C(X)$ that surpasses them all.*

Proof. $\Leftarrow$: Every $f_n$ will be bounded on any neighbourhood on which $g$ is bounded.
$\Rightarrow$: Let $f_1$, $f_2$, $\ldots$ be a sequence. We may assume, without loss of generality, that each $f_n > 1$. Using the SBP and the Lindelöf property, there is a countable cover $U_1$, $U_2$, $\ldots$ of $X$ such that for all $n$, $m \in \mathbf{N}$ each $f_n$ is bounded on each $U_m$. Since a Lindelöf space is paracompact, there is a partition of unity $\{t_n\}$ subordinate to the cover. In fact, we may refine the cover and suppose that $U_n = \operatorname{coz}(t_n)$ (see [Kelley (1955), 5W and 5Y]). Let $b_n$ be the sup of $f_n$ on $U_1 \cup U_2 \cup \cdots \cup U_{n-1}$. Define $h_n = f_1 + f_2 + \cdots + f_n$ and $g = \sum_{n\in\mathbf{N}} h_n t_n$. The local finiteness guarantees that this sum is actually finite in a neighbourhood of each point, so continuity is clear.

We next claim that $x \in \operatorname{coz}(t_m)$ implies that $f_n(x) \le b_n(x)h_m(x)$ for all $n$ and $m$. In fact, if $m < n$, then $f(x) \le b_n \le b_n h_m$ on $U_1 \cup U_2 \cup \cdots U_{n-1} \supseteq U_m = \operatorname{coz}(t_m)$. If $m \ge n$, then $f_n$ is one of the summands of $h_m$ so that $f_n \le h_m \le b_m h_m$. Note that the fact that each $f_n \ge 1$ everywhere implies the same for every $h_n$ and $b_n$.

We can now finish the proof. Given a point $x \in X$, let $N(x)$ denote the finite set of indices $n$ for which $t_n(x) \ne 0$. Then for all $m$,

$$
\begin{aligned}
b_n g(x) &= \sum_{n\in\mathbf{N}} b_n h_n(x) t_n(x) = \sum_{n\in N(x)} b_n h_n(x) t_n(x) \\
&\ge \sum_{n\in N(x)} f_m(x) t_n(x) = \sum_{n\in\mathbf{N}} f_m(x) t_n(x) = f_m(x)
\end{aligned}
$$

8.7. Example. Here is a nice application of Theorem 8.4. Say that a space satisfies the **open refinement condition or ORC** if the finite union closure of every ample $G_\delta$ cover has an open refinement. We explore this condition in some detail in [Barr, *et al.*, (to appear)]. All P-spaces and all locally compact spaces satisfy it and it is closed under finite products, closed subspaces and perfect images and preimages.

8.8. Theorem. *For Lindelöf spaces, ORC implies CNP.*

Proof. Let $X$ be Lindelöf and satisfy the ORC and let $f_1$, $f_2, \ldots$ be a sequence of functions in $C(X)$. For each compact set $A \in X$ and each $n \in \mathbf{N}$, let $b_n(A) = \sup_{x\in A} |f_n(x)|$. Let $U_n(A) = \{x \in X \mid |f_n(x)| < b_n(A) + 1\}$ and $U(A) = \bigcap_{n\in\mathbf{N}} U_n(A)$. Then $U(A)$ is a $G_\delta$ containing $A$. The cover by the set of $U(A)$, taken over all the compact subsets of $X$, is an ample $G_\delta$ cover. Each $f_n$ is bounded on each $U(A)$ and hence is bounded on the union of any finite number of them. Therefore each $f_n$ is bounded on each set in an open refinement of $\{U(A)\}$. ∎

## 9. Outgrowths

The **outgrowth** of a Tychonoff space $X$ is the space $\beta X - X$. In this section, we explore some of the ways a space and its outgrowth influence each other.

9.1. Lemma. *Let $K$ be a compactification of $X$. Then $K - X$ is countably compact if and only if $\beta X - X$ is.*

Proof. Let $\theta : \beta X \longrightarrow K$ be the canonical quotient map. If $\beta X - X$ is countably compact, then its image, $K - X$ clearly is too. To go the other way, we have from [Kelley (1955), 5E(a)] that if $\beta X - X$ is not countably compact, there is a sequence $S = \{p_1, p_2, p_3, \ldots\}$ that has no cluster point in $\beta X - X$. But then every cluster point in $\beta X$ of the sequence lies in $X$. Either $S$ has an infinite subsequence all of whose images in $K$ are the same or $S$ has an infinite subsequence all of whose in $K$ are distinct. We can suppose without loss of generality that $S$ has one of those properties. In the first case, if $x$ is any cluster point of $S$ in $X$, it is immediate that $\theta(S) = x$, a contradiction. In the second case, $\mathrm{cl}_{\beta X}(S) \subseteq S \cup X$ and hence $\mathrm{cl}_K(\theta(S)) \subseteq \theta(\mathrm{cl}_{\beta X}(S)) \subseteq \theta(S) \cup S$ which implies that $S$ has no limit point in $K - X$. ∎

The proofs in the following are exercises.

9.2. Theorem. *Of the following conditions on a space $X$ and a compactification $K$,*

1. *$X$ is a P-set in $K$;*
2. *The closure in $K - X$ of every $\sigma$-compact subset of $K - X$ is compact;*
3. *The closure in $K - X$ of every countable subset of $K - X$ is compact;*
4. *$K - X$ is countably compact;*
5. *$K - X$ is pseudocompact.*

*we have $1 \Leftrightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$.*

Usually, *the* outgrowth of a space $X$ is just $\beta X - X$. Here we will call any space of the form $K - X$ for any compactification of $X$ *an* outgrowth of $X$. We will say that $Y$ is *co-Lindelöf* if it is an outgrowth of a Lindelöf space.

9.3. Theorem. *If $Y$ is an outgrowth of a Lindelöf (resp. Alster's condition) space, then any outgrowth of $Y$ is Lindelöf (resp. Alster's condition). Conversely, if some outgrowth of $Y$ is Lindelöf (resp. Alster's condition), then $Y$ is an outgrowth of a Lindelöf (resp. Alster's condition) space.*

Proof. Suppose $X$ is a Lindelöf space and $K$ is a compactification of $X$ with $Y = K - X$. Then $L = \mathrm{cl}_K(Y)$ is a compactification of $Y$ and $L - Y = L \cap X$ is closed in $X$ and hence Lindelöf. The Alster's condition case goes the same way.

For the converse, suppose $L$ is a compactification of $Y$ with $L - Y$ Lindelöf. Let $\mathbf{N}^*$ be the one point compactification of $\mathbf{N}$ and $K = \mathbf{N}^* \times L$. Embed $Y$ as $\{\infty\} \times Y$ and let $X = K - Y$. Clearly $K$ is a compactification of $X$ with outgrowth $Y$. Finally, $X = (\mathbf{N} \times L) \cup \{\infty\} \times (L - Y)$ is the union of countably many Lindelöf spaces and is therefore Lindelöf. The argument with Alster's condition is similar. ∎

9.4. Theorem. *A space with a locally compact outgrowth is the union of a compact subset and a locally compact subset.*

Proof. Let $X \subseteq K$ be a compactification such that $Y = K - X$ is locally compact. Let $L = \mathrm{cl}_K(Y)$. Since $Y$ is locally compact it is open in $L$ and hence $L - Y$ is a compact subset of $X$. If $p \notin L - Y$, then $p$ has a $K$-neighbourhood that does not meet $Y$ and a $K$-closed $K$-neighbourhood inside it. Such a neighbourhood is a compact $K$-neighbourhood of $p$ inside $X$. ∎

9.5. Theorem. *A locally Alster's condition outgrowth of a Lindelöf CNP space is locally compact.*

Proof. Suppose that $X$ is a Lindelöf CNP space, $K$ is a compactification of $X$ and $Y = K - X$. Assume that each $p \in Y$ has an Alster's condition neighbourhood. Let $F$ denote the family of all $f \in C(K)$ that vanish nowhere on $X$. Then $\{\mathbf{Z}[f] \mid f \in F\}$ is readily seen to be an ample $G_\delta$ cover of $Y$, since for each compact set $A \subseteq Y$ there is an $f \in F$ that vanishes on $A$ (Smirnov). If $p \in Y$, there is a neighbourhood $U$ of $p$ that is covered by a countable family of $Z[f]$. This means that $U \subseteq \bigcup \mathbf{Z}[f_n]$. But then $\bigcap \mathrm{coz}(f_n)$ is a $G_\delta$ that contains $X$ and by CNP there is an open $V \subseteq K$ such that $X \subseteq U \subseteq \bigcap \mathrm{coz}(f_n)$ and then $K - V \supseteq U$ is a compact neighbourhood of $p$. ∎

## 10. SCZ spaces

We will say that a space satisfies the *SCZ* condition if every $\sigma$-compact subset is contained in a compact zeroset. This actually comprises two separate conditions:

SCZ-1. The closure of any $\sigma$-compact set is compact;

SCZ-2. Every compact set is contained in a compact zeroset.

10.1. Proposition. *A space that satisfies SCZ-1 is pseudocompact.*

Proof. Suppose $Y$ is such a space and $f \in C(Y)$ is unbounded. Choose points $p_1, p_2, \ldots, p_n, \ldots$ such that $|f(p_n)| > n$. The set $\{p_1, p_2, \ldots\}$ is discrete and not compact, but its closure is compact, and if $p$ is any point in its frontier, it is clear that $f(p)$ cannot be defined. ∎

10.2. Proposition. *If a space satisfies the CNP, any outgrowth satisfies SCZ-1.*

Proof. If $X$ satisfies the CNP and $K$ is a compactification, let $Y = K - X$. If $A$ is a $\sigma$-compact subset of $Y$, then $K - A$ is a $G_\delta$ set containing $X$. If $X$ satisfies the CNP, then there is an open set $U$ such that $X \subseteq U \subseteq K$ and then $K - U$ is a compact subset of $Y$ that contains $A$. ∎

10.3. Proposition. *If a space is Lindelöf, any outgrowth satisfies SCZ-2.*

Proof. If $X$ is Lindelöf and $K$ is a compactification, let $Y = K - X$. If $A$ is a compact set in $Y$, then $K - A$ is an open set that contains $X$. Since $X$ is Lindelöf, there is a cozero set $U$ such that $X \subseteq U \subseteq K - A$. Then $K - U$ is closed, hence compact, in $K$ and evidently $A \subseteq K - U \subseteq Y$. ∎

10.4. Proposition. *The outgrowth of any space that satisfies SCZ-1 satisfies the CNP.*

Proof. If $Y$ satisfies SCZ-1 and $K$ is a compactification, let $W = K - Y$. If $U = \bigcap_{n\in\mathbf{N}} U_n$ is a $G_\delta$ containing $W$, with each $U_n$ open, then $K - U = \bigcup(K - U_n)$ is a $\sigma$-compact set in $Y$ and hence contained in some compact set $A$. But then $W \subseteq K - A \subseteq U$. ∎

10.5. Proposition. *The outgrowth of any space that satisfies SCZ-2 is Lindelöf.*

Proof. If $Y$ satisfies SCZ-2 and $K$ is a compactification, let $W = K - Y$. We will first consider the case that $K = \beta Y$. It will suffice to show that any open subset of $K$ that contains $W$ contains a cozero-set containing $W$ since that will certainly be true of $\mathrm{cl}_K(W)$. If $U$ is open and $W \subseteq U$, $K - U$ is closed in $K$ and hence compact and $K - U \subseteq Y$. Then there is a function $f : Y \longrightarrow [0, 1]$ such that $K - U \subseteq \mathbf{Z}[f]$. Since $K = \beta Y$, $f$ extends to all of $K$. We claim that $\mathbf{Z}[f]$ does not meet $W$. For suppose that $p \in W$ with $f(p) = 0$. There is a function $g : K \longrightarrow [0, 1]$ such that $g(p) = 0$ and $g|(K - U) = 1$. Then $f + g$ vanishes nowhere on $Y$ since $g = 1$ wherever $f = 0$. But $1/(f + g)$ is bounded on $Y$ and hence bounded on $K = \mathrm{cl}_K(Y)$ and therefore $f + g$ cannot vanish anywhere on $W$, in particular at $p$. Thus $\mathbf{Z}[f]$ is a compact zero-set in $K$ that does not meet $W$ and then $W \subseteq K - \mathbf{Z}[f] \subseteq U$. This takes care of the case that $K = \beta Y$. For the general case, we know that $\theta : \beta Y \longrightarrow K$ is perfect and that $\theta^{-1}(K - Y) = \beta Y - Y$. If $\mathcal{U}$ is an open cover of $W = K - Y$, then $\{\theta^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of $\beta Y - Y$ from which we can extract a countable cover $\theta^{-1}(U_1), \theta^{-1}(U_2), \dots$ and it follows that $U_1, U_2, \dots$ is a cover of $K - Y$ that is a subcover of $\mathcal{U}$. ∎

For the rest of this section, $Y$ will be a space that satisfies part or all of SCZ, $K$ will be a compactification of $Y$, $W = K - Y$, and $L = \mathbf{N}^* \times K$. We will embed $K \longrightarrow L$ as $\{\infty\} \times K$ and similarly for $Y$ and $W$. We let $X = (\mathbf{N} \times K) \cup (\{\infty\} \times W) = L - \{\infty\} \times Y$.

10.6. Proposition. *If $W$ is Lindelöf, so is $X$.*

Proof. Since $\mathbf{N} \times K$ is $\sigma$-compact and $W$ is Lindelöf, it is obvious. ∎

10.7. Proposition. *If $W$ satisfies the CNP, so does $X$.*

Proof. Suppose $X \subseteq U = \bigcap U_n$ with each $U_n$ open in $L$. Then $W \subseteq \bigcap(K \cap U_n)$ and each $K \cap U_n$ is open in $K$. Since $W$ satisfies the CNP, there is an open $V \subseteq K$ such that $W \subseteq V \subseteq U \cap K$. But then $\mathbf{N}^* \times V$ and $\mathbf{N} \times K$ are open in $L$ and

$$X \subseteq (\mathbf{N}^* \times V) \cup \mathbf{N} \times K \subseteq U \qquad \blacksquare$$

10.8. Theorem. *The outgrowth of any Lindelöf CNP space satisfies SCZ; any space that satisfies the SCZ is the outgrowth of a Lindelöf CNP space.* $\blacksquare$

It does not follow that a space whose outgrowth satisfies SCZ is Lindelöf (although it is CNP). An outgrowth of any locally compact space, in particular of any discrete space, is compact and trivially satisfies SCZ, but the original space need not be Lindelöf. As for the CNP claim, any space $X = U \cup V$ where $U$ consists of the locally compact points and $V$ ????????????? is open. If an outgrowth satisfies SCZ, then its outgrowth is $V$, which therefore satisfies the CNP and it is immediate that the union of a CNP space and an open locally compact space is CNP.

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